A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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TRANSVERSALITY CONDITIONS FOR CONTROLS WITH BOUNDED VARIATION

In the present paper there has been obtained a necessary condition for the existence of extremum for the following optimization problem: find the extremum of the functional $\int_0^1 \Phi(x,u,t)dt$ under the conditions $\dot{x} = \varphi(x,u,t)$, $G^0(x(0)) = 0$, $G^1(x(1)) = 0$, where x is an absolutely continuous function, while u is function with bounded variation, taking values from a given set P.

INTRODUCTION

Let us consider the following problem:

$$I(x,u) = \int_0^1 \Phi(x,u,t)dt \to \min,$$

$$\dot{x} = \varphi(x,u,t),$$

$$G^0(x(0)) = 0, \quad G^1(x(1)) = 0,$$

$$u(t) \in P.$$

This is an optimization problem with non-fixed ends, one of the fundamental problems in optimization theory. The aim of the present paper is to give necessary conditions for the problem formulated above, with an additional constraint: the control u possesses a finite variation. Such conditions for a problem with fixed ends were given by the author in paper [4]. In papers [4] and [5], the purposefulness of investigating problems of this type was justified, as well as the literature concerning these problems was given.

Since we are going to make use of the Dubovitskii-Milyutin method, we shall briefly discuss its essence.

Let E be a linear-topological space, and Q_i, $i = 1, 2, \ldots, n$, subsets of this space. By F we shall denote a functional defined on E. Consider the following

Problem. Determine the minimal value of the functional F on a set

$$Q = \bigcap_{i=1}^{n} Q_i .$$

In order to be able to formulate the fundamental theorem od Dubovitskii-Milyutin, we shall give a few definitions and notations.

Definition 0.1. A set $Q \subset E$ is called an inequality constraint if $Q^o \neq \emptyset$ (Q^o - the interior of the set Q); otherwise, Q is called an equality constraint.

Definition 0.2. A vector $h \in E$ is called a direction of decrease of the functional F at the point $x_o \in E$ if there exist a neighbourhood U of the vector h and numbers $\varepsilon_o > 0$, $\alpha < 0$, such that, for any $\bar{h} \in U$ and any $\varepsilon \in (0, \varepsilon_o)$, the inequality

$$F(x_o + \varepsilon \bar{h}) < F(x_o) + \varepsilon \alpha$$

holds.

Definition 0.3. Let Q be an inequality constraint. The vector $h \in E$ is called a feasible direction for the set Q at the point x_o if there exist a neighbourhood U of the vector h and some $\varepsilon_o > 0$, such that, for any $\bar{h} \in U$ and $\varepsilon \in (0, \varepsilon_o)$, we have $x_o + \bar{h} \in Q$.

Definition 0.4. Let Q be an equality constraint. The vector $h \in E$ is called a direction tangent to Q at the point x_o if there exists $\varepsilon_o > 0$ such that, for any $\varepsilon \in (0, \varepsilon_o)$ there exists $x = x(\varepsilon) \in Q$ such that $x(\varepsilon) = x_o + \varepsilon h + r(\varepsilon)$, where $(1/\varepsilon)\, r(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

By K_d, K_f and K_t we shall denote, respectively, the set of directions of decrease of the functional F at the point x_o, the set of feasible directions for the set Q and the set of directions tangent to Q at the point x_o. It turns out that the sets K_d and K_f are open cones with apices at 0, whereas K_t is a cone.

Definition 0.5. The functional F is called regularly decreasing at the point $x_o \in E$ if the cone K_d is convex. Analogously, an inequality (equality) constraint is called regular if K_f (K_t) is a convex cone.

Let E^* stand for a space dual to E. The set of all functionals $f \in E^*$ such that $f(x) \geq 0$ for $x \in K$ will be denoted by K^*. It is easily noticed that K^* is a convex cone with apex at 0. This cone will be called a cone dual to K.

The basic theorem of the Dubovitskii-Milyutin method goes as follows:

Theorem 0.1. Let F be a functional defined on E, and $Q_i \subset E$, $i = 1, \dots, n+1$, subsets of the space E. Let $Q_i^o \neq \emptyset$ for $i = 1, \dots, n$ (Q_i - inequality constraints), and $Q_{n+1}^o \neq \emptyset$ (Q_{n+1} - an equality constraint). By K_o, K_i, $i = 1, \dots, n$, K_{n+1} we shall denote, successively: the cone of directions of decrease of the functional F at the point x_o, the cones of feasible directions and the cone of tangent directions. Assume that these cones are convex. If x_o an argument of the local minimum of the functional F on the set

$$Q = \bigcap_{i=1}^{n+1} Q_i,$$

then there exist functionals $f_o, f_1, \dots, f_{n+1}$ not vanishing simultaneously and such that

(0.1) $$\sum_{i=0}^{n+1} f_i = 0,$$

(0.2) $$f_i \in K_i^*, \quad i = 0, 1, \ldots, n+1.$$

Equation (0.1) is called the Euler-Lagrange equation.

1. FORMULATION OF THE PROBLEM

Denote by $V([a,b])$ a set of functions with finite variation on the interval $[a,b]$. In the case $[a,b] = [0,1]$, the set $V([0,1])$ will be denoted by V. For any arbitrary function $g \in V$, we shall define the functional by the following formula:

(1.1) $$\|g\| = |g(0)| + \bigvee_0^1 g,$$

where $\bigvee_0^1 g$ stands for a variation of the function g on the interval $[0,1]$. It is to be proved that the functional thus defined is a norm in the space V, and that the space V with the norm so defined is a Banach space (cf. [3], XII, § 3).

Denote by V^r a space of vector functions $g = (g^1, g^2, \ldots, g^r)$ defined on the interval $[0,1]$ and such that $g^i \in V$ for each $i = 1, \ldots, r$, that is, $V^r = \underbrace{V \times \ldots \times V}_{r \text{ times}}$.

The space V^r with the norm defined by the formula

(1.2) $$\|g\|_r = \sqrt{\sum_{i=1}^{r} \|g^i\|^2}$$

is a Banach space.

Definition 1.1. By vector of variation of the function $g = (g^1, \ldots, g^r)$ we shall mean the vector

$$\overset{1}{\underset{0}{\nabla}} g =: (\overset{1}{\underset{0}{\nabla}} g^1, \overset{1}{\underset{0}{\nabla}} g^2, \dots, \overset{1}{\underset{0}{\nabla}} g^r).$$

Let S be a compact subset of the positive cone of the space R^r. Let us recall that S is called normal with respect to each axis if and only if, together with any point $(s^1, \dots, s^r)$ belonging to S, also a point $(s^1, \dots, \lambda s^i, \dots, s^r)$ for each $\lambda \in [0,1]$, $i = 1, \dots, r$, belongs to S.

Let

$$U = \{u \in V^r : u(t) \in P, \ \overset{1}{\underset{0}{\nabla}} u \in S\},$$

where P is a closed convex subset of R^r with non-empty interior, while S is convex and compact in R^r, normal with respect to each axis.

The set U will be called a set of admissible controls, and its elements - admissible controls.

From the assumptions on the sets P and S, as well as from the fact that convergence in norm in the space V^r implies uniform convergence, follows at once

Lemma 1.2. The set U of admissible controls is a closed convex set with a non-empty interior U^o.

Consider the following

Problem I. Find the minimum of the functional

$$I(x,u) = \int_0^1 \Phi(x(t),u(t),t)dt, \tag{1.3}$$

under the conditions

$$\dot{x}(t) = \varphi(x(t),u(t),t), \tag{1.4}$$

$$G^o(x(0)) = 0, \quad G^1(x(1)) = 0, \tag{1.5}$$

$$u(\cdot) \in U, \tag{1.6}$$

where $\Phi : R^n \times R^r \times [0,1] \to R$, $\varphi : R^n \times R^r \times [0,1] \to R^n$, $x(\cdot) \in W^n_{11}$, $G^o : R^n \to R^k$, $G^1 : R^n \to R^1$, $k,1 \le n$. W^n_{11} stands for a space of absolutely continuous functions with norm

$$\|x\| = |x(0)| + \int_0^1 |\dot{x}(t)|dt. \tag{1.7}$$

On the functions Φ and φ we shall assume:

1^o $\Phi(x,u,\cdot)$ - measurable with fixed $(x,u) \in R^n \times R^r$,

2^o there exist Gâteaux derivatives with respect to x and u which will be denoted by $\nabla\Phi_x$, $\nabla\Phi_u$,

3^o for each compact set Ω contained in $R^n \times R^r$, there exists an $M > 0$ such that the moduli of $\nabla\Phi_x$ and $\nabla\Phi_u$ are bounded by M for each $(x,u) \in \Omega$,

4^o one of the following conditions is satisfied:

a) $\nabla\Phi_x$ is continuous with respect to u for $(x,u) \in R^n \times R^r$, $t \in [0,1]$,

b) $\nabla\Phi_u$ is continuous with respect to x for $(x,u) \in R^n \times R^r$,

5^o $\varphi(x,u,\cdot)$ - measurable with fixed $(x,u) \in R^n \times R^r$,

6^o there exist Fréchet derivatives continuous in the set $R^n \times R^r \times [0,1]$ with respect to (x,u), with fixed $t \in [0,1]$,

7^o there exist some $\varepsilon > 0$ as well as $r_1(\cdot)$ and $r_2(\cdot) \in L_1$, such that

$$|\varphi_x(x,u,t)| \le r_1(t), \quad |\varphi_u(x,u,t)| \le r_2(t)$$

for any $t \in [0,1]$ and any x and u satisfying the conditions

$$|x - x_o(t)| \le \varepsilon \quad \text{and} \quad |u - u_o(t)| \le \varepsilon,$$

where $x_o(\cdot)$ and $u_o(\cdot)$ are fixed functions,

8^o $G^o(\cdot)$ and $G^1(\cdot)$ are regular mappings, i.e. of class C^1, and the rows of matrics $G^{o'}(x)$ and $G^{1'}(x)$ are linearly independent vectors for each $x \in R^n$.

To the problem thus formulated we shall apply the Dubovitskii-Milyutin method.

All considerations will be carried out in the space $E = W^n_{11} \times V^r$ with norm

$$\|(x,u)\| = \sqrt{\|x\|^2_w + \|u\|^2_w}, \tag{1.8}$$

where $\|x\|_w$ and $\|u\|_v$ are defined by formulas (1.7) and (1.2), respectively.

It turns out that if we adopted $E = W^n_{11} \times L^r_\infty$, then the set of those (x,u) for which u satisfies condition (1.6), after replacing the space V^r by L^r_∞ in the definition of the set U, would have no interior points. Constraint (1.6) would therefore be (besides constraint (1.4)) an equality one, whereas the Dubovitskii-Milyutin method admits only one constraint of this type.

The adoption of the space $E = W^n_{11} \times V^r$ with the topology defined by norm (1.8) allows us to find, on account of *Lemma 1.2*, that constraint (1.6) is an inequality one, and so, to *Problem I* the Dubovitskii-Milyutin method may be applied.

2. ANALYSIS OF THE FUNCTIONAL

Consider in the space $E = W^n_{11} \times V^r$ a functional defined by the formula

$$I(x,u) = \int_0^1 \Phi(x(t),u(t),t)dt. \tag{2.1}$$

We shall prove the following

Lemma 2.1. If assumptions 1°-4° are satisfied, then there exists the Gâteaux derivative of the functional I, expressed by the formula

$$I(x_o,u_o)(\bar{x},\bar{u}) = \int_0^1 [(\nabla\Phi_x(x_o(t),u_o(t),t),\bar{x}(t)) + \tag{2.2}$$

$$+(\nabla\Phi_u(x_o(t),u_o(t),t),\bar{u}(t))]\,dt,$$

and the cone K_o of directions of decrease at the point $(x_o(\cdot),$ $u_o(\cdot))$ is convex and defined by

$$K_o = \{(\bar{x},\bar{u}) \in E : \nabla I(x_o,u_o)(\bar{x},\bar{u}) < 0\}. \tag{2.3}$$

P r o o f. Consider the quotient

$$\frac{I(x_o + \varepsilon\bar{x}, u_o + \varepsilon\bar{u}) - I(x_o,u_o)}{\varepsilon} = \int_0^1 f(t,\varepsilon)\,dt, \tag{2.4}$$

where

$$f(t,\varepsilon) = \frac{\Phi(x_o(t) + \varepsilon\bar{x}(t), u_o(t) + \varepsilon\bar{u}(t),t) - \Phi(x_o(t),u_o(t),t)}{\varepsilon}.$$

We shall calculate the limit of the function $f(t,\varepsilon)$ as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} f(t,\varepsilon) =$$

$$= \lim_{\varepsilon\to 0}\frac{\Phi(x_o(t) + \varepsilon\bar{x}(t), u_o(t) + \varepsilon\bar{u}(t),t) - \Phi(x_o(t),u_o(t),t)}{\varepsilon} =$$

$$= \lim_{\varepsilon\to 0}\frac{\Phi(x_o(t) + \varepsilon\bar{x}(t), u_o(t) + \varepsilon\bar{u}(t),t) - \Phi(x_o(t),u_o(t) + \varepsilon\bar{u}(t),t)}{\varepsilon} +$$

$$+ \lim_{\varepsilon\to 0}\frac{\Phi(x_o(t),u_o(t) + \varepsilon\bar{u}(t),t) - \Phi(x_o(t),u_o(t),t)}{\varepsilon} =$$

$$= \lim_{\varepsilon\to 0}\left[\frac{(\nabla\Phi_x(x_o(t),u_o(t) + \varepsilon\bar{u}(t),t),\varepsilon\bar{x}(t))}{\varepsilon} + \frac{o(\varepsilon)}{\varepsilon}\right] +$$

$$+ \lim_{\varepsilon\to 0}\left[\frac{(\nabla\Phi_u(x_o(t),u_o(t),t),\varepsilon\bar{u}(t))}{\varepsilon} + \frac{o(\varepsilon)}{\varepsilon}\right].$$

Making use of assumption 4^oa) (the continuity of the function $\nabla\Phi_x$ with respect to u), we get

(2.5) $$\lim_{\varepsilon \to 0} f(t,\varepsilon) = (\nabla\Phi_x(x_o(t),u_o(t),t),\bar{x}(t)) + + (\nabla\Phi_u(x_o(t),u_o(t),t),\bar{u}(t)).$$

In the case when assumption 4^ob) is satisfied, we would obtain formula (2.5) in an analogous way by adding and subtracting $\Phi(x_o(t) + \varepsilon\bar{x}(t),u_o(t),t)$.

Let us still observe that, in virtue of the mean value theorem, we have

$$|f(t,\varepsilon)| \leq \left|\frac{1}{\varepsilon}\int_0^1 (\nabla\Phi_x(x_o(t) + \varepsilon s\bar{x}(t),u_o(t) + \varepsilon\bar{u}(t),t),\varepsilon\bar{x}(t))ds\right| +$$

$$+ \left|\frac{1}{\varepsilon}\int_0^1 (\nabla\Phi_u(x_o(t) + \varepsilon\bar{x}(t),u_o(t) + \varepsilon s\bar{u}(t),t),\varepsilon\bar{u}(t))ds\right|$$

and, after successive application of the Schwarz inequality, assumption 3^o and the Cauchy inequality, we get

$$|f(t,\varepsilon)| \leq M\sqrt{2}\|\bar{x}(\cdot),\bar{u}(\cdot)\|.$$

Consequently, the assumptions of Lebesque's theorem on passing with a limit under the integral sign are satisfied. Performing this passing on the right-hand side of (2.4), we obtain a formula for the directional derivative of the functional I in the direction $(\bar{x},\bar{u})$.

This derivative is linear, therefore convex and continuous; it is a Gâteaux derivative, which proves the verity of (2.2). From *Lemma 7.1* and *Theorem 7.2* [1] the convexity of the cone of directions of decrease of the functional I and formula (2.3) follow, which concludes the proof.

3. ANALYSIS OF EQUALITY CONSTRAINTS

Let the differential equation

$$\frac{d\bar{x}(t)}{dt} = A(t)\bar{x}(t) + B(t)\bar{u}(t), \quad t \in [0,1], \tag{3.1}$$

with the initial condition

$$\bar{x}(0) = 0 \tag{3.2}$$

be given, where $\bar{x}(t) \in R^n$, $\bar{u}(t) \in R^r$, the matrices $A(\cdot)$, $B(\cdot)$ are integrable. With equation (3.1) we shall associate the set

$$D = \{z \in R^n : z = \bar{x}(1), \bar{x}(t) \text{ satisfies (3.1) with condition (3.2)}, \bar{u}(\cdot) \text{ - any element of } V^r\}.$$

Definition 3.1. System (3.1) with condition (3.2) is said to be fully controllable in the space V^r if $D = R^n$. The following assertion is true:

Assertion 3.2. If, for each non-zero function $\Psi(t)$ being a solution to the equation

$$\frac{d\Psi(t)}{dt} = -A^*(t)\ \Psi(t), \tag{3.3}$$

the condition $B^*(t)\ \Psi(t) \not\equiv 0$ (to be more precise, $B^*(t)\ \Psi(t)$ is different from zero on the set of positive measure) is satisfied, then system (3.1) with condition (3.2) is fully controllable in the space V^r.

Consider in the space $E = W^n_{11} \times V^r$ a set

$$Q = \{(x,u) \in E : \dot{x}(t) = \tag{3.4}$$

$$= \varphi(x(t),u(t),t),\ G^0\ (x(0)) = 0,\ G^1(x(1)) = 0\}.$$

We shall prove the following

Lemma 3.3. If assumptions 1°–8° are satisfied, and the system

$$\frac{d\bar{x}(t)}{dt} = \varphi_x(x(t),u(t),t)\bar{x}(t) + \varphi_u(x(t),u(t),t)\bar{u}(t) \tag{3.5}$$

with the initial condition $\bar{x}(0) = 0$ is fully controllable in the space V^r, then the cone K of directions tangent to the set Q at the point (x_o,u_o) is a subspace of the form

$$K = \{(\bar{x},\bar{u}) \in E : \frac{d\bar{x}(t)}{dt} = \varphi_x(x_o(t),u_o(t),t)\bar{x}(t) + \varphi_u(x_o(t),u_o(t),t)\bar{u}(t),\ G^{o\prime}(x_o(0))\bar{x}(0) = 0,\ G^{1\prime}(x_o(1))\bar{x}(1) = 0\}. \tag{3.6}$$

P r o o f. It can be shown that, under the assumptions imposed on the function φ, the mapping $\bar{\varphi} : W^n_{11} \times V^r \to L^n_1$ defined by the formula

$$[\bar{\varphi}(x(\cdot),u(\cdot))](t) = \varphi(x(t),u(t),t) \quad \text{for each } t \in [0,1] \tag{3.7}$$

is Fréchet differentiable at the point (x_o,u_o) and its derivative is expressed by

$$[\bar{\varphi}'(x_o(\cdot),u_o(\cdot))(\bar{x}(\cdot),\bar{u}(\cdot))](t) = \varphi_x(x_o(t),u_o(t),t)\bar{x}(t) + \varphi_u(x_o(t),u_o(t),t)\bar{u}(t). \tag{3.8}$$

Consider a mapping $P : W^n_{11} \times V^r \to L^n_1 \times R^k \times R^l$ defined by the formula

$$[P(x(\cdot),u(\cdot))](t) = (\dot{x}(t) - \varphi(x(t),u(t),t), G^o(x(0)), G^1(x(1))). \tag{3.9}$$

In this situation, the set Q defined by (3.4) may be written down as follows:

$$Q = \{(x,u) \in E : P(x,u) = 0\}.$$

Taking into account formula (3.8) as well as those of § 0.2 [2], it can be ascertained that the mapping P is Fréchet differentiable at the point (x_o, u_o) and the derivative of this mapping is expressed by

$$(3.10)\qquad [P'(x_o(\cdot),u_o(\cdot))(\bar{x}(\cdot),\bar{u}(\cdot))](t) =$$

$$= (\dot{\bar{x}}(t) - \varphi_x(x_o(t),u_o(t),t)\bar{x}(t) -$$

$$- \varphi_u(x_o(t),u_o(t),t)\bar{u}(t), G^{o\prime}(x_o(0))\bar{x}(0), G^{1\prime}(x_o(1))\bar{x}(1)).$$

We shall now examine the regularity of the mapping P at the point (x_o, u_o) (i.e. we shall investigate whether the condition $\operatorname{Im} P'(x_o(\cdot),u_o(\cdot))\ (E) = L_1^n \times R^k \times R^l$ is satisfied). Let $(a(\cdot), A, B)$ be an arbitrary element of the space $L_1^n \times R^k \times R^l$. In virtue of assumption 8^o, for any $A \in R^k$, there exists some $A^1 \in R^n$ such that $G^{o\prime}(x_o(0))A^1 = A$ and, for any $B \in R^l$, there exists some $B^1 \in R^n$ such that $G^{1\prime}(x_o(1))B^1 = B$. Consider a differential equation

$$(3.11)\qquad \dot{z}(t) = \varphi_x(x_o(t),u_o(t),t)z(t) + a(t)$$

with the initial condition

$$(3.12)\qquad z(0) = A^1.$$

The function $\varphi_x(x_o(t),u_o(t),t)$ is integrable by assumption 7^o. Consequently, there exists a unique solution to equation (3.11), satisfying initial condition (3.12) (th. of § 0.4 [2]). Denote this solution by $\bar{z}(t)$.

By hypothesis, system (3.5) is fully controllable. So, there exist functions $\bar{u}(\cdot) \in V^r$ and $\bar{y}(\cdot) \in W_{11}^n$, such that the equation

$$(3.13)\qquad \frac{d\bar{y}(t)}{dt} = \varphi_x(x_o(t),u_o(t),t)\bar{y}(t) + \varphi_u(x_o(t),u_o(t),t)\bar{u}(t)$$

with the boundary conditions

$$(3.14)\qquad \bar{y}(0) = 0, \qquad \bar{y}(1) = B^1 - \bar{z}(1)$$

is satisfied.

Let us now put $\bar{x}(t) = \bar{y}(t) + \bar{z}(t)$. Evidently, $\bar{x}(\cdot) \in W^n_{11}$, After some easy transformations we get

$$(3.15) \qquad [P'(x_o(\cdot),u_o(\cdot))(\bar{x}(\cdot),\bar{u}(\cdot))](t) = (a(t),A,B).$$

We have thus found the element $(\bar{x},\bar{u}) \in E$ which is the pre-image of the element $(a(\cdot),A,B)$; that means the regularity of the mapping P at the point $(x_o(\cdot),u_o(\cdot))$. By the Lyusternik theorem, we obtain the proposition of the lemma.

4. THE INTEGRAL MAXIMUM CONDITION

Denote by H a Hamilton function

$$(4.1) \quad H(x,u,\psi,\lambda,t) = (\psi(t),\varphi(x(t),u(t),t)) - \lambda\Phi(x(t),u(t),t),$$

where $\psi(t)$ is an absolutely continuous function satisfying the conjugate equation

$$(4.2) \qquad \frac{d\psi(t)}{dt} = -\varphi^*_x(x(t),u(t),t)\,\psi(t) + \lambda\nabla\Phi_x(x(t),u(t),t),$$

and λ is some non-negative constant.

Under the assumptions imposed on the functions Φ and φ, there exists the Gâteaux derivative of the Hamilton function with respect to u which is expressed by the formula

$$(4.3) \quad \nabla H_u(x,u,\Psi,\lambda,t) = \varphi^*_u(x(t),u(t),t)\Psi(t) - \lambda\nabla\Phi_u(x(t),u(t),t).$$

The following theorem holds:

Theorem 4.1. Let assumptions 1^o-8^o be satisfied. If (x_o,u_o) is a solution to *Problem I,* then there exist

1) a constant $\lambda_o \geq 0$,

2) vectors $\lambda^o \in R^k$, $\lambda^1 \in R^1$, and

3) an absolutely continuous function ψ_o satisfying the equation

$$(4.4) \qquad \frac{d\psi_o(t)}{dt} = -\varphi^*_x(x_o(t),u_o(t),t)\psi_o(t) + \lambda_o\nabla\Phi_x(x_o(t),u_o(t),t)$$

with the transversality conditions

$$\psi_o(0) = -G^{o'*}(x_o(0))\lambda^o, \quad \psi_o(1) = G^{1'*}(x_o(1))\lambda^1, \tag{4.5}$$

such that there cannot be simultaneously $\lambda_o = 0$, $\lambda^o = 0$, $\lambda^1 = 0$, $\psi_o(t) \equiv 0$ for which the relationship

$$\int_0^1 (\nabla H_u(x_o, u_o, \psi_o, \lambda_o, t), u_o(t))dt = \max_{\bar{u} \in U} \int_0^1 (\nabla H_u(x_o, u_o, \psi_o, \lambda_o, t), \bar{u}(t))dt \tag{4.6}$$

is satisfied.

P r o o f. Let, us formerly, $E = W_{11}^n \times V^r$. Let us introduce the notations

$$Q_1 = \{(x,u) \in E : x(\cdot) \in W_{11}^n, u(\cdot) \in U\}, \tag{4.7}$$

$$Q_2 = \{(x,u) \in E : \frac{dx(t)}{dt} = \varphi(x(t), u(t), t), G^o(x(0)) = 0, G^1(x(1)) = 0\}. \tag{4.8}$$

With these notations *Problem I* may be formulated as follows. Find the minimum of the functional

$$I(x,u) = \int_0^1 \Phi(x(t), u(t), t)dt \tag{4.9}$$

on the set $Q = Q_1 \cap Q_2$.

We shall now proceed to a thorough analysis of the elements of our problems.

1) Analysis of the functional. It follows from *Lemma 2.1* that, if assumptions 1^o-4^o are satisfied, then the cone K_o of directions of decrease is convex and defined by formula (2.3). Assume additionally that $K_o \neq \emptyset$ (the possibility of rejecting this

assumption will be discussed in (6)). Under this assumption, in virtue of theorem 10.2 [1], we obtain that any functional f_o belonging to K_o^* is of the form

$$(4.10)\qquad f_o(\bar{x},\bar{u}) = -\lambda_o \int_0^1 [(\nabla \Phi_x(x_o(t),u_o(t),t),\bar{x}(t)) + + (\nabla \Phi_u(x_o(t),u_o(t),t),\bar{u}(t))]\,dt,$$

where $\lambda_o \geq 0$.

2) Analysis of the constraint Q_1. It follows from *Lemma 1.2* that the set U is closed and convex, and $U^o \neq \emptyset$ in the space V^r. Thus, the set $Q_1 = W_{11}^n \times U$ is also a closed, convex set, and $Q_1^o = W_{11}^n \times U^o \neq \emptyset$ in the space E. Let K_1 stand for the cone of feasible directions for Q_1 at the point (x_o,u_o). Then, if $f_1 \in K_1^*$, then $f_1 = (0,f_1')$, where $f_1' \in V^{r*}$ is a functional supporting the set U at the point u_o. This follows from theorem 10.5 [1].

3) Analysis of the constraint Q_2. Assume additionally that the condition for the full controllability of system (3.5) is satisfied.

It follows from *Lemma 3.3* that, if assumptions 1^o–8^o are satisfied and system (3.5) is fully controllable, then the cone K_2 of tangent directions is a subspace given by (3.6), that is, $K_2 = L_1 \cap L_2$ where

$$(4.11)\qquad L_1 = \{(\bar{x},\bar{u}) \in E : \frac{d\bar{x}(t)}{dt} = \varphi_x(x_o(t),u_o(t),t)\bar{x}(t) + + \varphi_u(x_o(t),u_o(t),t)\bar{u}(t)\},$$

$$(4.12)\qquad L_2 = \{(\bar{x},\bar{u}) \in E : G^{o'}(x_o(0))\bar{x}(0) = 0, G^{1'}(x_o(1))\bar{x}(1) = 0\}.$$

L_1 and L_2 are closed subspaces, and therefore weakly closed ones (cf. [1], § 2).

Note that, if $f_{21} \in L_1^*$, then

$$f_{21}(\bar{x},\bar{u}) = 0 \quad \text{for} \quad (\bar{x},\bar{u}) \in L_1. \tag{4.13}$$

It is easy to notice that, if $f_{22} \in L_2^*$, then it is of the form

$$f_{22}(\bar{x},\bar{u}) = (\lambda^0, G^{0'}(x_0(0))\bar{x}(0)) + (\lambda^1, G^{1'}(x_0(1))\bar{x}(1)), \tag{4.14}$$

where $\lambda^0 \in R^k$, $\lambda^1 \in R^l$ (L_2 is determined by k+1 linear functionals). Hence, since L_1^* and L_2 are weakly $*$-closed, and L_2^* - finite-dimensional, we get that

$$K_2^* = (L_1 \cap L_2)^* = L_1^* + L_2^*. \tag{4.15}$$

That is, finally, if $f_2 \in K_2^*$, then

$$f_2(\bar{x},\bar{u}) = f_{21}(\bar{x},\bar{u}) + f_{22}(\bar{x},\bar{u}), \tag{4.16}$$

where f_{21} and f_{22} satisfy conditions (4.13) and (4.14), respectively.

4) The Euler-Lagrange equation. It follows from the Dubovitskii-Milyutin theorem that there exist functionals f_0, f_1, $f_2 \in E^*$ not all zero, such that, for any $(\bar{x},\bar{u}) \in E$, the equation

$$f_0(\bar{x},\bar{u}) + f_1(\bar{x},\bar{u}) + f_2(\bar{x},\bar{u}) = 0 \tag{4.17}$$

is satisfied, where f_0 and f_2 are defined by formulas (4.10) and (4.16), respectively, and $f_1(\bar{x},\bar{u}) = f_1'(\bar{u})$ is a functional supporting the set U at the point $u_0(\cdot)$.

5) Analysis of the Euler-Lagrange equation. Let us take any $\bar{u}(\cdot) \in V^r$ and find some $\bar{x}(\cdot)$ such that $(\bar{x},\bar{u}) \in L_1$. By formulas (4.10), (4.13) and (4.14), we may write down the Euler-Lagrange equation in the form

$$f_1'(\bar{u}) = \lambda_0 \int_0^1 (\nabla \Phi_x(x_0(t),u_0(t),t),\bar{x}(t)) + \tag{4.18}$$

$$+ (\nabla \Phi_u(x_0(t),u_0(t),t),\bar{u}(t))\,dt -$$

$$- (\lambda^0, G^{0'}(x_0(0))\bar{x}(0)) - (\lambda^1, G^{1'}(x_0(1))\bar{x}(1)).$$

Let $\Psi_0(t)$ be a solution to conjugate system (4.4) with transversality conditions (4.5). After simple transformations we get

$$(4.19) \qquad f_1'(\bar{u}) = \int_0^1 (-\varphi_u^*(x_0(t),u_0(t),t)\Psi_0(t) + \lambda_0 \nabla \Phi_u(x_0(t),u_0(t),t),\bar{u}(t))dt,$$

where $\bar{u}(\cdot)$ is an arbitrary element of V^r, and f_1' is a functional supporting the set U at the point $u_0(\cdot)$, i.e.

$$(4.20) \qquad f_1'(u) \geq f_1^0(u_0) \quad \text{for each} \quad \bar{u}(\cdot) \in U.$$

With the notations adopted, this inequality may be written down in the form of (4.6). The case $\lambda_0 = 0$, $\lambda^0 = 0$, $\lambda^1 = 0$ and $\Psi_0(t) \equiv 0$ is impossible since all the functionals f_i, $i = 0, 1, 2$, would then be identically zero.

b) Analysis of singular cases. In the course of the proof we assumed, in addition, two things: 1° $K_0 \neq \emptyset$ and 2° the condition for the full controllability of system (3.5) is satisfied.

If $K_0 = \emptyset$, then

$$(4.21) \qquad (\nabla\Phi_x(x_0(t),u_0(t),t),\bar{x}(t)) + (\nabla\Phi_u(x_0(t),u_0(t),t),\bar{u}(t)) = 0$$

for each $(\bar{x},\bar{u}) \in E$.

Putting $\lambda_0 = 1$ and $\Psi_0(1) = 0$ $(\lambda^1 = 0)$ and choosing $\bar{x}$ such that $(\bar{x},\bar{u}) \in L_1$ and $G^{0'}(x_0(0))\bar{x}(0)$, one can easily check that

$$(4.22) \qquad \int_0^1 (\nabla\Phi_x(x_0(t),u_0(t),t),\bar{x}(t))dt = -\int_0^1 (\varphi_u^*(x_0(t),u_0(t),t)\Psi_0(t),\bar{u}(t))dt,$$

that is

$$(4.23)\qquad \int_0^1 \left(-\varphi_u^*(x_o(t),u_o(t),t)\,\psi_o(t) + \nabla\Phi_u(x_o(t),u_o(t),t),\bar{u}(t)\right)dt = 0$$

for each $u(\cdot) \in V^r$, and consequently, (4.6) is satisfied.

Let us now suppose that the condition for the full controllability of system (3.5) is not satisfied. Putting $\lambda_o = 0$, we shall find some $\psi_o(t)$ being a non-zero solution to equation (4.4), such that $\varphi_u^*(x_o(t),u_o(t),t)\,\psi_o(t) \equiv 0$. And so, also in this case, $\nabla H_u \equiv 0$ for each $\bar{u}(\cdot) \in V^r$, i.e. condition (4.6) is satisfied. This completes the proof of the theorem.

Remark 1. Condition (4.6) of the proposition of theorem 4.1 is also satisfied in the case when boundary conditions (1.5) in the formulation of the problem are replaced by the conditions $x(0) \in S_o$, $x(1) \in S_1$, where S_o and S_1 are smooth manifolds. Transversality conditions are then as follows: $\psi_o(0)$ is transversal (orthogonal to a tangent space) to S_o at the point $x_o(0)$, $\psi_o(1)$ - transversal to S_1 at the point $x_o(1)$.

Remark 2. In particular, if the left end of the trajectory is fixed $x(0) = c \in R^n$ and the right one non-fixed, then the boundary condition should be adopted only for $\psi(1)$ and it has the form $\psi_o(1) = 0$.

Remark 3. One knows (cf. [1] and [2]) that, for the full class of admissible Pontryagin controls, the integral maximum principle is equivalent to the pointwise maximum principle. It is easy to notice that the class of admissible controls under consideration is not admissible in the sense of Pontryagin. It can be shown that the pointwise maximum principle is not satisfied in the class of controls considered.

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WARUNKI TRANSWERSALNOŚCI DLA STEROWAŃ O OGRANICZONYM WAHANIU

W pracy uzyskany jest całkowity warunek konieczny istnienia ekstremum w następującym zadaniu optymalizacyjnym: znaleźć ekstremum funkcjonału

$$\int_0^1 \Phi(x,u,t)\, dt$$

przy warunkach $\dot{x} = \varphi(x,u,t)$, $G^0(x(0)) = 0$, $G^1(x(1)) = 0$, gdzie x jest funkcją absolutnie ciągłą, a sterowanie u jest funkcją o wahaniu ograniczonym, przyjmującą wartości z danego zbioru P. Zadanie to jest rozważane bez założeń wypukłości funkcji φ i Φ. Dowód oparty jest na metodzie Dubowickiego-Milutina.