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A VARIATIONAL METHOD FOR GENERALIZED GEL'FER FUNCTIONS

To Professor Lech Włodarski on His 80th birthday

The note is devoted to a class $\mathcal{G}_n$ of functions f analytic and univalent in the unit disk U, satisfying in addition the conditions $f(0) = 1$ and, in the case $n = 1$: $O \notin f(U)$, in the case $n \geq 2$: if $w \in f(U)$, then $\varepsilon_j w \notin f(U)$, $\varepsilon_j = \exp \frac{2\pi i j}{n}$, for every $j = 1, \ldots, n-1$. Variational formulas are derived and, as applications, are given the estimations of some functions in the considered class of functions.

1. Introduction

Let $\mathcal{G}_n$, $n \geq 1$, be a class of functions f which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$, have a series development

$$f(z) = 1 + a_1 z + a_2 z^2 + \ldots$$

and satisfy the condition:

1^0 if $w \in f(U)$, then $\varepsilon_j w \notin f(U)$, $\varepsilon_j = \exp \frac{2\pi i j}{n}$, for every $j = 1, \ldots, n-1$, $n \geq 2$, $0 \notin f(U)$ for $n = 1$.

It is clear that $\mathcal{G}_2$ is the well known Gel'fer class $\mathcal{G}$ and $\mathcal{G}_1$ is the class of nonvanishing functions [8]. Moreover if f is in $\mathcal{G}_n$, then $f^{n/2}$, where $f^{n/2}(0) = 1$, is in $\mathcal{G}$ and if f is in $\mathcal{G}$, then $f^{2/n}$, where $f^{2/n}(0) = 1$, is in $\mathcal{G}_n$. The variational method developed by Hummel in [5] for Gel'fer class induces of course a variation in the class $\mathcal{G}_n$. However the calculations are such that it is almost as easy to obtain the variation directly in the class $\mathcal{G}_n$, using the technics developed by Hummel and Schiffer in [6].

2. Variational formulas

Let D be a domain with the following property:

2^0 if $w \in D$, then $\varepsilon_j w \notin D$, $j = 1, \ldots, n-1$, for $n \geq 2$ and $0 \notin D$ for $n = 1$.

Let Δ be a domain containing ∂D and such that $w \in \overline{\Delta}$ iff $\varepsilon_j w \in \overline{\Delta}$, $j = 1, \ldots, n-1$. Let $\Phi(w)$ be analytic in $\overline{\Delta}$ and let it satisfy the identity

$$\Phi(w) = \Phi(\varepsilon_j w), \quad j = 1, \ldots, n-1,$$

for $n \geq 2$ and all $w \in \overline{\Delta}$. Moreover let it be such that the function

$$\Psi(w, \omega) = \begin{cases} \frac{\Phi(w) - \Phi(\omega)}{w - \omega}, & w \neq \omega, \\ \Phi'(w), & w = \omega, \end{cases}$$

is defined, analytic and uniformly bounded in $\overline{\Delta} \times \overline{\Delta}$. It can be proved that the function

$$w^*(w) = w \exp \varepsilon e^{i\alpha} \Phi(w),$$

where $\varepsilon > 0$, $\alpha \in \mathcal{R}$, for ε sufficiently small and for all α is univalent in Δ and maps the boundary ∂D onto the boundary of a new domain D^* also having the property 2^0.

Let w_0 be any point such that $\varepsilon_j w_0 \notin \partial D$, $j = 0, \ldots, n-1$, and set

$$\Phi(w) = \frac{w^n - 1}{w^n - w_0^n} \tag{1}$$

It is easily shown that $\Phi(w)$ satisfies the requirements given above, and hence induces a variation for generalized Gel'fer functions.

Let $f \in \mathcal{G}_n$ and let

(i) $\varepsilon_j w_0 \notin \overline{f(U)}$, $j = 0, ..., n-1$ or

(ii) $w_0 \in f(U)$.

In both cases the function (1) has the required properties for $D = f(U)$. In the case (i) the composition $w^* \circ f$ gives us the varied function

$$f^*(z) = f(z) + \varepsilon f(z) e^{i\alpha} \frac{f^n(z) - 1}{f^n(z) - w_0^n} + o(\varepsilon). \tag{2}$$

In the case (ii), basing on Goluzin's method of constructing variations of functions of the class S ($f \in S$ if f is analytic and univalent in U and $f(0) = f'(0) - 1 = 0$) [4] p. 98, we obtain the varied function in the form

$$\begin{aligned} f^*(z) = f(z) &+ \varepsilon f(z) e^{i\alpha} \frac{f^n(z) - 1}{f^n(z) - f^n(\zeta)} \\ &- \varepsilon e^{i\alpha} z f'(z) \frac{f^n(\zeta) - 1}{n\zeta^2 f^{n-2}(\zeta) f'^2(\zeta)} \frac{\zeta}{z - \zeta} \\ &+ \varepsilon e^{-i\alpha} z f'(z) \frac{\overline{f^n(\zeta) - 1}}{n\bar{\zeta}^2 \overline{f^{n-2}(\zeta)}\, \overline{f'^2(\zeta)}} \frac{\bar{\zeta} z}{1 - \bar{\zeta} z} \\ &+ o(\varepsilon), \end{aligned} \tag{3}$$

where $f(\zeta) = w_0$.

Other useful varied functions we can obtain by transformations in the z-plane. Let $\omega(z)$ be univalent in U, $\omega(U) \subset U$, $\omega(0) = 0$, then $f \circ \omega \in \mathcal{G}_n$. In particular, putting $\omega(z) = e^{\pm i\varepsilon} z$, $\varepsilon > 0$, we have a varied function

$$f^*(z) = f(e^{\pm i\varepsilon} z) = f(z) \pm \varepsilon i z f'(z) + o(\varepsilon). \tag{4}$$

Putting next $\omega(z) = (1 - \varepsilon) z$, $0 < \varepsilon < 1$, we have a varied function

$$f^*(z) = f((1 - \varepsilon) z) = f(z) - \varepsilon z f'(z) + o(\varepsilon). \tag{4'}$$

Putting finally $\omega(z) = k_\alpha^{-1}((1 - \varepsilon) k_\alpha(z))$, where $k_\alpha(z) = z(1 + e^{-i\alpha} z)^{-2}$, $0 < \varepsilon < 1$, $\alpha \in \mathcal{R}$, we have the slit variation

$$f^*(z) = f(z) - \varepsilon z f'(z) \frac{e^{i\alpha} + z}{e^{i\alpha} - z} + o(\varepsilon). \tag{5}$$

3. Schiffer equation

Let Ψ be a continuous complex valued functional over $\mathcal{G}_n$, having a Gâteaux complex derivative. That means

$$\Psi(f+\varepsilon h) = \Psi(f) + \varepsilon \Lambda_f(h) + o(\varepsilon) \tag{6}$$

for any $f \in \mathcal{G}_n$, h analytic in U, and any $\varepsilon > 0$, where Λ_f is continuous linear functional in h.

The class $\mathcal{G}_n$ is not compact in the topology of uniform convergence on compact subsets. However it becomes compact by addition of the function $f(z) \equiv 1$. Thus the problem of maximizing $\operatorname{Re}\Psi$ in $\mathcal{G}_n \cup \{1\}$, if of course Ψ is defined also for $f = 1$, always has a solution in $\mathcal{G}_n \cup \{1\}$ and this solution is in $\mathcal{G}_n$, if we verify that $f = 1$ is not maximal.

Suppose now that $f \in \mathcal{G}_n$ is locally maximal for $\operatorname{Re}\Psi$, that means $\operatorname{Re}\Psi(f^*) \leq \operatorname{Re}\Psi(f)$ for all ,,nearby" $f^* \in \mathcal{G}_n$ (,,nearby" in the sense of uniform convergence on compact subsets). Using the varied functions (2), (3), (4), (4') and (5) we can prove

Theorem 1. *Let Ψ be a complex valued functional defined and continuous over $\mathcal{G}_n$, having a complex Gâteaux derivative Λ_f as defined in (6).*

If $f \in \mathcal{G}_n$ is locally maximal for $\operatorname{Re}\Psi$, then f has following properties.

(a) *$w = f(\zeta)$ satisfies the differential equation*

$$\frac{n\zeta^2 w^{n-2} w'^2}{w^n - 1}\Lambda_f\left(f(z)\frac{f^n(z)-1}{f^n(z)-w^n}\right) = \Lambda_f\left(zf'(z)\frac{\zeta}{z-\zeta}\right) - \overline{\Lambda_f\left(zf'(z)\frac{\overline{\zeta}z}{1-\overline{\zeta}z}\right)} \tag{7}$$

in some ring $P = \{\zeta : r < |\zeta| < 1\}$.

(b) $\operatorname{Im}\Lambda_f(zf'(z)) = 0$, $\operatorname{Re}\Lambda_f(zf'(z)) \geq 0$.

(c) *The right-hand side of (9) is an analytic function in the ring $P_1 = \{\zeta : r < |\zeta| < \frac{1}{r}\}$, real and non-positive on ∂U.*

(d) *f maps U onto a domain whose boundary was made up of analytic arcs which lie on trajectories of the quadratic differential*

$$\text{(8)} \qquad \frac{n\,w^{n-2}}{w^n-1}\Lambda_f\left(f(z)\frac{f^n(z)-1}{f^n(z)-w^n}\right)dw^2.$$

(e) *If $\Lambda_f\left(f(z)\frac{f^n(z)-1}{f^n(z)-w^n}\right)$ is a rational function $\not\equiv$const., then the set $\mathcal{C}\setminus\bigcup_{j=0}^{n-1}\varepsilon_j f(U)$ has no interior points, where $\varepsilon_j f(U)=\{w:\exists_{z\in U} w=\varepsilon_j f(z)\}$.*

(f) *If the points $\varepsilon_j w_0$, $j=0...n-1$, are not in $f(U)$, then at least one of them is on the boundary $\partial f(U)$. The points 0 and ∞ are also on this boundary.*

The function $g(z)=\frac{1+f^n(z)}{1-f^n(z)}$ is univalent in U and maps U onto a domain which closure is the entire complex sphere.

Proof.

Ad (a) From the varied formula (3) and the formula (6) it follows

$$\begin{aligned}\operatorname{Re}\Psi(f^*) = \operatorname{Re}\Psi(f) + \varepsilon\operatorname{Re}\Bigg\{e^{i\alpha}\Bigg[\Lambda_f\Bigg(&\frac{f^n(z)-1}{f^n(z)-f^n(\zeta)}\\ &- zf'(z)\frac{f^n(\zeta)-1}{n\zeta^2 f^{n-2}(\zeta)f'^2(\zeta)}\frac{\zeta}{z-\zeta}\Bigg)\\ &+\overline{\Lambda_f\left(zf'(z)\frac{\overline{f^n(\zeta)}-1}{n\overline{\zeta}^2\,\overline{f^{n-2}(\zeta)}\;\overline{f'^2(\zeta)}}\frac{\overline{\zeta}z}{1-\overline{\zeta}z}\right)}\Bigg]\Bigg\}+o(\varepsilon).\end{aligned}$$

Furthermore, because α is arbitrary and f makes $\operatorname{Re}\Psi$ a local maximum, we have

$$\text{(9)}\quad \Lambda_f\left(\frac{n\zeta^2 f^{n-2}(\zeta)f'^2(\zeta)}{f^n(\zeta)-1}f(z)\frac{f^n(z)-1}{f^n(z)-f^n(\zeta)} - zf'(z)\frac{\zeta}{z-\zeta}\right) = \overline{\Lambda_f\left(zf'(z)\frac{\overline{\zeta}z}{1-\overline{\zeta}z}\right)}.$$

Taking account of the representation of continuous linear functional in the set of analytic functions in U [1], we shall extend the functional

Λ_f to a continuous linear functional on the class of functions meromorphic in U and having the poles in the ring P. As a consequence we obtain (7), where ζ is arbitrary in P.

Ad (b) The varied formulas (4) and (4') give

$$\operatorname{Re}\Psi(f^*) = \operatorname{Re}\Psi(f) + \varepsilon \operatorname{Re}\{\Lambda_f(i\,z\,f'(z))\} + o(\varepsilon),$$

and

$$\operatorname{Re}\Psi(f^*) = \operatorname{Re}\Psi(f) - \varepsilon \operatorname{Re}\{\Lambda_f(zf'(z))\} + o(\varepsilon).$$

Since f realizes the maximum of $\operatorname{Re}\Psi$ and ε is an arbitrary real number or an arbitrary positive number, then (b).

Ad (c),(d) The varied formula (5) shows that for $\zeta \in \partial U$

$$\operatorname{Re}\Psi(f^*) = \operatorname{Re}\Psi(f) - \varepsilon \operatorname{Re}\left\{\Lambda_f\left(zf'(z)\frac{\zeta+z}{\zeta-z}\right)\right\} + o(\varepsilon).$$

Since f is maximal for $\operatorname{Re}\Psi$ and ε is an arbitrary positive number, then

$$\operatorname{Re}\left\{\Lambda_f\left(zf'(z)\frac{\zeta+z}{\zeta-z}\right)\right\} \le 0$$

and hence and by (b) it follows that the left-hand side of (7) is analytic for $w = f(\zeta)$ on ∂U and that

$$\operatorname{Re}\left\{\frac{n\zeta^2 f^{n-2}(\zeta) f'^2(\zeta)}{f^n(\zeta)-1}\Lambda_f\left(f(z)\frac{f^n(z)-1}{f^n(z)-f^n(\zeta)}\right)\right\} \le 0$$

on ∂U. Consequently, the boundary $\partial f(U)$ must lie on the trajectories of the quadratic differential (8).

Ad (e) Suppose now that the set $E = \mathcal{C} \setminus \bigcup_{j=0}^{n-1} \varepsilon_j f(U)$ has an interior point. Hence, there exists a disk $K \subset E$. Let $w \in K$. If we apply (6) and the varied formula (2), we obtain

$$\operatorname{Re}\Psi(f^*) = \operatorname{Re}\Psi(f) + \varepsilon \operatorname{Re}\left\{e^{i\alpha}\Lambda_f\left(f(z)\frac{f^n(z)-1}{f^n(z)-w^n}\right)\right\} + o(\varepsilon).$$

f is maximal, then $\Lambda_f\left(f(z)\frac{f^n(z)-1}{f^n(z)-w^n}\right) = 0$ for $w \in K$, that contradicts our assumption.

Ad (f) The univalence of g is the consequence of the condition 1^o. The rest of the properties are obvious.

Example. To illustrate the theorem given above, we shall now find an estimate for the functional $\Psi(f) = f'(0)$. We find without any difficulties that any $f \in \mathcal{G}_n$ which locally maximizes $\operatorname{Re} f'(0)$ must satisfy the differential equation

$$\frac{n^2 w^{n-2} w'^2}{(1 - w^n)^2} = \frac{1}{\zeta^2}.$$

Integrating this equation we receive that

$$f(\zeta) = \left(\frac{1+\zeta}{1-\zeta}\right)^{\frac{2}{n}} = 1 + \frac{4}{n}\zeta + \dots,$$

and then

$$\max_{f \in \mathcal{G}_n} \operatorname{Re} f'(0) = \frac{4}{n}. \tag{10}$$

4. Goluzin and Grunsky inequalities

As an important application of the Theorem 1, consider the problem of maximizing the functional defined as follows.

Suppose that L denote a continuous and linear functional defined in the set $H(U)$ of analytic functions in U and let $L(1) = 0$. Let $\varphi(z, u)$ be analytic in $U \times U$ and $\psi(z, u) = \varphi(z, \overline{u})$. We define

$$L^2(\varphi) = L(L(\varphi)), \quad |L|^2(\psi) = L(\overline{L(\psi)}),$$

where we compose L succesively with the function of the first remaining variable, see [7], p. 114. For L^2 the order of composition is not important by general formula of the continuous and linear functional defined in $H(U)$. For $|L|^2$ we note that $|e^{i\alpha} L|^2 = |L|^2$ and $\operatorname{Im} |L|^2(\psi) = 0$ if $\psi(z, u) = \overline{\psi(\overline{z}, \overline{u})}$.

Theorem 2. *Assume that there exists the function which maximizes in $\mathcal{G}_n$ the real part of the functional*

$$\Psi(f) = \lambda^2 \log \frac{n\, f'(0)}{4} + 2\lambda L \left(\log \frac{f^k(z) - 1}{z(f^k(z) + 1)}\right) + L^2 \left(\log \frac{f^k(z) - f^k(u)}{(z-u)(f^k(z) + f^k(u))}\right),$$

where $k = \frac{n}{2}$, $\lambda \in \mathcal{R}$, then

$$\max_{f \in \mathcal{G}_n} \operatorname{Re} \Psi(f) = |L|^2 \left(\log(1 - z\overline{u})\right). \tag{11}$$

Remark. The functional Ψ is defined and continuous in $\mathcal{G}_n$ but not in $\mathcal{G}_n \cup \{1\}$, then it is possible that the function maximal for $\operatorname{Re} \Psi$ does not exist.

Proof. Let f be the maximal function for $\operatorname{Re} \Psi$. The complex Gâteaux derivative Λ_f of Ψ is

$$\Lambda_f(h) = \lambda^2 \frac{h'(0)}{f'(0)} + 2\lambda L \left(n \frac{f^{k-1}(z) h(z)}{f^n(z) - 1}\right) + L^2 \left(n f^{k-1}(z) f^{k-1}(u) \frac{f(u)h(z) - f(z)h(u)}{f^n(z) - f^n(u)}\right).$$

Applying the Theorem 1 we obtain for $f(\zeta)$, $\zeta \in P = \{\zeta : r < |\zeta| < 1\}$

$$\frac{n^2 \zeta^2 f^{n-2}(\zeta) f'^2(\zeta)}{(1 - f^n(\zeta))^2} \left(\lambda + (1 - f^n(\zeta)) L \left(\frac{f^k(z)}{f^n(z) - f^n(\zeta)}\right)\right)^2 = -B(\zeta), \tag{12}$$

where

$$B(\zeta) = \Lambda_f \left(z\, f'(z) \frac{\zeta}{z - \zeta}\right) - \overline{\Lambda_f \left(z\, f'(z) \frac{\overline{\zeta} z}{1 - \overline{\zeta} z}\right)},$$

is analytic in the ring $P_1 = \{\zeta : r < |\zeta| < \frac{1}{r}\}$ and $B(\zeta) \leq 0$ on ∂U. Hence, the left hand side of (12) has an analytic continuation on the same ring and it is a square of the function

$$(13)\quad \Phi(\zeta) = \frac{n\zeta f^{k-1}(\zeta)f'(\zeta)}{1-f^n(\zeta)}\left(\lambda + (1-f^n(\zeta))L\left(\frac{f^k(z)}{f^n(z)-f^n(\zeta)}\right)\right)$$

which is analytic in the ring P. By (12) we see that there exists a branch of square root $B^*(\zeta)$ of $-B(\zeta)$ in P and it can be proved that $B^*(\zeta)$ has an analytic continuation over ∂U. The same is true for the function $\Phi(\zeta)$, so it is analytic in the ring $P_2 = \{\zeta : r < |\zeta| \leq 1\}$ and, what follows from the inequality $-B(\zeta) \geq 0$ on ∂U, real on ∂U. By adding to (13) the function $-L\left(\frac{\zeta}{z-\zeta}\right) + L\left(\frac{1}{1-\overline{\zeta}z}\right)$, which is analytic in P_2 and real on ∂U, we receive the function

$$X(\zeta) = \Phi(\zeta) - L\left(\frac{\zeta}{z-\zeta}\right) + \overline{L\left(\frac{1}{1-\overline{\zeta}z}\right)}$$

analytic in $\overline{U}$, real on ∂U and $X(0) = \Phi(0) = -\lambda$. By the Schwarz reflection principle, X extends to a bounded analytic function in $\mathcal{C}$. Then $X(\zeta) = X(0) = -\lambda$ for each ζ. We divide this identity by ζ and write it as follows

$$(14)\quad \frac{\partial}{\partial\zeta}\left(\lambda \log\frac{f^k(\zeta)-1}{\zeta(f^k(\zeta)+1)} + L\left(\log\frac{f^k(z)-f^k(\zeta)}{(z-\zeta)(f^k(z)+f^k(\zeta))}\right) + \overline{L(\log(1-\overline{\zeta}z))}\right) = 0.$$

If we integrate (14) from 0 to ζ, we obtain

$$(15)\quad \lambda \log\frac{f^k(\zeta)-1}{\zeta(f^k(\zeta)+1)} + L\left(\log\frac{f^k(z)-f^k(\zeta)}{(z-\zeta)(f^k(z)+f^k(\zeta))}\right) + \overline{L(\log(1-\overline{\zeta}z))} = c,$$

where

$$(16)\quad c = \lambda \log\frac{n f'(0)}{4} + L\left(\log\frac{f^k(z)-1}{z(f^k(z)+1)}\right).$$

It is easily seen that $\operatorname{Re} c = 0$. Indeed, by Theorem 1 (e) the set $\mathcal{C} \setminus \bigcup_{j=0}^{n-1} \varepsilon_j f(U)$ has no interior points, then there exists a point ω which is common for the boundaries $\partial\varepsilon_\mu f(U)$ and $\partial\varepsilon_\nu f(U)$, where $\varepsilon_\mu^k = -1$ and $\varepsilon_\nu^k = 1$. Hence, there exist two sequences (z'_m) and (z''_m), z'_m, $z''_m \in U$ such that $\varepsilon_\mu f(z'_m) \to \omega$, $\varepsilon_\nu f(z''_m) \to \omega$ as $m \to \infty$. We may assume that $z'_m \to z'$, $z''_m \to z''$, z', $z'' \in \partial U$. Putting in (15) succesively $\zeta = z'_m$ and $\zeta = z''_m$, letting $m \to \infty$ and adding side by side the equalities obtained in such a way, we conclude that c is pure imaginary. Multiplying now (16) by λ, applying L to both sides of (15), adding side by side the equalities obtained in such a way we have (11). The maximal function satisfies the identity (15) with (16) where $\operatorname{Re} c = 0$.

Theorem 3. *If $\lambda \neq 0$ real, then every function f of $\mathcal{G}_n$ satisfies the inequality*

$$\operatorname{Re} \Psi(f) \leq -|L|^2(\log(1 - \bar{z}u)). \tag{17}$$

This inequality is exact in the sense that there exists the function in $\mathcal{G}_n$ for which the inequality (17) becomes equality.

Proof. First we observe that the functional $\operatorname{Re} \Psi$ is bounded from above in $\mathcal{G}_n$. This follows from (10) and from the general form of the continuous linear functional in $H(U)$ by using the inequality of Goluzin [4] ($|\log \frac{g(z)}{z} + \log(1 - |z|^2)| \leq \log \frac{1+|z|}{1-|z|}$ for $g \in S$).

Let $M = \sup_{f \in \mathcal{G}_n} \operatorname{Re} \Psi(f)$. Then there exists a sequence (f_m), $f_m \in \mathcal{G}_n$, almost uniformly convergent in U, such that $\operatorname{Re} \Psi(f_m) \to M$. Let $f_m \to f$. If $f \in \mathcal{G}_n$, then $\operatorname{Re} \Psi(f) = M$ and f is a maximal function for $\operatorname{Re} \Psi$. Suppose now that $f = 1$. Then $f'_m(0) \to 0$ and the first term in $\operatorname{Re} \Psi(f_m)$ has a limit $-\infty$. Let $F_m = \frac{2}{n}(f'_m(0))^{-1}(f_m^k - 1)$. We see that $F_m \in S$ and, without loss of generality, we may suppose that $F_m \to F \in S$ almost uniformly in U. Substituting $\frac{n}{2} f'(0) F_m + 1$ in the place of f_m^k in $\operatorname{Re} \Psi(f_m)$, we get that the last two terms in $\operatorname{Re} \Psi(f_m)$ converge to finite limits

$$\operatorname{Re} \left\{ 2\lambda \left(\log \frac{F(z)}{z} \right) \right\} \quad \text{and} \quad \operatorname{Re} \left\{ L^2 \left(\log \frac{F(z) - F(u)}{z - u} \right) \right\}$$

respectively. Hence, $M = -\infty$, which is impossible.

We shall now examine the case when $\lambda = 0$. Getting $\lambda \to 0$ in (17), we assert that

$$\text{(18)} \quad \operatorname{Re}\left\{L^2\left(\log\frac{f^k(z)-f^k(u)}{(z-u)(f^k(z)+f^k(u))}\right)\right\} \leq -|L|^2(\log(1-\overline{z}u))$$

for every $f \in \mathcal{G}_n$. To prove that the inequality (18) is exact, we first remind that for each $F \in S$ the inequality

$$\text{(19)} \qquad \operatorname{Re}\left\{L^2\left(\log\frac{F(z)-F(u)}{z-u}\right)\right\} \leq -|L|^2(\log(1-\overline{z}u))$$

holds and there is a function of S for which (19) becomes equality, [7], p.114. Let $\tilde{F}$ be such a function. Next we observe that $\tilde{F}$ can be represented as a limit of a sequence (F_m) of bounded functions belonging to S. From the other hand, for each bounded function F_m and for the constant b_m, $|b_m|$ sufficiently small, the function $f_m(z) = (1+b_mF_m(z))^{2/n}$ belongs to $\mathcal{G}_n$. If we assume that $b_m \to 0$ then, by the continuity of L, we have

$$\begin{aligned}
&\operatorname{Re}\left\{L^2\left(\log\frac{f_m^k(z)-f_m^k(u)}{(z-u)(f_m^k(z)+f_m^k(u))}\right)\right\}\\
&\quad = \operatorname{Re}\left\{L^2\left(\log\frac{b_m(F_m(z)-F_m(u))}{(z-u)(2+b_m(F_m(z)+F_m(u))}\right)\right\}\\
&\quad\quad \to \operatorname{Re}\left\{L^2\left(\log\frac{\tilde{F}(z)-\tilde{F}(u)}{z-u}\right)\right\} = -|L|^2(\log(1-\overline{z}u)).
\end{aligned}$$

Hence, it is obvious that the inequality (18) can not be improved.

We have thus proved

Theorem 4. *For every function $f \in \mathcal{G}_n$ the following inequality*

$$\operatorname{Re}\left\{L^2\left(\log\frac{f^k(z)-f^k(u)}{(z-u)(f^k(z)+f^k(u))}\right)\right\} \leq -|L|^2\log(1-\overline{z}u)$$

holds. This inequality can not be improved.

Remark. By replacing L by $e^{i\alpha}L$, $\alpha \in \mathcal{R}$, we have a second version of inequality (18):

$$\text{(18')} \qquad \left|L^2\left(\log\frac{f^k(z)-f^k(u)}{(z-u)(f^k(z)+f^k(u))}\right)\right| \leq -|L|^2\log(1-\overline{z}u).$$

To illustrate the theorems given above, we shall find the estimations for some functionals defined in $\mathcal{G}_n$.

(A) Let $\lambda = 1$, $L(h) = 0$. Then for each $f \in \mathcal{G}_n$

$$|f'(0)| \le \frac{4}{n}$$

and the maximal function is

$$f(z) = \left(\frac{1+z}{1-z}\right)^{\frac{2}{n}}.$$

The same result was obtained directly from the equation (7).

(B) Let $L(h) = \sum_{\mu=1}^{N} \lambda_\mu (h(z_\mu) - h(0))$, where $z_1, ..., z_N$ are arbitrary points in U and $\lambda_1, ..., \lambda_N$ arbitrary complex numbers, $\lambda \in \mathcal{R}$. Then for each $f \in \mathcal{G}_n$

$$\text{(20)} \quad \operatorname{Re}\left\{\lambda^2 \log \frac{n f'(0)}{4} + 2\lambda\left(\sum_{\mu=1}^{N} \lambda_\mu \log \frac{f^k(z_\mu) - 1}{z_\mu (f^k(z_\mu) + 1)} - \log \frac{n f'(0)}{4} \sum_{\mu=1}^{N} \lambda_\mu\right) + \sum_{\mu,\nu=1}^{N} \lambda_\mu \lambda_\nu \log \frac{f^k(z_\mu) - f^k(z_\nu)}{(z_\mu - z_\nu)(f^k(z_\mu) + f^k(z_\nu))} - 2\sum_{\mu=1}^{N} \lambda_\mu \cdot \sum_{\mu=1}^{N} \lambda_\mu \log \frac{f^k(z_\mu) - 1}{z_\mu (f^k(z_\mu) + 1)} + \log \frac{n f'(0)}{4} \left(\sum_{\mu=1}^{N} \lambda_\mu\right)^2\right\} \le - \sum_{\mu,\nu=1}^{N} \lambda_\mu \overline{\lambda_\nu} \log(1 - z_\mu \overline{z_\nu}).$$

We define the differential quotient $\frac{h(z)-h(u)}{z-u}$ as $h'(z)$ when $u = z$. Putting in (20) $\lambda = \sum_{\mu=1}^{N} \lambda_\mu$, where $\sum_{\mu=1}^{N} \lambda_\mu$ is real, we have the inequality

$$\text{(21)} \quad \operatorname{Re}\left\{\sum_{\mu,\nu=1}^{N} \lambda_\mu \lambda_\nu \log \frac{f^k(z_\mu) - f^k(z_\nu)}{(z_\mu - z_\nu)(f^k(z_\mu) + f^k(z_\nu))}\right\} \le - \sum_{\mu,\nu=1}^{N} \lambda_\mu \overline{\lambda_\nu} \log(1 - z_\mu \overline{z_\nu}).$$

It is analogous to the Goluzin inequality for the class S, [4], p. 128. For the special case when $N = 1$, $\lambda_1 = 1$, $z_1 = z$, we have the inequality

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{4}{n(1-|z|^2)}. \tag{22}$$

(C) Let $\lambda = 0$, $L(h) = \lambda_1 h'(z)$, λ_1 - an arbitrary complex number, $z \in U$ arbitrary but fixed. Then for every $f \in \mathcal{G}_n$ we have the inequality

$$\mathrm{Re}\left\{\lambda_1^2\{f(z), z\} + \lambda_1^2(k^2 + \frac{1}{2})\left(\frac{f'(z)}{f(z)}\right)^2\right\} \leq 6\frac{|\lambda_1|^2}{(1-|z|^2)^2}.$$

Taking in account that λ_1 is arbitrary, we have the inequality

$$\left|\{f(z), z\} + (k^2 + \frac{1}{2})\left(\frac{f'(z)}{f(z)}\right)^2\right| \leq 6\frac{1}{(1-|z|^2)^2}, \tag{23}$$

where $\{f(z), z\} = \frac{d}{dz}\left(\frac{f''(z)}{f'(z)}\right) - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$ denote a Schwarzian derivative for f at the point z.

Let $\{\lambda_m\}_{m=1}^{\infty}$ be a sequence of complex numbers such that

$$\lim_{m->\infty} \sup|\lambda_m|^{\frac{1}{m}} < 1$$

and λ arbitrary real. By Toeplitz theorem [6] p. 36, there exists a functional $L \in H'(U)$ such that $L(z^m) = \lambda_m$, $m = 1, \ldots,$ $L(1) = 0$. Let

$$\sum_{p,q=0}^{\infty} a_{pq}\, z^p\, u^q = \log\frac{f^k(z) - f^k(u)}{(z-u)(f^k(z) + f^k(u))},$$

then for every $f \in \mathcal{G}_n$ we have by (11) the inequality

(24)

$$\mathrm{Re}\left\{\lambda^2 \log\frac{n\, f'(0)}{4} + 2\lambda\sum_{p=1}^{\infty}\lambda_p a_{p0} + \sum_{p,q=1}^{\infty}\lambda_p\lambda_q a_{pq}\right\} \leq \sum_{p=1}^{\infty}\frac{|\lambda_p|^2}{p}.$$

It is analogous to the weak Grunsky inequality for the class S [2], p. 122. By the Toeplitz theorem mentioned above we observe that (24) represents a different form of the inequality (17).

By (24) for $\lambda = 0$ we can obtain like in [7], p. 119, the inequality

$$\sum_{p=1}^{N} p \left| \sum_{q=1}^{N} a_{pq}\, \lambda_q \right|^2 \le \sum_{p=1}^{N} \frac{|\lambda_p|^2}{p}, \tag{25}$$

for all $\lambda_1, \dots, \lambda_N \in \mathcal{C}$, which is analogous to the strong Grunsky inequality in S, [2], p. 123.

From (25), applying the Cauchy-Schwartz inequality, we have

$$\sum_{p=1}^{N} \sum_{q=1}^{N} a_{pq}\, \lambda_p\, \mu_q \le \sum_{p=1}^{N} \frac{|\lambda_p|^2}{p} \cdot \sum_{q=1}^{N} \frac{|\mu_q|^2}{q}, \tag{26}$$

where λ_p and μ_q are arbitrary complex numbers. (26) is analogous to the generalized weak Grunsky inequality [7], p. 124.

Remark. The inequalities (24) are not only necessary but also sufficient for $f \in H(U)$ to be in $\mathcal{G}_n$.

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METODA WARIACYJNA DLA UOGÓLNIONYCH FUNKCJI GEL'FER'A

Praca jest poświęcona klasom $\mathcal{G}_n$ funkcji f analitycznych i jednolistnych w kole jednostkowyn U, spełniających dodatkowo warunki $f(0) = 1$ i, w przypadku $n = 1$: $O \notin f(U)$, w przypadku $n \geq 2$: jeśli $w \in f(U)$, to $\varepsilon_j w \notin f(U)$, $\varepsilon_j = \exp \frac{2\pi i j}{n}$, dla $j = 1, \ldots, n-1$. Uzyskano wzory wariacyjne i zastosowano je do oszacowania pewnych funkcjonałów w rozważanych klasach.

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