A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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OPERATOR EQUATIONS AT RESONANCE WITH UNBOUNDED NONLINEARITIES

To Professor Lech Włodarski on His 80th birthday

The existence of solutions of the equation $Lx = Nx$, where L is a Fredholm linear operator of index zero and N is a nonlinear continuous map, is established. N is sublinear Landesman-Lazer ones. The results are applied to boundary value problems with nonlinearities also involving derivatives. The resonance can be multidimensional.

1. Introduction

Most of boundary value problems: $Pu = N(u)$, $Bu = 0$, where P is a linear differential operator, N is a superposition operator and B - a boundary linear operatop, can be transformed to $x = P^{-1}N(x)$ in an appropriate function space. There are a lot of topological and variational techniques to find a fixed point of $P^{-1}N$, mainly, when this operator is compact. However, if the linear problem $Pu = 0$, $Bu = 0$, has nontrivial solutions, then P^{-1} does no exist and we cannot use this method; the system is said to be at resonance.

The resonance problem was first studied by Landesman and Lazer [11] with $N(u)(x) = f(x, u(x))$, f bounded and having limits as $u \to \pm\infty$. Then, there started to appear papers by several authors with attempts to examine an unbounded nonlinearity f. The growth of f as $|u|$ tends to infinity is sufficiently slow and the Landesman-Lazer condition is replaced (or strengthened) by the monotonicity of $f(\cdot,t)$ [6], [20], [15] or another assumptions [8], [10]. The authors often confine themselves to the case of second order ordinary differential equations not involving the first derivative. Moreover, the multiplicity of the resonance, i.e. the dimension of the space of solutions $Pu = 0$, $Bu = 0$, is usually 1. This excludes, for instance, the periodic boundary value problem; $u'' + m^2u = f(t,u)$, $u(0) - u(2\pi) = 0 = u'(0) - u'(2\pi)$; $m = 1,2,\dots$ (cf. [9],[4]).

Here, we consider all these problems without the above restrictions in an abstract framework closely related to our previous paper [19]. Let X, Y and Z be Banach spaces. Consider the following equation:

$$L(\lambda_0)y = N(Jy) \tag{1.1}$$

where $L : \mathbb{R} \supset \langle\lambda_0, \lambda_1\rangle \to L(Y,Z)$ is a continuous map with $L(\lambda)$ being a linear homeomorphism for $\lambda \neq \lambda_0$, $N : X \to Z$ is a nonlinear continuous map transforming bounded sets onto bounded ones and $J : Y \to X$ - a completely continuous linear injection. Suppose that the inverse operator $G(\lambda) = L(\lambda)^{-1} \in L(Z,Y)$, $\lambda \neq \lambda_0$, has the form

$$G(\lambda) = G_0(\lambda) + \sum_{j=1}^{n} c_j(\lambda)\langle u_j(\lambda), \cdot\rangle w_j(\lambda) \tag{1.2}$$

where $G_0 : \langle\lambda_0, \lambda_1\rangle \to L(Z,Y)$, $u_j : \langle\lambda_0, \lambda_1\rangle \to Z^*$, $w_j : \langle\lambda_0, \lambda_1\rangle \to Y$, $c_j : (\lambda_0, \lambda_1\rangle \to \mathbb{R}$, $j = 1, \dots, n$, are continuous and $\lim_{\lambda\to\lambda_0^+} |c_j(\lambda)| = \infty$ for any j.

Let

$$\operatorname{Im} L(\lambda_0) = \bigcap_{j=1}^{n} \ker u_j(\lambda_0), \tag{1.3}$$

$$\ker L(\lambda_0) = \operatorname{Lin}\{w_j(\lambda_0) : j = 1, \dots, n\} \tag{1.4}$$

with $w_j(\lambda_0), j = 1, \dots, n$, being linearly independent. It follows that $w_j(\lambda), j = 1, \dots, n$ and $u_j(\lambda), j = 1, \dots, n$ are linearly independent for λ sufficiently close to λ_0. We assume without loss of generality that all $\lambda \in \langle\lambda_0, \lambda_1\rangle$ have this property. Suppose that

$$L(\lambda_0)G_0(\lambda_0)z = z, \quad z \in \operatorname{Im} L(\lambda_0). \tag{1.5}$$

It is equivalent to the system

$$\lim_{\lambda \to \lambda_0} c_j(\lambda)\langle u_j(\lambda), z\rangle L(\lambda) w_j(\lambda) = 0$$

for $j = 1, \dots, n$ and $z \in \operatorname{Im} L(\lambda_0)$, which can easily be verified in applications.

It is obvious that equation (1.1) is equivalent to the system

$$\langle u_j(\lambda_0), N(Jy)\rangle = 0, \quad j = 1, \dots, n, \tag{1.6}$$

$$y = G_0(\lambda_0)N(Jy) + \sum_{j=1}^{n} C_j w_j(\lambda_0) \tag{1.7}$$

where $C_1, \dots, C_n$ are arbitrary real constants. The multiplicity of the resonance equals n.

We consider separately two cases: sublinear

$$\limsup_{\|x\| \to \infty} \frac{\|N(x)\|}{\|x\|} = 0, \tag{1.8}$$

and of linear growth

$$\limsup_{\|x\| \to \infty} \frac{\|N(x)\|}{\|x\|} =: \gamma > 0. \tag{1.9}$$

2. THE SUBLINEAR CASE

We shall need the following lemma in both cases.

Lemma. *If there exists a sequence (y_k) of solutions of equations*

$$L(\lambda_k)y_k = t_k N(Jy_k)$$

where $\lambda_k \to \lambda_0^+$, $t_k \to 1$, *such that* $(Jy_k) \subset Y$ *is bounded, then equation (1.1) has a solution.*

Proof. Denoting $x_k = Jy_k$, we have $x_k = t_k JG(\lambda_k)N(x_k)$ and, by (1.2), both $(t_k JG_0(\lambda_k)N(x_k))$, and $(t_k \sum_j c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle Jw_j(\lambda_k))$ are bounded sequences. Due to the linear independence of $Jw_j(\lambda)$, $j = 1, \ldots, n$ all sequences

$$(t_k c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle)_k, \quad j = 1, \ldots, n,$$

are bounded. We can choose convergent subsequences (we do not change the indices for simplicity)

$$\begin{aligned} t_k JG_0(\lambda_k)N(x_k) &\to x_0, \\ t_k c_j(\lambda_k)\langle u_j(\lambda_k), N(x_k)\rangle &\to C_j, \quad j = 1, \ldots, n. \end{aligned}$$

Hence $x_k \to x_0 + \sum_{j=1}^n C_j Jw_j(\lambda_0) =: x$. Moreover, for every j, $\langle u_j(\lambda_0), N(x)\rangle = 0$ and $x_0 = JG_0(\lambda_0)N(x)$. Therefore $x = Jy$ for some $y \in Y$, and

$$y = G_0(\lambda_0)N(Jy) + \sum_{j=1}^n C_j w_j(\lambda_0).$$

This means that y is a solution of (1.6) – (1.7), thus, equivalently, of (1.1).

Let $N : X \to Z$ satisfy condition (1.8). This holds, for example, if

$$\|N(x)\| \le a\|x\|^p + b, \quad x \in X,$$

for some positive constants a, b and $p < 1$. Let $\alpha_j = +1$ if $\lim_{\lambda\to\lambda_0} c_j(\lambda) = +\infty$ and $\alpha_j = -1$ if this limit equals $-\infty$, $j = 1, \ldots, n$.

Theorem 1. *If for any sequence* $(x_\nu) \subset X$ *with properties* $\|x_\nu\| \to \infty$, $\|x_\nu\|^{-1}x_\nu \to \sum C_j Jw_j(\lambda_0)$ *there exists* $j_1 \in \{1, \ldots, n\}$ *such that* $C_{j_1} \neq 0$ *and*

$$C_{j_1}\alpha_{j_1}\langle u_{j_1}(\lambda_0), N(x_\nu)\rangle \le 0 \tag{2.1}$$

for sufficiently large ν, then equation (1.1) has a solution.

Proof. Take any sequence $\lambda_k \to \lambda_0^+$, $\lambda_k \neq \lambda_0$, and consider a sequence of equations

$$L(\lambda_k)y_k = N(Jy_k)$$

or, equivalently

$$x_k = JG(\lambda_k)N(x_k).$$

Fix $k \in N$ and choose $R_k > 0$ such that

$$\frac{\|N(x)\|}{\|x\|} \leq \|JG(\lambda_k)\|^{-1}$$

for $\|x\| \geq R_k$. In particular, for $\|x\| = R_k$

$$\|JG(\lambda_k)N(x)\| \leq \|x\| = R_k$$

and, by the Rothe theorem [12], solutions $x_k = Jy_k$ exist. Due to the Lemma, we should only show that (x_k) cannot be unbounded.

Suppose it is unbounded. We may assume without loss of generality that $\|x_k\| \to \infty$. Then

$$\begin{aligned}\|x_k\|^{-1}x_k = {} & JG_0(\lambda_k)(\|x_k\|^{-1}N(x_k)) \\ & + \sum_j c_j(\lambda_k)\langle u_j(\lambda_k), \|x_k\|^{-1}N(x_k)\rangle Jw_j(\lambda_k),\end{aligned}$$

where $\|x_k\|^{-1}N(x_k) \to 0$. Repeating the arguments from the proof of the Lemma, we get convergent subsequences

$$c_j(\lambda_k)\langle u_j(\lambda_k), \|x_k\|^{-1}N(x_k)\rangle \to C_j, \quad j = 1, \dots, n \tag{2.2}$$

and, hence

$$\|x_k\|^{-1}x_k \to \sum_{j=1}^{n} C_j Jw_j(\lambda_0).$$

Thus, for sufficiently large k and each j, $\langle u_j(\lambda_0), N(x_k)\rangle$ has a constant sign. By (2.2), $\alpha_j\langle u_j(\lambda_0), N(x_k)\rangle$ has the same sign as C_j for $j = 1, \dots, n$ and this contradicts (2.1).

Corollary 1. *If N is bounded and there exists limits $N(C_1, \dots, C_n) = \lim_{\nu\to\infty} N(x_\nu)$ for any $\|x_\nu\| to\infty$, $\|x_\nu\|^{-1}x_\nu \to \sum C_j J w_j(\lambda_0)$, independent of the choice of the sequence (x_ν), then the sufficient condition can be written in the form*

$$C_{j_1}\alpha_{j_1}\langle u_{j1}(\lambda_0), N(C_1, \dots, C_n)\rangle < 0.$$

as it is in [19]. For $n = 1$ and N being a superposition operator it is equivalent to the Landesman-Lazer condition.

3. The nonlinearity with linear growth

Let us introduce two families of linear subspaces of Y:

$$\overline{Y}_\lambda = \mathrm{Lin}\{w_j(\lambda) : j = 1, \dots, n\} \quad \widetilde{Y}_\lambda = \mathrm{Im}\, G_0(\lambda),$$

where $\lambda \in \langle \lambda_0, \lambda_1\rangle$. Changing slightly the regular part $G_0(\lambda)$ and the irregular one in (1.2) if it is necessary, one may assume that

$$Y = \widetilde{Y}_\lambda \oplus \overline{Y}_\lambda. \tag{3.1}$$

we shall use the obvious notation $y = \widetilde{y}_\lambda + \overline{y}_\lambda$ for $y \in Y$ and $\lambda \in \langle \lambda_0, \lambda_1\rangle$.

Theorem 2. *Let N satisfy (1.9) with γ such that*

$$\gamma\|\|JG_0(\lambda)\| < \sigma(1+\sigma)^{-1} \tag{3.2}$$

for a positive constant σ. Assume that there exists $\delta > 0$ and $r > 0$ such that, for all j's, either

$$C_j^{-1}\langle u_j(\lambda), N(J\widetilde{y}_\lambda + \sum_i C_i J w_i(\lambda))\rangle > \delta \tag{3.3}$$

or

$$C_j^{-1}\langle u_j(\lambda), N(J\widetilde{y}_\lambda + \sum_i C_i J w_i(\lambda))\rangle < -\,\delta \tag{3.4}$$

or

$$C_j\alpha_j\langle u_j(\lambda), N(J\widetilde{y}_\lambda + \sum_i C_iJw_i(\lambda))\rangle \le 0 \tag{3.5}$$

for any $\lambda \in (\lambda_0, \lambda_1\rangle$, $\widetilde{y}_\lambda \in \widetilde{Y}_\lambda$, $C_1, \ldots, C_n \in \mathbb{R}$ *provided that* $|C_i| \le |C_j| \ge r$ *for* $i \ne j$, $\|J\widetilde{y}_\lambda\| \le \sigma\|J\overline{y}_\lambda\| = \sigma\|\sum C_iJw_j(\lambda)\|$.
Then equation (1.1) has a solution.

Proof. We may assume without loss of generality that

$$\begin{aligned} |c_j(\lambda)| &> 2\delta^{-1}, \quad j = 1, \ldots, n, \\ \gamma\|JG_0(\lambda)\| &< \sigma(1+\sigma)^{-1}, \end{aligned} \tag{3.6}$$

for every $\lambda \in (\lambda_0, \lambda_1\rangle$. Take $\varepsilon > 0$ such that

$$(\gamma+\varepsilon)\|JG_0(\lambda)\| \le \sigma(1+\sigma)^{-1}, \qquad \lambda \in \langle\lambda_0, \lambda_1\rangle,$$

and $R > 0$ such that

$$\|N(x)\| \le (\gamma+\varepsilon)\|x\|, \qquad \|x| \ge R,$$

and consider a homotopy $H : X \times \langle 0, 1) \to X$

$$H(x,t) = \begin{cases} tJG_0(\lambda_1)N(x) + \frac{1}{2}\sum_j c_j(\lambda_1)\langle u_j(\lambda_1), N(x)\rangle Jw_j(\lambda_1), \\ \qquad\qquad \text{for } \ t \in \langle 0, \frac{1}{2}\rangle, \\ tJG(2\lambda_1(1-t) + \lambda_0(2t-1))N(x), \\ \qquad\qquad \text{for } \ t \in (\frac{1}{2}, 1). \end{cases}$$

Clearly, H is continuous. We shall show that all fixed points of $H(\cdot, t)$, $t \in \langle 0, 1)$, if they exist, are contained in one ball. Let $t \in (\frac{1}{2}, 1)$ and $\lambda = \lambda(t) = 2\lambda_1(1-t) + \lambda_0(2t-1)$. by (3.1) the equation $x = H(x,t)$ is equivalent to the system

$$\widetilde{y}_\lambda = tG_0(\lambda)N(J\widetilde{y}_\lambda + \sum_i d_iJw_i(\lambda)) \tag{3.7}$$

$$d_j = tc_j(\lambda)\langle u_j(\lambda), N(J\widetilde{y}_\lambda + \sum_i d_iJw_i(\lambda))\rangle, \quad j = 1, \ldots, n. \tag{3.8}$$

Suppose that a solution $x = J\widetilde{y}_\lambda + J\overline{y}_\lambda$ satisfies $\|x\| \geq R$. Then

$$\begin{aligned}\|J\widetilde{y}_\lambda\| &\leq \|JG_0(\lambda)\|(\gamma+\varepsilon)\|x\| \\ &\leq \sigma(1+\sigma)^{-1}(\|J\widetilde{y}_\lambda\| + \|J\overline{y}_\lambda\|)\end{aligned}$$

so

$$\|J\widetilde{y}_\lambda\| \leq \sigma \|J\overline{y}_\lambda\|.$$

On the other hand, $|d_j| \geq r$ cannot satisfy (3.8) due to (3.3) or (3.4) or (3.5) and (3.6). Hence, all solutions to system (3.7) – (3.8) satisfy

$$\|x\| = \|Jy\| < \max\{R, (\sigma+1)r\, n \max_{\lambda,j} \|Jw_j(\lambda)\|\} =: R_0.$$

For $t \in \langle 0, \frac{1}{2}\rangle$, the reasoning leading to the estimate is almost the same. The corresponding system has the form

$$\begin{aligned}\widetilde{y}_{\lambda_1} &= tG_0(\lambda_1)N(J\widetilde{y}_{\lambda_1} + \sum_i d_i Jw_i(\lambda_1)) \\ d_j &= \frac{1}{2}c_j(\lambda_1)\langle u_j(\lambda_1), N(J\widetilde{y}_{\lambda_1} + \sum_i d_i Jw_i(\lambda_1))\rangle, \quad j = 1, \ldots, n.\end{aligned}$$

Our next step is to study the mapping $H(\cdot, 0) : X \to X$. If we show that the Leray-Schauder degree $\deg_{LS}(I - H(\cdot,0), B(0,R_0), 0) \neq 0$ where I is the identity and $B(0, R_0)$ - a ball in Xcentred at 0 with radius R_0, then equation (1.1) will have a solution by the Lemma. But $H(\cdot,))$ is finite dimensional, thus this degree equals the Brouwer degree (cf. [12]) of this mapping resticted to $J\overline{Y}_{\lambda_1}$. At last, it is equal to the degree

$$\deg(I - g : (-R_0, R_0)^n, 0)$$

where $g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$ is given by the formulae

$$g_j(d_j, \ldots, d_n) = \frac{1}{2}c_j(\lambda_1)\langle u_j(\lambda_1), N\left(\sum_i d_i Jw_i(\lambda_1)\right)\rangle,$$

$j = 1, \ldots, n$. As we have shown above,

$$g_j(d_j, \ldots, d_{j-1}, \pm R_0, d_{j+1}, \ldots, d_n) \neq \pm R_0,$$

hence g has no fixed points on the boundary of the cube $\langle -R_0, R_0 \rangle^n$ and the sign of g_j on the whole face $d_j = +R_0$ (resp. $d_j = -R_0$) is constant. Moreover, by one of the conditions (3.3), (3.4) or (3.5),

$$g_j(d_1, \ldots, +R_0, \ldots, d_n) g_j(d_1, \ldots, -R_0, \ldots, d_n) \leqq 0. \tag{3.10}$$

We shall define $h_j : \langle -R_0, R_0 \rangle^n \times \langle 0, 1 \rangle \to \mathbb{R}$, $j = 1, \ldots, n$, by putting $d = (d_1, \ldots, d_n)$:

$$h_j(d, 0) = d_j - g_j(d) \quad \text{for every } d,$$
$$h_j(d, s) = (1 - s)(d_j - g_j(d)) - s d_j \quad \text{for } s \in \langle 0, 1 \rangle$$
$$\text{and } d = (d_1, \ldots, \pm R_0, \ldots, d_n)$$

if g_j is nonnegative on the face $d_j = +R_0$, or

$$h_j(d, s) = (1 - s)(d_j - g_j(d)) + s d_j \text{ for } s \in \langle 0, 1 \rangle$$
$$\text{and } d = (d_1, \ldots, \pm R_0, \ldots, d_n)$$

if g_j is nonpositive on the face $d_j = +R_0$, and

$$h_j(d, 1) = -d_j \quad \text{for every } d$$

in the first case or

$$h_j(d, 1) = +d_j \quad \text{for every } d$$

in the second one. If g_j vanishes on the whole face $d_j = +R_0$, the choice of the case is arbitrary. Then, we can extend h_j to $\langle -R_0, R_0 \rangle^n \times \langle 0, 1 \rangle$ continuously. Set $h = (h_1, \ldots, h_n)$. By (3.9) and (3.10), h is a homotopy which has no zeros on the boundary of $\langle -R_0, R_0 \rangle^n$. But $h(\cdot, 0) = I - g$ and $h(\cdot, 1)$ is an antipodal mapping that has an odd degree due to Borsuk's Antipodensatz [12]. This ends the proof.

Remark. In the most important case, X is a Hilbert space and subspaces $J\widetilde{Y}_\lambda$ and $J\overline{Y}_\lambda$ are orthogonal. Then condition (3.2) can be weakened:

$$\gamma\|JG_0(\lambda_0)\| < \sigma(1+\sigma^2)^{-\frac{1}{2}}. \tag{3.11}$$

In fact, it is used only to get the estimate

$$\begin{aligned}\|J\widetilde{Y}_\lambda\| &\leqq (\gamma+\varepsilon)\|JG_0(\lambda)\|\|J\widetilde{Y}_\lambda + J\overline{Y}_\lambda\| \\ &\leqq \sigma(1+\sigma^2)^{\frac{-1}{2}}(\|J\widetilde{Y}_\lambda\|^2 + \|J\overline{Y}_\lambda\|^2)^{\frac{1}{2}}\end{aligned}$$

implying

$$\|J\widetilde{Y}_\lambda\|^2 = \sigma^2\|J\overline{Y}_\lambda\|^2$$

which is needed in order to apply (3.3) – (3.5).

4. Applications to elliptic BVPs – sublinear case

The typical examples of Fredholm operators are elliptic partial differential operators acting on Sobolev spaces restricted by some coercive boundary conditions. We confine ourselves to the Dirichlet boundary value problems for simplicity, although we can apply all results to many other problems. Let P be a uniformly elliptic partial differential operator in an open bounded set $\Omega \subset \mathbb{R}^k$:

$$Pu = \sum_{|x|\leq 2m} a_\alpha(x)D^\alpha u \tag{4.1}$$

(where a_α are sufficiently smooth bounded real functions on Ω and the boundary of Ω is sufficiently regular [1]). Denote by $H^{2m}(\Omega)$ the space of functions which have all derivatives up to order $2m$ sitting in the space $L^2(\Omega)$, and by $H_0^m(\Omega)$ – the closure of the space of all smooth functions with compact (in Ω) support in $H^m(\Omega)$. Then the operator $P : H^{2m}(\Omega) \cap H_0^m(\Omega) \to L^2(\Omega)$ is a Fredholm operator of index 0. Suppose it is not invertible and denote by $\omega_1, \ldots, \omega_n$ the orthonormal base of its kernel $\ker P$ (the orthogonality with respect to the L^2 -scalar product), and by $\theta_1, \ldots, \theta_n$ – the orthonormal base of im P. Put $Q\omega_j = \theta_j, \quad j = 1, \ldots, n, \quad Q|\ker P^\perp = 0$, and extend Q linearly to the whole space $Y = H^{2m}(\Omega) \cap H_0^m(\Omega)$. It is easy to see that the family of operators $L(\lambda) : Y \to Z = L^2(\Omega)$ given by the formula $L(\lambda) = P + \lambda Q, \quad \lambda \in \mathbb{R}$, is admissible for our scheme with $\widetilde{Y}_\lambda = \ker P^\perp, \ \overline{Y}_\lambda = \ker P$,

$$G_0(\lambda) = (P|\ker P^\perp)^{-1} pr_{\operatorname{Im} P},$$

$$c_j(\lambda) = \lambda^{-1}, \ \ \omega_j(\lambda) = \omega_j, \ \ \langle u_j(\lambda), z\rangle = \int_\Omega \theta_j z,$$

for $j = 1, \ldots, n, \quad z \in Z$, where $pr_{\operatorname{Im} P}$ stands for the orthogonal projector on $\operatorname{Im} P$.

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^k \times \cdots \times \mathbb{R}^{k^l}$ be a Carathéodory function, i.e. $f(x, \cdot)$ is continuous for a.e. $x \in \Omega$, $f(\cdot, u_0, u_1, \ldots, u_1) : \Omega \to \mathbb{R}$ is measurable. Assume that $l < 2m$ and f satisfies the following growth condition: for any $M > 0$, there exist $a_M \in L^2(\Omega)$ and constants $b_{p+1}, \ldots, b_l \geqq 0$ where $p = 2m - [(k+2)/2]$, such that

$$|f(x, u_0, \ldots, u_l)| \leqq a_M(x) + b_{p+1}\|u_{p+1}\|^\rho + \cdots + b_l\|u_l\|^\rho \tag{4.2}$$

($0 \leqq \rho \leqq 1$ is a fixed constant). It is known [7] that under these assumption the superposition operator

$$N(u)(x) = f(x, u(x), D_1 u(x), \ldots, D_l u(x))$$

where $D_s u = (D^\alpha u)_{|\alpha|=s}, \quad s = 1, \ldots, l$, maps the space $X = H^l(\Omega)$ into $Z = L^2(\Omega)$. Let $J : Y \to X$ be the inclusion map which is completely continuous [1]. The nonlinear operator N is sublinear if $\rho < 1$ and

$$\lim_{M\to\infty} M^{-1}\|a_M\|_{L^2} = 0. \tag{4.3}$$

We look for a solution of the BVP:

$$Pu = f(x, u, D_1 u, \ldots, D_l u), \quad u \in H^m(\Omega). \tag{4.4}$$

Assume that, for any $(C_1,\dots,C_n)\in\mathbb{R}^n\setminus\{0\}$ and $|\alpha|\leqq l$

$$\mu_k\{x\in\Omega:\sum_{j=1}^{n}C_jD^\alpha\omega_j(x)=0\}=0 \tag{4.5}$$

where μ_k stands for the Lebesgue measure in $\mathbb{R}^k$ (comp. [7]). Introduce, for $i_p\in\{\pm1\}^{k^p}$, $p=0,1,\dots,l$, the following limits

$$\hat f(x;i_0,i_1,\dots,i_k)=\limsup_{\substack{u_0\to i_0\infty\\ \dots\dots\\ u_l\to i_l\infty}} f(x,u_0,\dots,u_l)$$

$$\check f(x;i_0,\dots,i_l)=\liminf_{\substack{u_0\to i_0\infty\\ \dots\dots\\ u_l\to i_l\infty}} f(x,u_0,\dots,u_l)$$

where $(\varepsilon_1,\dots,\varepsilon_s)\infty=(\varepsilon_1\infty,\dots,\varepsilon_s\infty)$, and sets

$$A^{j+}_{i_0},\dots,i_l(C_1,\dots,C_n)=\{x\in\Omega:C_j\theta_j(x)>0,$$

$$\operatorname{sgn}\sum_i C_iD_p\omega_i(x)=i_p,\quad p=0,1,\dots,l\}$$

and similarly $A^{j-}_{i_0},\dots,i_l(C_1,\dots,C_n)$ with $C_j\theta_j(x)<0$.

Theorem 3. *Under the above assumption, if, for each* $(C_1,\dots,C_n)\in\mathbb{R}^n\setminus\{0\}$ *there exists* $j\in\{1,\dots,n\}$ *such that*

$$C_j\sum_{i_0,\dots,i_l}\Big(\int_{A^{j+}_{i_0,\dots,i_l}(C_1,\dots,C_n)}\theta_j\hat f(\cdot;i_0,\dots,i_l)+\int_{A^{j-}_{i_0,\dots,i_l}(C_1,\dots,C_n)}\theta_j\check f(\cdot;i_0,\dots,i_l)\Big)<0, \tag{4.6}$$

then problem (4.4) has a solution.

Proof. Taking an arbitrary sequence $(u^{(\nu)})_{\nu\in N} \subset X$ such that $\|u^{(\nu)}\| \to \infty$ and $\|u^{(\nu)}\|^{-1}u^{(\nu)} \to \sum C_j\omega_j$, and using (4.5), we get

$$\limsup_{\nu\to\infty} \int_\Omega C_j\theta_j f(\cdot, u^{(\nu)}, D_1u^{(\nu)}, \ldots, D_lu^{(\nu)})$$

$$\leqq \int\limits_{\bigcup_{i_0,\ldots,i_l} A^{j+}_{i_0,\ldots,i_l}(C1,\ldots,C_n)} C_j\theta_j \limsup_{\nu\to\infty} f(\cdot, u^{(\nu)}, \ldots, D_lu^{(\nu)})+$$

$$\int\limits_{\bigcup_{i_0,\ldots,i_l} A^{j-}_{i_0,\ldots,i_l}(C1,\ldots,C_n)} C_j\theta_j \liminf_{\nu\to\infty} f(\cdot, u^{(\nu)}, \ldots, D_lu^{(\nu)}).$$

Obviously, $\limsup_{\nu\to\infty} f(\cdot, u^{(\nu)}, \ldots, D_lu^{(\nu)}) \leqq \hat{f}(\cdot; i_0, \ldots, i_l)$, $\liminf_{\nu\to\infty} f(\cdot, u^{(\nu)}, \ldots, D_lu^{(\nu)}) \geqq \check{f}(\cdot; i_0, \ldots, i_l)$, and, by (4.6), we obtain condition (2.1).

The inequality (4.6) can be reversed (take $\lambda \to 0-$ so $\alpha_j = -1$) but with replacing $\hat{f}$ and $\check{f}$. The functions $\hat{f}$ and $\check{f}$ can be infinite – the left-hand side of (4.6) can be equal $-\infty$.

If the elliptic operator (4.1) is selfadjoint, we have $\theta_j = \omega_j, j = 1, \ldots, n$, and we can replace the left-hand side of (2.1)by the sum over $j_1 = 1, 2, \ldots, n$. Here, this means

$$\int_\Omega \omega f(\cdot, u^{(\nu)}, \ldots, D_lu^{(\nu)}) \leqq 0 \tag{4.7}$$

where $\|u^{(\nu)}\| \to \infty$, $\|u^{(\nu)}\|^{-1}u^{(\nu)} \to \omega \in \mathrm{Lin}\{\omega_1, \ldots, \omega_n\}$. Thus, we have the following sufficient condition for the solvability of (4.4):

Corollary 2. *If P is selfadjoint and, for any solution ω of the linear homogeneous problem $Pu = 0$, $u = H^m_0(\Omega)$,*

$$\int\limits_{\{x:\omega(x)>0\}} \omega\hat{f} + \int\limits_{\{x:\omega(x)<0\}} \omega\check{f} < 0 \tag{4.8}$$

where

$$\hat{f}(x) = \limsup_{\substack{u_0 \to \operatorname{sgn} \omega(x)\infty \\ u_1 \to \operatorname{sgn} D_1\omega(x)\infty \\ \cdots\cdots\cdots \\ u_l \to \operatorname{sgn} D_l\omega(x)\infty}} f(x, u_0, \dots, u_l)$$

and $\check{f}$ *is defined analogously with* lim inf *instead* lim sup.

If f does not depend on derivatives and $n = 1$, the sufficient condition has the form

$$\int_{A_+} \omega_1 f_+ + \int_{A_-} \omega_1 f^- > 0 > \int_{A_+} \omega_1 f^- + \int_{A_-} \omega_1 f_+ \tag{4.9}$$

where

$$A_+ = \{x : \omega_1(x) > 0\}, \quad A_- = \{x : \omega_1(x) < 0\},$$
$$f_+(x) = \liminf_{u \to +\infty} f(x, u), \quad f^- = \limsup_{u \to -\infty} f(x, u).$$

This result is closely related to [3].

The case when $\hat{f} = \check{f}$ (there exist limits) was intensively studied starting from the paper by Nirenberg [17]. He considered the linear Fredholm operator of nonnegative index but (as he noticed) his assumptions can be verified practically only for the index equal to 0. Therefore, our results generalizes the Nirenberg theorem in a sense . Moreover, Corollary 2 implies the first theorem on resonant problem obtained by Landesman and Lazer [11] and its generalization by Williams [23].

For the case f depended on derivatives, one can get less restrictive assumptions by using Corollary 1 directly.

Corollary 3. *Let* $n = 1$ *and* f *be sublinear. Suppose that, for any* $(u^{(\nu)}) \subset H^l(\Omega)$ *with properties* $\|u^{(\nu)}\| \to \infty$, $\|u^{(\nu)}\|^{-1} u^{(\nu)} \to \omega_1$ *(resp.* $\to -\omega_1$*), there exists an* L^2*-limit*

$$f_+(x) = \lim_{\nu \to \infty} f\left(x, u^{(\nu)}(x), \dots, D_l u^{(\nu)}(x)\right)$$

(*resp.* $f_-(x) = \lim_{\nu\to\infty} f\left(x, u^{(\nu)}(x), \dots, D_l u^{(\nu)}(x)\right)$.)

If the numbers

$$(4.10)\qquad \int_{A_+} \omega_1 f_+ + \int_{A_-} \omega_1 f_-; \quad \int_{A_+} \omega_1 f_- + \int_{A_-} \omega_1 f_+$$

have the opposite sings, then BVP (4.4) has a solution.

Example. Let us apply the above result to a typical problem

$$(4.11)\qquad \begin{aligned} u'' + u &= \arctan(u + u') + h(x), \\ u(0) &= u(\pi) = 0, \end{aligned}$$

where $h \in L^2(0, \pi)$. Here $Pu = u''$, $\lambda_0 = -1$, $\omega_0(x) = \sin x$, $f(x, u, u') = \arctan(u + u') + h(x)$,

$$f_\pm(x) = \begin{cases} h(x) \pm \frac{\pi}{2} & \text{for} \quad x \in (0, \frac{3}{4}\pi), \\ h(x) \mp \frac{\pi}{2} & \text{for} \quad x \in (\frac{3}{4}\pi, \pi), \end{cases}$$

$A_+ = (0, \pi)$, $A_- = \emptyset$. Condition (4.10) is equivalent to

$$(4.12)\qquad \left| \int_0^\pi \sin x h(x) dx \right| < \sqrt{2}\pi.$$

Therefore, nonlinear problem (4.11) has a solution provided that the square integrable function h satisfies inequality (4.12).

The functions f_+ and f_- can be infinite. It is only important that the sums in (4.10) are not of the form $\infty - \infty$. For example,

$$f(x, u, u') = \sqrt{\max(0, u + \sqrt{|u'|})} + h(x)$$

is a Carethéodory function ($h \in L^2$) and, for the problem

$$(4.13)\qquad u'' + u = f(x, u, u'), \quad u(0) = u(\pi) = 0,$$

we have $f_+(x) = +\infty$, $f_-(x) = h(x)$. Therefore, the condition guaranteeing the solvability of (4.13) is the following:

$$\int_0^\pi \sin x \cdot h(x) dx < 0.$$

Example. Let us consider the periodic problem

$$(4.14)\qquad \begin{aligned} &u'' + u = p(x)g_1(u^+) + r(x)g_2(u^-),\\ &u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi), \end{aligned}$$

where $p, r : \langle -\pi, \pi\rangle \to \mathbb{R}$ are bounded measurable functions, $g_1, g_2 : \langle 0, \infty) \to \mathbb{R}$ are continuous functions with the following properties:

$$\begin{aligned} &g_1(0) = g_2(0) = 0,\\ &\lim_{u\to\infty} \frac{g_j(u)}{u} = 0, \qquad\qquad j = 1,2,\\ &\lim_{u\to+\infty} g_j(u) = \gamma_j \in (0,\infty\rangle, \quad j = 1,2. \end{aligned}$$

Here, $u^+ = \max(0,u)$, $u^- = -\min(0,u)$, $u = u^+ - u^-$. Since the multiplicity of the resonance equals 2 – $w_1(x) = \sin x$, $w_2(x) = \cos x$ – we apply theorem 3 (with $\hat{f} = \check{f}$). It is easily seen that

$$f(x;+1) = \gamma_1 p(x), \quad f(x;-1) = \gamma_2 r(x), \quad f(x;0) = 0.$$

Therefore, we have to study the signs of numbers

$$b_1 =: C_1\gamma_1 \int\limits_{A_+} p(x)\sin x dx + C_1\gamma_2 \int\limits_{A_-} r(x)\sin x dx,$$

$$b_2 =: C_2\gamma_1 \int\limits_{A_+} p(x)\cos x dx + C_2\gamma_2 \int\limits_{A_-} r(x)\cos x dx,$$

where $A_\pm = x \in \langle -\pi, \pi\rangle : \operatorname{sgn}(C_1 \sin x + C_2 \cos x) = \pm 1$. Suppose, for example, that $p(x) \leqq 0 \leqq r(x)$ a.e. x and at least one of these inequalities is sharp on a set of positive measure. Then one of $b_j's$ must be negative since their sum is negative, and we have a solution of (4.14). Our assumption on p and r is not necessary. One can prove that one of $b_j's$ is negative also for the case: $p(x) = r(x) = 0$ for $x \in \langle -\pi, 0)$, $\gamma_1 = \infty$, $\gamma_2 = +\infty$, and

$$\int\limits_0^\pi p(x)\sin x dx < 0.$$

The results can easily be generalized for the right-hand side depending on derivatives

$$f(x,u,u') = p(x)g_1(u^+,u') + r(x)g_2(u^-,u')$$

where $g_j : \langle 0,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous, $g_j(0,u') = 0$, $\lim_{u\to+\infty} g_j(u,u') = \gamma_j \in (0,\infty >$ for $j = 1,2$ and $u' \in \mathbb{R}$. The calculations are exactly the same.

5. Applications continued – general case

Now, we consider elliptic BVP (4.4) in the case when f has a linear growth, i.e. condition (4.2) holds with $\rho = 1$. We shall apply theorem 2, and the main difficulty lies in finding conditions that guarantee inequalities (3.3) – (3.5) to be satisfied. We choose inequality (3.5) which seems to be the simplest one and we consider only a resonance with a one-dimensional eigenspace spanned by $\omega_1 = \omega$ and with the nonlinearity independent of derivatives. Moreover, let P be selfadjoint, thus im $L(0)\perp\omega$.

Let us suppose that f satisfies the condition

$$\check{a} \leqq \big(f(x,u) - b(x)\big)u^{-1} \leqq \hat{a} \tag{5.1}$$

for $|u| > M$ (M is a positive constant), where $\check{a}, \hat{a} > 0$ and $b \in L^2(\Omega)$. The remaining assumptions and notations of section 4 are kept valid. We should show (taking $\lambda \to \lambda_0^-$, we have $\alpha_0 = -1$) that

$$C\int_\Omega \omega(x)N(C\omega + \widetilde{u})(x)dx \geqq 0 \tag{5.2}$$

for sufficiently large $|C|$ and $\widetilde{u}$ orthogonal to ω in L^2-sense, $\|\widetilde{u}\| \leqq \sigma C\|\omega\|$ where σ is connected with γ and $\|JG_0(\lambda_0)\|$ by inequality (3.11). Obviously, $\hat{a}$ can be chosen arbitrarily close to γ.

Introduce two functions:

$$y(x) = C\omega(x) + \widetilde{u}(x), \qquad x \in \Omega,$$

and $\chi : \Omega \to \mathbb{R}$ measurable and such that

$$\chi(x) = \big(f(x, y(x)) - b(x)\big)y(x)^{-1}$$

for $x \in \Omega$ which satisfy $|y(x)| > M$, and $\check{a} \leqq \chi(x) \leqq \hat{a}$ for all x. Then the left-hand side of (5.2) can be rewritten in the form

$$\int_\Omega \omega(x)N(C\omega + \widetilde{u})(x)dx =$$

$$\int_{|y| \leqq M} \omega(x)f(x, y(x))dx - \int_{|y|>M} \omega(x)b(x)dx$$

$$- \int_{|y| \leqq M} \omega(x)y(x)\chi(x)dx + \int_\Omega \omega(x)y(x)\chi(x)dx.$$

The first three summands are bounded independently of C, while, for the last one, we have the following estimates:

for $C > 0$

$$\int_\Omega \omega(x)y(x)\chi(x)dx \geqq C\check{a}\int_\Omega \omega^2 + \check{a}\int_{\omega\widetilde{u}>0} \omega\widetilde{u} + \hat{a}\int_{\omega\widetilde{u}<0} \omega\widetilde{u}$$

$$= C\check{a}\int_\Omega \omega^2 + (\hat{a} - \check{a})\int_{\omega\widetilde{u}<0} \omega\widetilde{u} \geqq C(\check{a} - \sigma(\hat{a} - \check{a}))\|\omega\|^2;$$

for $C < 0$

$$\int_\Omega \omega y\chi \leqq C\check{a}\|\omega\|^2 + (\hat{a} - \check{a})\int_{\omega\widetilde{u}>0} \omega\widetilde{u} \leqq C(\check{a} - \sigma(\hat{a} - \check{a}))\|\omega\|^2.$$

Now, inequality (5.2) will be satisfied if $\check{a} - \sigma(\hat{a} - \check{a}) \geqq 0$. Hence by (3.11), we need

$$\gamma\|JG_0(\lambda_0)\| < \check{a}\left(\check{a}^2 + (\hat{a} - \check{a})^2\right)^{-\frac{1}{2}}. \tag{5.3}$$

Simple calculations give us the norm of the linear operator $JG_0(\lambda_0) : L^2(\Omega) \to L^2(\Omega)$

$$\|JG_0(\lambda_0)\| = \left(\min_{s>0} |\lambda_0 - \lambda_s|\right)^{-1} \tag{5.4}$$

and the constant γ

$$\gamma = \limsup_{M\to\infty} M^{-1} \|a_M\|_{L^2}.$$

Therefore, we prove the following

Theorem 4. (cf. [10]) *Let us consider BVP (4.3) where $L(\lambda_0) = P - \lambda_0 I$, P is an elliptic partial differential operator and λ_0 is its simple eigenvalue. If the nonlinearity f does not depend on derivatives and has a linear growth $\gamma \in (0, \infty)$ and, for sufficiently large $|u|$, satisfies (5.1) with $\check{a}$ and $\hat{a}$ such that inequality (5.3) holds, then the BVP has a solution.*

Example. Consider the second-order ordinary defferential equation with two-point null conditions

$$u'' + m^2 u = f(x, u),$$

$$u(0) = u(\pi) = 0,$$

where $m \in \mathbb{N}$. Let f be a Carathéodory function with property (5.1). Here, by (5.4), $\|JG_0(m^2)\| = (2m-1)^{-1}$ for $m > 1$ and $\|JG_0(m^2)\| = \frac{1}{3}$ for $m = 1$. Thus, condition (5.3) has the form

$$\gamma < (2m-1)\check{a}(\check{a}^2 + (\hat{a} - \check{a})^2)^{-\frac{1}{2}}, m > 1,$$

$$\gamma < 3\check{a}(\check{a}^2 + (\hat{a} - \check{a})^2)^{-\frac{1}{2}}, \qquad m = 1.$$

In particular, let

$$f(x, u) = \hat{a}u^+ - \check{a}u^- + g(x, u)$$

where $\check{a}, \hat{a}$ are positive constants and g is a Carathéodory function which is sublinear.

Then the problem with the jumping nonlinearity is solvable provided that

$$\max(\check{a},\hat{a})^2 + (\max(\check{a},\hat{a})\min(\check{a},\hat{a})^{-1} - 1)^2 < (2m-1)^2$$

for $m > 1$ and < 9 for $m = 1$ (cf. [4]).

Remark. Assumption (5.1) can be replaced by

$$-\hat{a} \leqq (f(x,u) - b(x)u^{-1} \leqq -\check{a}$$

for $|u| > M$; $\check{a}, \hat{a} > 0$. Theorem 4 with its proof and Example will be changed slightly (now $\lambda \to \lambda_0^+$, so $\alpha_0 = +1$ and we should prove the inequality opposite to (5.2)).

Although assumption (5.3) on γ is much more restrictive than those in recent papers [8], [10], [3], we have no conditions on the behaviour of f on the set $\Omega \times \langle -M, M\rangle$, such as $uf(x,u) \leqq 0$ which is global in the works mentioned.

We shall give an example of application of our abstract results, where the differential operator is not selfadjoint and the spaces X, Y, Z are, actually, Banach ones (even nonreflexive). Moreover, the nonlinearity is given by a more general function f. The problem has the Neumann form

$$u'' + p(\lambda_0)u' + r(\lambda_0)u = f(x,u), \qquad u'(0) = u'(1) = 0, \tag{5.6}$$

where p and r are analytic functions of real parameter λ in a neighbourhood of λ_0, and $f : \langle 0,1\rangle \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x,u)| \leqq a|u| + b(x) \tag{5.7}$$

where $b \in L^1(0,1)$. The assumption on f guarantee that the superposition operator $Nu(x) = f(x,u(x))$ maps $L^1(0,1)$ into itself. We need a resonance problem, so the linear homogeneous BVP

$$u'' + p(\lambda_0)u' + r(\lambda_0)u = 0, \qquad u'(0) = u'(1) = 0, \tag{5.8}$$

should have a nontrivial solution. This means that either $r(\lambda_0) = 0$ or $p^2(\lambda_0) - 4r(\lambda_0) + 4k^2\pi^2 = 0$ for some nonnegative integer k. We shall consider the second case with $k > 0$ and with the assumption that $p^2 - 4r + 4k^2\pi^2$ does not vanish identically.

Put

$$\omega_1(x;\lambda) = \exp\left(-\frac{1}{2}p(\lambda)x\right)\left(\cos d(\lambda)x + \frac{p(\lambda)}{2d(\lambda)}\sin d(\lambda)x\right),$$

$$\omega_2(x;\lambda) = \exp\left(-\frac{1}{2}p(\lambda)x\right)\sin d(\lambda)x,$$

where

$$d(\lambda) = \frac{1}{2}\left(4r(\lambda) - p^2(\lambda)\right)^{\frac{1}{2}}.$$

These functions form a fundamental system for the linear differential operator

$$L(\lambda)u = u'' + p(\lambda)u' + r(\lambda)u$$

such that $\omega_1(\cdot;\lambda_0)$ spans the one-dimensional space of solutions to (5.8). The standard calculations (see [2] for instance) show that the Green function of the problem is equal, for any $\lambda \neq \lambda_0$, to

$$\begin{aligned}\widetilde{G}(x,y;\lambda) = &\left[4d\left(p^2 + 4d^2\right)^{-1}\sin^{-1} d\left(d\cos d - \frac{p}{2}\sin d\right)\right.\\ &\left.\omega_1(y) - \omega_2(y)\right]e^{py}\omega_2(x) + K(x,y;\lambda)\end{aligned}$$

where

$$K(x,y;\lambda) = \begin{cases} d^{-1}e^{py}\det\begin{bmatrix}\omega_1(y) & \omega_2(y)\\ \omega_1(x) & \omega_2(x)\end{bmatrix} & \text{for } y < x,\\ 0 & \text{for } y > x,\end{cases}$$

and we have omitted the argument λ.

Let $X = Z = L^1(0,1)$ and $Y = \{u \in W^{2,1}(0,1) : \lim_{x\to 0+} u'(x) = \lim_{x\to 0+} u'(x) = 0\}$ where $W^{2,1}(0,1)$ is the Sobolew space of all functions $u : \Omega \to \mathbb{R}$ whose all derivatives (in the distribution sense) up to order 2 are integrable ($\in L^1(0,1)$). It easy to see that $L(\lambda)$ maps homeomorphically Y onto Z for $\lambda \neq \lambda_0$. Put

$$c_1(\lambda) = 4d^2\left(p^2 + 4d^2\right)^{-1} \operatorname{ctg} d,$$

$u_1(\lambda)$ - a linear continuous functional on Z given by a bounded function $e^{py}\omega_1(y)$, $w_1(\lambda)(x) = \omega_1(x;\lambda_0)$ and $G_0(\lambda) : Z \to Y$ - a linear integral operator with the following kernel:

$$\begin{aligned}\widetilde{G}_0(x,y;\lambda) = &- [2pd^2(p^2+4d^2)^{-1}\omega_1(y;\lambda) + \omega_2(y;\lambda)]e^{py}\omega_1(x;\lambda)\\ &+ K(x,y;\lambda) + c_1(\lambda)e^{py}\omega_1(y;\lambda)(\omega_1(x,\lambda) - \omega_1(x;\lambda_0)).\end{aligned}$$

This kernel has a limit as $\lambda \to \lambda_0$ iff there exists a limit

$$\lim_{\lambda\to\lambda_0} \sin^{-1} d(\lambda)\left[e^{-\frac{1}{2}p(\lambda)x}\cos d(\lambda)x - e^{-\frac{1}{2}p(\lambda)x}\cos k\pi x\right].$$

If we denote the multiplicity of λ_0 as a zero of an analytic function h by $m(h;\lambda_0)$, then the last limit exists iff

$$(5.9)\quad m(4r - p^2 - 4k^2\pi^2;\lambda_0) \leqq m\left(p - p(\lambda_0);\lambda_0\right) = m(r - r(\lambda_0);\lambda_0)$$

(the last equality always holds if $p(\lambda_0) \neq 0$). Under this assumption, all the functions have continuous extensions as $\lambda \to \lambda_0$. Hence, we can consider BVP (5.6) in our abstract framework.

Condition (5.7) gives

$$\lim_{\|u\|\to\infty} \sup \|Nu\|/\|u\| = \gamma \leqq a$$

where the norms are from $L^1(0,1)$. If $\gamma = 0$, we can apply Theorem 1 and get the following result:

BVP (5.6) with a sublinear nonlinearity ($\gamma = 0$) has a solution if the numbers

$$\int_{A_+} e^{p_0 y}\omega_1(y;\lambda_0)f_+(y)dy + \int_{A_-} e^{p_0 y}\omega_1(y;\lambda_0)f_-(y)dy,$$

$$\int_{A_-} e^{p_0 y}\omega_1(y;\lambda_0)f_-(y)dy + \int_{A_+} e^{p_0 y}\omega_1(y;\lambda_0)f_+(y)dy$$

are of opposite sings, where

$$f_\pm(y) = \lim_{u\to\pm\infty} f(y,u),$$

$A_+ = \{y : \omega_1(y;\lambda_0) > 0\}$, $A_- = \{y : \omega_1(y;\lambda_0) < 0\}$, $p_0 = p(\lambda_0)$. Similarly as in (4.9), one can replace the limits in the definition of $f_\pm$ by $\limsup$ and $\liminf$.

When $\gamma > 0$, we shall assume condition (5.1) with $b \in L^1(0,1)$. As in the selfadjoint case, one should study the sign of

$$\int_0^1 e^{py}\omega_1(y;\lambda)N(Cw_1(\lambda_0)+\widetilde{u})(y)dy$$

for large $|C|$ or, equivalently, of

$$R = \int_0^1 e^{py}\omega_1(y;\lambda)\big(C\omega_1(y;\lambda_0)+\widetilde{u})(y)\big)\,\chi(y)dy$$

where $\chi(y) \in \langle \check{a}, \hat{a}\rangle$ and $\|\widetilde{u}\| \leqq \sigma|C|\,\|w_1(\lambda_0)\|$. For $C > 0$, we have

$$\begin{aligned} R &\geqq C\min(1,e^p)\check{a}\|\omega_1(\cdot;\lambda)\|_{L^2}\|w_1(\lambda_0)\|_{L^2} \\ &\quad - \max(1,e^p)\sup_y|\omega_1(y;\lambda)|\,\hat{a}\int|u| \\ &\geqq C\|w_1(\lambda_0)\|_{L^2}(\min(1,e^p)\,\check{a}\,\|\omega_1(\cdot;\lambda)\|_{L^2} \\ &\quad - \max(1,e^p)\,\hat{a}\,\sigma\sup_y|\omega_1(y;\lambda)|) \end{aligned}$$

and the following condition

$$\min(1,e^p)\,\check{a}\,\|\omega_1(\cdot;\lambda)\|_{L^2} \leqq \max(1,e^p)\,\hat{a}\,\sigma\sup_y|\omega_1(y;\lambda)|$$

is needed to obtain $R > 0$. The same condition is obtained for $C < 0$. Inequality (5.10) should be satisfied for all λ from a neighbourhood of λ_0, but the left-hand side is a continuous function of λ, thus we can put $\sup|w_1(\lambda_0)|$ and $\|w_1(\lambda_0)\|_{L^2}$ in (5.10) with the sharp inequality $>$. After simple though toilsome calculations we get

$$\sup|w_1(\lambda_0)| = \max\left(1, e^{-\frac{1}{2}p_0}\right),$$

$$\|w_1(\lambda_0)\|_{L^2}^2 = \left(\frac{1}{2p_0} + \frac{2p_0}{p_0^2+4k^2\pi^2}\right)(1-e^{-p_0}) > \frac{1-e^{-p_0}}{2p_0},$$

$$\|JG_0(\lambda_0)\| = \int_0^1 \sup_y \widetilde{G}_0(x,y;\lambda_0)dx < \left(\frac{3}{2}|\xi| + \frac{2}{9}|p_0| + \frac{11}{3} + \frac{14}{|p_0|}\right)e^{\frac{1}{2}|p_0|}$$

where $\xi = \lim\limits_{\lambda\to\lambda_0}(p(\lambda)-p(\lambda_0))/(4r(\lambda)-p^2(\lambda)-4k^2\pi^2)$. Using (3.2), we get the following restriction for the solvability of nonlinear problem (5.6) in the case $\gamma > 0$:

$$\gamma\left(\frac{3}{2}|\xi| + \frac{2}{9}|p_0| + \frac{11}{3} + \frac{14}{|p_0|}\right)\left(\hat{a}\sqrt{2|p_0|} + \check{a}e^{-|p_0|}\sqrt{|1-e^{-p_0}|}\right)$$
$$< \check{a}e^{-\frac{3}{2}|p_0|}\sqrt{|1-e^{-p_0}|}.$$

Our estimation were not subtle and can be strengthened, but the last condition will then become more and more complicated.

One can consider a more general problem in the same way:

$$u^{(m)} + p_{m-1}(x,\lambda_0)u^{(m-1)} + \cdots + p_1(x,\lambda_0)u' + p_0(x,\lambda_0)u = f(x,u),$$
$$B_i(u) = 0, \qquad i = 1,\ldots,m,$$

where $B_1,\ldots,B_m$ are linear operators acting on $u^{(j)}(a), u^{(j)}(b)$, $j = 0,1,\ldots,m-1$; $p_0,\ldots,p_{m-1}$ are functions sufficiently smooth w.r.t. x and analytic w.r.t. λ. The linear homogeneous problem is supposed to have a nontrivial solution for $\lambda = \lambda_0$ and not to have such a solution for λ close to λ_0. This general problem was studied in [19] in the case of a bounded continuous f.

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REZONANSOWE RÓWNANIA OPERATOROWE Z NIEOGRANICZONYMI CZĘŚCIAMI NIELINIOWYMI

Udowodnione jest istnienie rozwiązań równań nieliniowych postaci $Lx = N(x)$, gdzie L jest operatorem liniowym indeksu 0, a N odwzorowaniem nieliniowym ciągłym subliniowym lub o wzroście liniowym. Zakładane są warunki uogólniające warunki Landesmana-Lazera. Rezultaty abstrakcyjne zastosowano do problemów brzegowych, w których część nieliniowa zależy także od pochodnych, a rezonans może być wielowymiarowy.

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