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AN INTEGRAL FORMULA FOR A RIEMANNIAN MANIFOLD WITH TWO ORTHOGONAL DISTRIBUTIONS

We derive global results concerning integrals of curvatures for closed oriented Riemannian manifolds.

Let D be a distribution on a Riemannian manifold $(M, \langle\ ,\ \rangle)$. The second fundamental form B of D is defined in the following way: $B(X,Y)$ is the normal component of the field $\frac{1}{2}(\nabla_X Y + \nabla_Y X)$ where X, Y are two tangent vector fields to D (see [1]) and ∇ is the Levi-Civita connection on M. The trace H of the form B is called the mean curvature vector of D. Let D_1, D_2 be two orthogonal distributions on M such that $\dim D_1 + \dim D_2 \geq \dim M$. Let us put $D_3 = D_1 \cap D_2$. In this paper we consider the mean curvature vectors H_k of D_k, $k = 1,2,3$, and calculate the quantity $\operatorname{div} H_1 + \operatorname{div} H_2 - \operatorname{div} H_3$. Applying the Green theorem for M closed and oriented, we derive global resuls concerning integrals of curvatures.

The results obtained generalize a theorem proved in [2] for two complementary orthogonal distributions.

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Throughout the paper, manifolds, fields, metrics etc. are assumed to be C^∞-differentiable.

Let $D^\perp$ denote the orthogonal complement of a distribution D. We say that a distribution D_1 is orthogonal to a distribution D_2 if the intersection $D_1 \cap (D_1 \cap D_2)^\perp$ is orthogonal to the intersection $D_2 \cap (D_1 \cap D_2)^\perp$. Suppose that $e_1, \dots, e_m$ is a local orthonormal frame on M and assume that

(i) e_i is tangent to $D_2^\perp$ for $i = 1, 2, \dots, \dim D_2^\perp$,

(ii) e_α is tangent to $D_1^\perp$ for $\alpha = \dim D_2^\perp + 1, \dots, \dim D_1^\perp + \dim D_2^\perp$,

(iii) e_j is tangent to $D_1 \cap D_2$ for $j = \dim(D_1 \cap D_2)^\perp + 1, \dots, \dim M$.

If v is a vector tangent to M, we write

$$v = v^{\top 1} + v^{\top 2} + v^{\top 3}$$

where $v^{\top 1}$ is tangent to $D_2^\perp$, $v^{\top 2}$ is tangent to $D_1^\perp$ and $v^{\top 3}$ is tangent to $D_1 \cap D_2$.

By $v^{\perp k}$ and $v^{\top D_k}$, $k = 1, 2, 3$, we denote, respectively, the components of v orthogonal and tangent to D_k. The integrability tensors T_k of D_k are defined by the formula

$$T_k(X_k, Y_k) = \frac{1}{2}[X_k, Y_k]^{\perp k}$$

for vector fields X_k, Y_k tangent to D_k.

Let B_k be the second fundamental form of D_k. The mean curvature vectors H_k of D_k are given by

$$\begin{aligned} H_1 &= \sum_i B_1(e_i, e_i) + \sum_j B_1(e_j, e_j) \\ &= \sum_i (\nabla_{e_i} e_i)^{\perp 1} + \sum_j (\nabla_{e_j} e_j)^{\perp 1}, \end{aligned}$$

$$\begin{aligned} H_2 &= \sum_\alpha B_2(e_\alpha, e_\alpha) + \sum_j B_2(e_j, e_j) \\ &= \sum_\alpha (\nabla_{e_\alpha} e_\alpha)^{\perp 1} + \sum_j (\nabla_{e_j} e_j)^{\perp 2}, \end{aligned}$$

$$H_3 = \sum_j B_3(e_j, e_j) = \sum_j (\nabla_{e_j} e_j)^{\perp 3}.$$

Therefore

$$\begin{aligned} \operatorname{div} H_1 &= -|H_1|^2 + \sum_\alpha \langle \nabla_{e_\alpha} H_1, e_\alpha \rangle \\ &= -|H_1|^2 + \sum_{\alpha,i} \langle \nabla_{e_\alpha} (\nabla_{e_i} e_i)^{\top 2}, e_\alpha \rangle \\ &\quad + \sum_{\alpha,j} \langle \nabla_{e_\alpha} (\nabla_{e_j} e_j)^{\top 2}, e_\alpha \rangle, \end{aligned}$$

$$\begin{aligned} \operatorname{div} H_2 &= -|H_2|^2 + \sum_i \langle \nabla_{e_i} H_2, e_i \rangle \\ (1) \qquad &= -|H_2|^2 + \sum_{\alpha,i} \langle \nabla_{e_i} (\nabla_{e_\alpha} e_i)^{\top 1}, e_i \rangle \\ &\quad + \sum_{i,j} \langle \nabla_{e_i} (\nabla_{e_j} e_j)^{\top 1}, e_i \rangle, \end{aligned}$$

$$\begin{aligned} \operatorname{div} H_3 &= -|H_3|^2 + \sum_i \langle \nabla_{e_i} H_3, e_i \rangle + \sum_\alpha \langle \nabla_{e_\alpha} H_3, e_\alpha \rangle \\ &= -|H_3|^2 + \sum_{i,\alpha,j} \big(-\langle (\nabla_{e_j} e_j)^{\top 2}, \nabla_{e_i} e_i \rangle \\ &\quad - \langle (\nabla_{e_j} e_j)^{\top 1}, \nabla_{e_\alpha} e_\alpha \rangle \\ &\quad + \langle \nabla_{e_i} (\nabla_{e_j} e_j)^{\top 1}, e_i \rangle + \langle \nabla_{e_\alpha} (\nabla_{e_j} e_j)^{\top 2}, e_\alpha \rangle \big). \end{aligned}$$

It follows from (1) that

$$\begin{aligned} \operatorname{div} H_1 + \operatorname{div} H_2 &= -|H_1|^2 - |H_2|^2 + \sum_{i,\alpha,j} \big(\langle \nabla_{e_\alpha} (\nabla_{e_i} e_i)^{\top 2}, e_\alpha \rangle \\ &\quad + \langle \nabla_{e_\alpha} (\nabla_{e_j} e_j)^{\top 2}, e_\alpha \rangle + \langle \nabla_{e_i} (\nabla_{e_\alpha} e_\alpha)^{\top 1}, e_i \rangle \\ &\quad + \langle \nabla_{e_i} (\nabla_{e_j} e_j)^{\top 1}, e_i \rangle \big) = \operatorname{div} H_3 - |H_1|^2 \end{aligned}$$

$$-|H_2|^2+|H_3|^2+\sum_{i,\alpha,j}\big(\langle(\nabla_{e_j}e_j)^{\mathsf{T}2},\nabla_{e_i}e_i\rangle$$
$$+\langle(\nabla_{e_j}e_j)^{\mathsf{T}1},\nabla_{e_\alpha}e_\alpha\rangle+\langle\nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\mathsf{T}2},e_\alpha\rangle$$
$$+\langle\nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}1},e_i\rangle\big).$$

Now, we calculate the last component of the above sum by applying the definition of the curvature tensor R:

(2)
$$\sum_{i,\alpha}\big(\langle\nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\mathsf{T}2},e_\alpha\rangle+\rangle\nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}1},e_i\rangle\big)$$
$$=\sum_{i,\alpha}\big(2\langle R(e_i,e_\alpha)e_\alpha,e_i\rangle+\langle\nabla_{e_\alpha}\nabla_{e_i}e_\alpha,e_i\rangle+\langle\nabla_{e_i}\nabla_{e_\alpha}e_i,e_\alpha\rangle$$
$$+2\langle\nabla_{[e_\alpha,e_i]}e_i,e_\alpha\rangle-\langle\nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\mathsf{T}1},e_\alpha\rangle-\langle\nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\mathsf{T}3},e_\alpha\rangle$$
$$-\langle\nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}2},e_i\rangle-\langle\nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}3},e_i\rangle\big).$$

Then we observe that

(3)
$$\sum_{i,\alpha}\langle\nabla_{e_i}\nabla_{e_\alpha}e_i,e_\alpha\rangle=\sum_{i,\alpha}\big(-e_i\langle e_i,\nabla_{e_\alpha}e_\alpha\rangle-\langle\nabla_{e_\alpha}e_i,\nabla_{e_i}e_\alpha\rangle\big),$$
$$\sum_{i,\alpha}\langle\nabla_{e_\alpha}\nabla_{e_i}e_\alpha,e_i\rangle=\sum_{i,\alpha}\big(-e_\alpha\langle e_\alpha,\nabla_{e_i}e_i\rangle-\langle\nabla_{e_i}e_\alpha,\nabla_{e_\alpha}e_i\rangle\big)$$

and

(3')
$$e_\alpha\langle e_\alpha,\nabla_{e_i}e_i\rangle=e_\alpha\langle e_\alpha,(\nabla_{e_i}e_i)^{\mathsf{T}2}\rangle=\langle\nabla_{e_\alpha}e_\alpha,(\nabla_{e_i}e_i)^{\mathsf{T}2}\rangle+\langle e_\alpha,\nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\mathsf{T}2}\rangle,$$
$$e_i\langle e_i,\nabla_{e_\alpha}e_\alpha\rangle=e_i\langle e_i,(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}1}\rangle=\langle\nabla_{e_i}e_i,(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}1}\rangle+\langle e_i,\nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\mathsf{T}1}\rangle.$$

Let us put

$$K(D_1, D_2) = \sum_{i,\alpha} \langle R(e_i, e_\alpha)e_\alpha, e_i\rangle,$$

$$H_{11} = \sum_i \big((\nabla_{e_i} e_i)^{\top 2} + (\nabla_{e_i} e_i)^{\top 3}\big),$$

$$H_{22} = \sum_\alpha \big((\nabla_{e_\alpha} e_\alpha)^{\top 1} + (\nabla_{e_\alpha} e_i)^{\top 3}\big).$$

Comparing equalities (1), (2) and (3), (3'), we have

$$\begin{aligned} &\operatorname{div} H_1 + \operatorname{div} H_2 - \operatorname{div} H_3 = -|H_1|^2 - |H_2|^2 + |H_3|^2 \\ &+K(D_1, D_2) + \sum_{i,\alpha,j} \big(\langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha\rangle + \langle (\nabla_{e_i} e_i)^{\top 3}, \nabla_{e_\alpha} e_\alpha\rangle \\ (4) \qquad &-\langle \nabla_{e_i} e_\alpha, \nabla_{e_\alpha} e_i\rangle + \langle (\nabla_{e_\alpha} e_\alpha)^{\top 1}, \nabla_{e_j} e_j\rangle \\ &+\langle (\nabla_{e_i} e_i)^{\top 2}, \nabla_{e_j} e_j\rangle\big) + \langle H_{11}, H_{22}\rangle + \langle H_{11}, H_3\rangle + \langle H_{22}, H_3\rangle. \end{aligned}$$

Since $[X, Y] = \nabla_X Y - \nabla_Y X$ for arbitrary vector fields on M, it follows that

$$\begin{aligned} &\langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha\rangle = \sum_{p,\beta,j} \big(\langle \nabla_{e_\alpha} e_i, e_p\rangle\langle \nabla_{e_p} e_i, e_\alpha\rangle \\ &+\langle \nabla_{e_\alpha} e_i, e_\beta\rangle\langle \nabla_{e_\beta} e_i, e_\alpha\rangle + \langle \nabla_{e_\alpha} e_i, e_j\rangle\langle \nabla_{e_j} e_i, e_\alpha\rangle \\ (5) \qquad &-\langle \nabla_{e_i} e_\alpha, e_p\rangle\langle \nabla_{e_p} e_i, e_\alpha\rangle - \langle \nabla_{e_i} e_\alpha, e_\beta\rangle\langle \nabla_{e_\beta} e_i, e_\alpha\rangle \\ &-\langle \nabla_{e_i} e_\alpha, e_j\rangle\langle \nabla_{e_j} e_i, e_\alpha\rangle\big) = \sum_{p,\beta,j} \big(\langle \nabla_{e_\alpha} e_i, (\nabla_{e_i} e_\alpha)^{\top 1}\rangle \\ &+\langle \nabla_{e_\alpha} e_\beta, (\nabla_{e_\beta} e_\alpha)^{\top 1}\rangle + \langle \nabla_{e_\alpha} e_j, (\nabla_{e_j} e_\alpha)^{\top 1}\rangle \\ &+\langle \nabla_{e_i} e_p, (\nabla_{e_p} e_i)^{\top 2}\rangle + \langle \nabla_{e_i} e_\alpha, (\nabla_{e_\alpha} e_i)^{\top 2}\rangle \end{aligned}$$

$$+\langle \nabla_{e_i} e_j, (\nabla_{e_j} e_i)^{\top 2} \rangle).$$

From (5) we have

$$\begin{aligned}\langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle - \langle \nabla_{e_i} e_\alpha, \nabla_{e_\alpha} e_i \rangle = -\langle \nabla_{e_\alpha} e_i, (\nabla_{e_i} e_\alpha)^{\top 3} \rangle \\ + \langle \nabla_{e_\alpha} e_\beta, (\nabla_{e_\beta} e_\alpha)^{\top 1} \rangle + \langle \nabla_{e_\alpha} e_j, (\nabla_{e_j} e_\alpha)^{\top 1} \rangle \\ + \langle \nabla_{e_i} e_p, (\nabla_{e_p} e_i)^{\top 2} \rangle + \langle \nabla_{e_i} e_j, (\nabla_{e_j} e_i)^{\top 2} \rangle.\end{aligned}$$

Now, we shall introduce some notation:

$$2B_1(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 2} + (\nabla_{e_\delta} e_\gamma)^{\top 2},$$
$$2T_1(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 2} - (\nabla_{e_\delta} e_\gamma)^{\top 2}$$

for $\gamma, \delta \in \{1, \ldots, \dim D_2^\perp\} \cup \{\dim D_1^\perp + \dim D_2^\perp + 1, \ldots, m\} = A_1$,

$$2B_2(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 1} + (\nabla_{e_\delta} e_\gamma)^{\top 1},$$
$$2T_2(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 1} - (\nabla_{e_\delta} e_\gamma)^{\top 1}$$

for $\gamma, \delta \in \{\dim D_1^\perp + 1, \ldots, m\} = A_2$,

$$2B_{12}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 3} + (\nabla_{e_\delta} e_\gamma)^{\top 3},$$
$$2T_{12}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 3} - (\nabla_{e_\delta} e_\gamma)^{\top 3}$$

for $\gamma, \delta \in \{1, \ldots, \dim D_1^\perp + \dim D_2^\perp\} = A_3$,

$$\begin{aligned}2B_3(e_j, e_q) &= (\nabla_{e_j} e_q)^{\perp 3} + (\nabla_{e_q} e_j)^{\perp 3}, \\ 2T_3(e_j, e_q) &= (\nabla_{e_j} e_q)^{\perp 3} - (\nabla_{e_q} e_j)^{\perp 3}, \\ 2B_{11}(e_i, e_p) &= (\nabla_{e_i} e_p)^{\top D_2} + (\nabla_{e_p} e_i)^{\top D_2}, \\ 2T_{11}(e_i, e_p) &= (\nabla_{e_i} e_p)^{\top D_2} - (\nabla_{e_p} e_i)^{\top D_2}, \\ 2B_{22}(e_\alpha, e_\beta) &= (\nabla_{e_\alpha} e_\beta)^{\top D_1} + (\nabla_{e_\beta} e_\alpha)^{\top D_1}, \\ 2T_{22}(e_\alpha, e_\beta) &= (\nabla_{e_\alpha} e_\beta)^{\top D_1} - (\nabla_{e_\beta} e_\alpha)^{\top D_1}.\end{aligned}$$

Let us notice that

$$\begin{aligned}\sum_{\gamma,\delta\in A_1} &\langle (\nabla_{e_\gamma} e_\delta)^{\top 2}, (\nabla_{e_\delta} e_\gamma)^{\top 2}\rangle \\ &= \sum_{\gamma,\delta\in A_1} (|B_1(e_\gamma, e_\delta)|^2 - |T_1(e_\gamma, e_\delta)|^2) \\ &= |B_1|^2 - |T_1|^2.\end{aligned}$$

For the remaining forms, analogous equalities are true. Hence

$$\begin{aligned}(6)\quad \sum_{i,\alpha} &(\langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha\rangle - \langle \nabla_{e_i} e_\alpha, \nabla_{e_\alpha} e_i\rangle) \\ &= \frac{1}{2}(|B_1|^2 + |B_2|^2 + |B_{11}|^2 + |B_{22}|^2 - |B_3|^2 - |B_{12}|^2 \\ &\quad - |T_1|^2 - |T_2|^2 - |T_{11}|^2 - |T_{22}|^2 + |T_3|^2 + |T_{12}|^2).\end{aligned}$$

Equalities (4) and (6) lead us to the formula

$$\begin{aligned}(7)\quad \operatorname{div} H_1 + \operatorname{div} H_2 - \operatorname{div} H_3 &= K(D_1, D_2) \\ &+ \sum_{i=1}^{2} (-|H_i|^2 + \frac{1}{2}(|B_i|^2 + |B_{ii}|^2 - |T_i|^2 - |T_{ii}|^2)) \\ &+ |H_3|^2 + \frac{1}{2}(-|B_3|^2 - |B_{12}|^2 + |T_3|^2 + |T_{12}|^2).\end{aligned}$$

Proposition. *If D_1, D_2 are two orthogonal distributions on a Riemannian manifold M, such that $\dim D_1 + \dim D_2 \geq \dim M$, then*

$$\begin{aligned}\operatorname{div} H_1 + \operatorname{div} H_2 - \operatorname{div} H_3 &= K(D_1, D_2) \\ &+ \sum_{i=1}^{2} (-|H_i|^2 + \frac{1}{2}(|B_i|^2 + |B_{ii}|^2 - |T_i|^2 - |T_{ii}|^2)) \\ &+ |H_3|^2 + \frac{1}{2}(-|B_3|^2 - |B_{12}|^2 + |T_3|^2 + |T_{12}|^2).\end{aligned}$$

where B_n, H_n, T_n, $n = 1, 2, 3$, denote, respectively, the second fundamental forms, mean curvature vectors and integrability tensors of

$D_1, D_2, D_3 = D_1 \cap D_2$; B_{ii}, T_{ii}, $i = 1, 2$, *are, respectively, the second fundamental forms and integrability tensors of* $(D_1 \cap D_2)^\perp \cap D_i$; B_{12}, T_{12} *are the second fundamental forms and integrability tensors of* $(D_1 \cap D_2)^\perp$.

This is a direct consequence of (7). Now, applying the Green theorem, we obtain the following result:

Theorem. *If* D_1, D_2 *are two orthogonal distributions on a closed oriented Riemannian manifold* M, *such that* $\dim D_1 + \dim D_2 \geq \dim M$, *then*

$$\int_M \Big(K(D_1, D_2) + \sum_{i=1}^{2} (-|H_i|^2 + \frac{1}{2}(|B_i|^2 + |B_{ii}|^2 - |T_i|^2 - |T_{ii}|^2))$$

$$+|H_3|^2 + \frac{1}{2}(-|B_3|^2 - |B_{12}|^2 + |T_3|^2 + |T_{12}|^2) \Big) \Omega = 0$$

where Ω *is the volume element on* M.

Corollary. *If* $\mathcal{F}_1, \mathcal{F}_2$ *are two orthogonal foliations on a closed oriented Riemannian manifold* M, *such that* $\dim F_1 + \dim F_2 \geq \dim M$, *then*

$$\int_M \Big(K(\mathcal{F}_1, \mathcal{F}_2) + \sum_{i=1}^{2} (-|H_i|^2 + \frac{1}{2}(|B_i|^2 + |B_{ii}|^2 - |T_{ii}|^2))$$

$$+ |H_3|^2 + \frac{1}{2}(-|B_3|^2 - |B_{12}|^2 + |T_{12}|^2) \Big) \Omega = 0$$

where Ω *is the volume element on* M.

This follows immediately from our theorem.

References

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WZÓR CAŁKOWY DLA ROZMAITOŚCI RIEMANNOWSKIEJ Z DWIEMA DYSTRYBUCJAMI ORTOGONALNYMI

Praca zawiera globalne wyniki dotyczące całek z krzywizn na zorientowanej zwartej rozmaitości riemannowskiej bez brzegu.

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