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ESTIMATION OF THE FUNCTIONAL $A_2 \cdot A_3$ IN THE CLASS OF BOUNDED SYMMETRIC UNIVALENT FUNCTIONS

Denote by $S_R(M)$, $M > 1$, the family of functions $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ holomorphic and univalent in the disc $E = \{z : |z| < 1\}$, with real coefficients and satisfying the condition $|F(z)| \leq M$, $z \in E$.
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In the paper there has been obtained a sharp estimation of the functional $H(F) = A_2 \cdot A_3$ in the classes $S_R(M)$, $M > 1$.

Introduction

Let S_R stand for the family of functions

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n \tag{0.1}$$

holomorphic and univalent in the disc $E = \{z : |z| < 1\}$, with real coefficients.

Denote by $S_R(M)$, $M > 1$, the subfamily of the former, consisting of bounded functions, i.e. of those satisfying the condition

$$|F(z)| \leq M \quad \text{for} \quad z \in E$$

The problems connected with the estimation of coefficients in the classes defined above were dealt with by many mathematicians.

In the family $S_R(M)$ well known are, among others, the following sharp estimations:

1) for each function $F \in S_R(M)$ [5],

$$|A_2| \le 2(1 - \frac{1}{M}) \quad \text{for} \quad M > 1; \tag{0.2}$$

2) for each function $F \in S_R(M)$ ([7], [9], [3]),

$$|A_3| \begin{cases} 1 - M^{-2} & \text{when} \quad 1 < M \le e \\ 1 + 2\lambda^2 - 4\lambda M^{-1} + M^{-2} & \text{when} \quad e \le M < +\infty \end{cases} \tag{0.3}$$

where the λ occurring in (0.3) in the greater root of the equation

$$\lambda \lg \lambda = -M^{-1}$$

3) for each function $F \in S_R(M)$, n even and M sufficiently large [8], [2], [10], [11]),

$$A_n \le P_n(M) \tag{0.4}$$

where $P_n(M)$ is the n-th coefficient in the Maclaurin expansion of a Pick function $w = \mathcal{P}(z, M)$ defined by the equation

$$\frac{w}{(1 - \frac{w}{M})^2} = \frac{z}{(1 - z)^2}, \quad z \in E, \quad \mathcal{P}(0, M) = 0 \tag{0.5}$$

Moreover, this function realizes the equality in estimation (0.2).

From estimation (0.3) it follows that the Pick function does not realize the maximum of the functional $H(F) = A_3$, $F \in S_R(M)$, $M > 1$.

The above-mentioned results justify the purposefulness of the investigation of the functional of the form

$$H(F) = A_2 \cdot A_3 \tag{0.6}$$

defined in the family $S_R(M)$. It is worth noting that (0.6) is not a linear functional.

To estimate the functional $H(F) = A_2 \cdot A_3$, the variational method is made use of in the paper. In particular, we use the general equation for extremal functions, obtained by I. Dziubiński [1].

1. The equation for functions extremal with respect to the functional $A_2 \cdot A_3$ defined in the class $S_R(M)$

Consider the functional

$$H(F) = A_2 \cdot A_3 \tag{1.1}$$

defined on the family $S_R(M)$, $M > 1$. Since functional (1.1) is continuous, and the family $S_R(M)$ compact, there exists a function $F^* \in S_R(M)$ of the form

$$w = F^*(z) = z + \sum_{n=2}^{\infty} A_n^* z^n \tag{1.2}$$

for which the functional attains its maximum. It is easy to notice that $H(F^*) > 0$ and, thereby, $A_2^* \neq 0$ and $A_3^* \neq 0$. Moreover, in virtue of [1], each of the extremal functions satisfies the following differential-functional equation:

$$\left[\frac{F^{*\prime}(z)}{F^*(z)}\right]^2 \mathfrak{m}\,(F^*(z)) = \frac{1}{z^2}\mathfrak{n}(z), \quad 0 < |z| < 1 \tag{1.3}$$

where

$$\mathfrak{m}\,(w) = \sum_{p=2}^{3} D_{p-1}^*\left[\left(\frac{w}{M}\right)^{p-1} + \left(\frac{w}{M}\right)^{1-p}\right] - 2\mathfrak{P}^*$$

$$\mathfrak{n}(z) = \sum_{p=1}^{3} E_{p-1}^*\,(z^{p-1} + z^{1-p}) - 2\mathfrak{P}^*$$

$$D_1^* = 2M^{-2}(A_3^* + 2A_2^*), \quad D^*_2 = 2M^{-3}A_2^*$$

$$E_0^* = 3M^{-1}A_2^*A_3^*, \quad E_1^* = 2M^{-1}\left[A_3^* + 2A_2^{*2}\right] \tag{1.4}$$

$$E_2 = 2M^{-1}A_2^*, \quad \mathfrak{P}^* = \min_{0\le x\le 2\pi}\left[D_1^* \cos x + D_2^* \cos 2x\right]$$

The functions $\mathfrak{m}(w)$ and $\mathfrak{n}(z)$ take real-negative values on the circles $|w| = M$ and $|z| = 1$, respectively, and either of them has at least one double zero on the respective circle just mentioned. It can easily be observed that if w_0 is a zero of the function $\mathfrak{m}(w)$, so are the numbers w_0, $\frac{M^2}{w_0}$, $\frac{M^2}{\overline{w}_0}$; analogously if z_0 is a zero of the function $\mathfrak{n}(z)$, so are the numbers $\overline{z}_0$, $\frac{1}{z_0}$, $\frac{1}{\overline{z}_0}$.

Let us denote

$$\beta = \frac{M(A_3^* + 2A_2^*)}{A_2^*} \tag{1.5}$$

Then formulae (1.4) will take the following form:

$$\mathfrak{m}(w) = \frac{2A_2^*\left[w^4 + \beta M w^3 - \frac{\mathfrak{p}^* M^5}{A_2^*} w^2 + \beta M^3 w + M^4\right]}{M^5 w^2} \tag{1.6}$$

$$\mathfrak{n}(z) = \frac{2A_2^*\left[z^4 + \frac{\beta}{M} z^3 + \frac{3H(F^*) - \mathfrak{p}^* M}{A_2^*} z^2 + \frac{\beta}{M} z + 1\right]}{M z^2} \tag{1.7}$$

$$\mathfrak{p}^* = \frac{2}{M^3} \min_{0 \le x < 2\pi} A_2^*\left[\cos 2x + \beta \cos x\right] . \tag{1.8}$$

Lemma 1. Each function (1.2) extremal with respect to functional (1.1) satisfies the equation:

I. The case $A_2^* > 0$.

A.

$$\left(\frac{zw'}{w}\right)^2 \frac{A_2^* (w^2 + \frac{1}{2}\beta M w + M^2)^2}{M^4 w^2} = \frac{A_2^*(z^2 + \frac{\beta}{2M} z + 1)^2}{z^2}$$

if $0 < \beta \le 4$;

B.

$$\left(\frac{zw'}{w}\right)^2 \frac{A_2^*(w + M)^2\left[w^2 + (\beta - 2) M w + M^2\right]}{M^4 w^2} =$$

$$= \frac{A_2^*(z^2 + \frac{\beta}{2M} z + 1)^2}{z^2}$$

if $4 < \beta < 4M$;

C.

$$\left(\frac{zw}{w}\right)^2 \frac{A_2^*(w + M)^2\left[w^2 + (\beta - 2)Mw + M^2\right]}{M^4 w^2} =$$

$$= \frac{A_2^*(z + 1)^2\left[z^2 + (\frac{\beta}{M} - 2)z + 1\right]}{z^2}$$

if $\beta \geq 4M$.

II. The case $A_2^* < 0$.

D.

$$\left(\frac{zw}{w}\right)^2 \frac{A_2^*(w - M)^2\left[w^2 + M(\beta - 2)w + M^2\right]}{M^4 w^2} =$$

$$= \frac{A_2^*(z + 1)^2\left[z^2 + (\frac{\beta}{M} - 2)z + 1\right]}{z^2}$$

if $\beta < 0$;

E.

$$\cdot\left(\frac{zw}{w}\right)^2 \frac{A_2^*(w - M)^2 (w + M)^2}{M^4 w^2} =$$

$$= \frac{A_2^*(z + 1)^2 (z - 1)^2}{z^2}$$

if $\beta = 0$,

F.

$$\left(\frac{zw}{w}\right)^2 \frac{A_2^*(w - M)^2\left[w^2 + M(\beta + 2)w + M\right]^2}{M^4 w^2} =$$

$$= \frac{A_2^*(z - 1)^2\left[z^2 + (\frac{\beta}{M} + 2)z + 1\right]}{z^2}$$

if $\beta > 0$.

Besides,

$$
(1.9) \qquad H(F^*) = \begin{cases} \frac{2}{3} A_2^* \left(1 - \frac{1}{M^2}\right) & \text{in case A} \\ \frac{2}{3} A_2^* \left[1 + \frac{1}{M^2} \left(\frac{\beta^2}{8} - \beta + 1\right)\right] & \text{in case B} \\ \frac{2}{3} A_2^* \left[\frac{1-\beta}{M^2} + \frac{\beta}{M} - 1\right] & \text{in case C} \\ -\frac{2}{3} A_2^* \left(1 - \frac{1}{M}\right)\left(1 + \frac{1}{M} - \frac{\beta}{M}\right) & \text{in case D} \\ -\frac{2}{3} A_2^* \left(1 - \frac{1}{M^2}\right) & \text{in case E} \\ -\frac{2}{3} A_2^* \left(1 - \frac{1}{M}\right)\left(1 + \frac{1}{M} + \frac{\beta}{M}\right) & \text{in case F} \end{cases}
$$

P r o o f. Consider case I. Since $A_2^* > 0$ and $A_2^* \cdot A_3^* > 0$, therefore from (1.5) it follows that, in this case, $\beta > 0$.

Let us calculate φ^*. For the purpose, put

$$u(x) = \beta \cos x + \cos 2x$$

We then have $u'(x) = -\beta \sin x - 2 \sin 2x$, whence $u'(x_i) = 0$ if $x_1 = 0$, $x_2 = \pi$ or $\cos x_3 = -\frac{\beta}{4}$. Since $u''(x) = -\beta \cos x - 4 \cos 2x$, therefore:

a) if $0 < \beta < 4$, then $u_{min} = u(x_3)$,

b) if $\beta > 4$, then $u_{min} = u(\pi)$.

Finally, in virtue of the above and (1.8), we obtain

$$
(1.10) \qquad \varphi^* = \begin{cases} \frac{-2A_2^*}{M^3}\left(1 + \frac{\beta^2}{8}\right) & \text{if } 0 < \beta \le 4, \ A_2^* > 0 \\ \frac{2A_2^*}{M^3}(1 - \beta) & \text{if } \beta > 4, \ A_2^* > 0 \end{cases}
$$

Let us substitute the values determined in (1.10) into (1.6). After some simple transformations we shall get

$$(1.11)\qquad \mathfrak{m}(w)=\begin{cases} \dfrac{2A_2^* \left(w^2+\frac{1}{2}\beta Mw+M^2\right)^2}{M^5w^2} & \text{if } 0<\beta\le 4 \\[2ex] \dfrac{2A_2^* (w+M)^2\left[w^2+(\beta-2)Mw+M^2\right]}{M^5w^2} & \text{if } \beta>4 \end{cases}$$

We shall next examine the function $\mathfrak{n}(z)$ in the case under consideration. From the properties of this function - mentioned at the beginning of the section - and from formula (1.7) it follows that it can only be the form

$$(1.12)\qquad \mathfrak{n}(z)=\frac{2A_2^*(z-\varepsilon)^2\left[z^2+\delta\left(r+\frac{1}{r}\right)z+1\right]}{Mz^2},$$

$$\varepsilon,\delta=\pm 1,\quad 0<r\le 1$$

or

$$(1.13)\qquad \mathfrak{n}(z)=\frac{2A_2^*(z-\sigma)^2\,(z-\bar{\sigma})^2}{Mz^2},$$

$$|\sigma|=1,\quad \sigma\ne\pm 1$$

Since the function $\mathfrak{n}(z)$ is non-negative on the circle $|z|=1$, therefore, in the case of equation (1.12), the inequality

$$\mathfrak{n}(e^{iy})=\frac{4A_2^*}{M}(\cos y-\varepsilon)\left[2\cos y+\delta\left(r+\frac{1}{r}\right)\right]\ge 0,$$

$$0\le y<2\pi$$

should hold, whence

$$(1.14)\qquad \varepsilon=1 \text{ and } \delta=-1$$

or

$$(1.14')\qquad \varepsilon=-1,\quad \delta=1$$

By comparing the coefficients at the same powers z^3 and z^2 in the numerators of formulae (1.7) and (1.12), we obtain

(1.15) $$\frac{\beta}{M} = -2\varepsilon + \delta(r + \frac{1}{r})$$

(1.16) $$\frac{3H(F^*) - \wp^* M}{A_2^*} = 2 - 2\varepsilon\delta\,(r + \frac{1}{r})$$

Case (1.14) is not possible because $\beta > 0$ and the other, i.e. (1.14′), remains valid. Then, by (1.15),

(1.17) $$r + r^{-1} = \frac{\beta}{M} - 2$$

Equality (1.17) is possible only for $\beta \geq 4M$. Hence, in view of (1.12), we get the right-hand side of equation C, in juxtaposition with (1.11), gives us equation C.

A suitable formula for $H(F^*)$ follows then from (1.14′) (1.16), (1.17), and (1.10).

Proceeding analogously when the function $\pi(z)$ is defined by formula (1.13), we have

(1.18) $$\frac{\beta}{M} = -4\cos\varkappa \quad \text{where} \quad \varkappa = \operatorname{Arg}\sigma$$

(1.19) $$\frac{3H(F^*) - \wp^* M}{A_2^*} = 2 + 2\cos\varkappa$$

whence

$$\pi(z) = \frac{2A_2^*(z^2 + \frac{\beta}{2M}z + 1)^2}{Mz^2}$$

From (1.11) it follows that equation (1.3) is of form A when $0 < \beta \leq 4$, and of form B if $4 < \beta < 4M$. Whereas from (1.19), (1.18) and (1.10) we obtain suitable formulae (1.9).

Proceeding analogously in case II, i.e. when $A_2^* < 0$, we get forms D, E and F of equation (1.3) as well as suitable formulae for $H(F^*)$ defined in (1.9).

So, we have shown that equation (1.3) - with appropriateness to the values of A_2^* and β - takes one of the six forms. We have also expressed the upper bound of the functional $H(F)$ by means of β, A_2^* and M. Consequently, it is necessary to determine the unknown quantities A_2^*, β by M and to find the intervals of variability of M. For the purpose, we shall next integrate each particular differential-functional equation A - F in order to obtain equations for the unknown quantities and other auxiliary parameters, and carry out an appropriate discussion.

2. Integration of equations for extremal functions

Let us successively consider each particular equation:

1. Equation A. In this case - by lemma 1 - we have

$$\frac{zw'}{w}\,\frac{w^2+\frac{1}{2}\beta Mw+M^2}{M^2w}=\varepsilon\,\frac{z^2+\frac{\beta}{2M}z+1}{z}, \tag{2.1}$$

$$\varepsilon=\pm 1$$

Since $F^*(o)=0$, $F^{*\prime}(o)=1$, therefore $\varepsilon=1$.

Integrating both sides of equation (2.1) in any simply connected sets which do not contain zero and are contained in the discs $|w|<M$ and $|z|<1$, respectively, we have

$$\frac{w}{M^2}+\frac{\beta}{2M}\log\frac{w}{z}-\frac{1}{w}=z-\frac{1}{z}+C \tag{2.2}$$

where the branch of the logarithm is so chosen that, for $z=0$, it takes the value 0, and C is a constant.

Since on the circle $|z|=1$ there is an arc γ which is transformed by the function $w=F^*(z)$ onto an arc γ' of the circle $|w|=M$ (cf. [1]), therefore, after substituting $z=e^{ix}\varepsilon\gamma$ and, respectively, $w=Me^{iy}\varepsilon\gamma'$ in equation (2.2), we get

$$\operatorname{re} C=\frac{\beta}{2M}\log M \tag{2.3}$$

Expanding the left-hand side of (2.2) in a power series in a neighbourhood of the point $z=0$ and comparing the absolute terms and the coefficients at z, we have

$$A_2^*=C \tag{2.4}$$

(2.5) $$\frac{1}{M^2} + \frac{\beta}{2M} A_2^* + A_3^* - A_2^{*2} = 1$$

Making use of the fact that A_2^* is real and of (2.3), (2.4), we obtain

(2.6) $$A_2^* = \frac{\beta}{2M} \log M$$

After determining the A_3^* from (1.9, A) and after substituting it, together with (2.6), into (2.5), we get

(2.7) $$\beta^2 \log M (1 - \log M) = \frac{4}{3}(M^2 - 1)$$

This equality makes sense only for $M \in (1, e)$. After taking account of the fact that $\beta > 0$, from (2.7) we have

(2.7′) $$\beta = \frac{2}{3\sqrt{3}} \sqrt{\frac{M^2 - 1}{\log M(1 - \log M)}}$$

Let us substitute the above relation into (2.6) and, next, (2.6) into (1.9, A). We shall then obtain the formula for $H(F^*)$ in case A.

From the condition $\beta \le 4$ we infer that (2.7′) can hold only for $M \in (1, M_0>$ where M_0 is the only root of the equation

(2.8) $$12 \log M(1 - \log M) + 1 - M^2 = 0$$

We have thus proved

Lemma 2. If in the family $S_R(M)$, $M \in (1, M_0>$, there is an extremal function $w = F^*(z)$ satisfying equation A, then it fulfils the equation

(2.9) $$\frac{w}{M^2} + \frac{\beta}{2M} \log \frac{w}{z} - \frac{1}{w} = z - \frac{1}{z} + \frac{\beta}{2M} \log M$$

and the equality

(2.10) $$H(F^*) = \frac{2}{3\sqrt{3}} \sqrt{\left(1 - \frac{1}{M^2}\right)^3 \frac{\log M}{1 - \log M}}$$

takes place, where M_0 is the only root of equation (2.8). For $M > M_0$, there is no extremal function satisfying equation A.

2. Equation B. It can be represented in the form

(2.11) $$\left(\frac{zw'}{w}\right)^2 \frac{(w+M)^2 (w+\tau)\left(w+\frac{M^2}{\tau}\right)}{M^4 w^2} = \frac{\left(z^2+\frac{\beta}{2M}z+1\right)^2}{z^2}$$

where $-\tau$ $(0<\tau<M)$ is one of the roots of the equation $w^2 + (\beta-2)Mw + M^2 = 0$.

From [6] (p. 660) and from (2.11) it follows that the point $w = -\tau$ must be a boundary point of the domain $F^*(E)$, or else, the right-hand side of (2.11) would have a root at some interior point of the disc E. Since $F^*(o) = 0$, $F^{*'}(o) = 1$, therefore, by (2.11), we have

(2.12) $$\frac{zw'}{w} \frac{(w+M)\left(w+\frac{M^2}{\tau}\right)\sqrt{\dfrac{w+\tau}{w+\frac{M^2}{\tau}}}}{M^2 w} = \frac{z^2+\frac{\beta}{2M}z+1}{z^2}$$

the branch of the root

(2.13) $$P(w) = \sqrt{\frac{w+\tau}{w+\frac{M^2}{\tau}}}$$

being so chosen that, for $w = 0$, it takes the value $\frac{\tau}{M}$.

Integrating both sides of equation (2.12), after using (2.13) and after some simple transformations, we have

(2.14) $$\frac{(M+\tau)^2}{2M^2\tau}\log\frac{\tau-MP(w)}{\tau+MP(w)} - \frac{(M+\tau)^2}{2M^2\tau}\log\frac{1-P(w)}{1+P(w)} -$$

$$-\frac{M^2-\tau^2}{M^2}\cdot\frac{P(w)}{M^2P(w)-1} - \frac{M^2-\tau^2}{\tau M^2}\,\frac{p(w)}{P^2(w-1)} =$$

$$= z + \frac{\beta}{2M} \log z - \frac{1}{z} + c$$

where the branch of the root is chosen as above, the branches of the logarithms - so that log 1 = 0.

Since there exists a point $z_0 = e^{i\theta}$, $\theta \in \langle 0, 2\pi)$ such that $\lim_{\substack{z \to z_0 \\ z \in E^0}} F^*(z) = \tau$ [6], from equation (2.14) we have

$$\text{(2.15)} \qquad \text{re}\{C\} = 0$$

Let us expand the left-hand side of (2.14) in a power series in a neighbourhood of the point z = 0 and compare the absolute terms. Making use of (2.15), we get

$$\frac{\beta}{2M} \log \frac{\beta}{4M} + A_2^* - \frac{\beta}{2M} + \frac{2}{M} = 0$$

From the above relation we have

$$\text{(2.16)} \qquad A_2^* = \frac{\beta - 4}{2M} - \frac{\beta}{2M} \log \frac{\beta}{4M}$$

Substituting (2.16) into (1.9, B), we obtain H(F*) depending only on β. In order to determine the H(F*), we have to determine the β. With that end in view, let us compare the coefficients at z after expanding the left-hand side of equation (2.14) in a Taylor series in a neighbourhood of zero; we shall then get

$$\text{(2.17)} \qquad \frac{\beta}{2M}\left[A_2^* - \frac{\beta - 2}{2M}\right] + \frac{\beta}{2M^2} + A_3^* - A_2^{*2} + \frac{\beta^2 - 4\beta}{8M^2} + \frac{\beta - 2}{2M^2} = 1$$

After determining the A_3^* from (1.9, B) and after substituting it into (2.17) we have

$$\text{(2.18)} \qquad A_2^* = \frac{\beta}{4M} + \sqrt{\frac{\beta^2 + 16\beta - 16(M^2 + 1)}{48M^2}}$$

or

$$\text{(2.18')} \qquad A_2^* = \frac{\beta}{4M} - \sqrt{\frac{\beta^2 + 16\beta - 16(M^2 + 1)}{48M^2}}$$

Juxtaposing (2.16) with (2.18), (2.18´), we obtain two possible equations which should be satisfied by the β in the case considered:

$$\frac{\beta - 4}{2M} - \frac{\beta}{2M}\log\frac{\beta}{4M} = \frac{\beta}{4M} + \sqrt{\frac{\beta^2 + 16\beta - 16(M^2 + 1)}{48M^2}} \tag{2.19}$$

or

$$\frac{\beta - 4}{2M} - \frac{\beta}{2M}\log\frac{\beta}{4M} = \frac{\beta}{4M} - \sqrt{\frac{\beta^2 + 16\beta - 16(M^2 + 1)}{48M^2}} \tag{2.19´}$$

where $4 < \beta < 4M$.

At present, we shall be concerned with the problem of the existence of roots of the above equations, as well as with the uniqueness of solutions.

Denote

$$\frac{\beta}{4M} = \lambda \quad \text{and} \quad \frac{1}{M} = T \tag{2.20}$$

From the conditions imposed upon β and M we obtain

$$T < \lambda < 1 \quad \text{and} \quad 0 < T < 1 \tag{2.21}$$

Let us substitute (2.20) into equations (2.19) and (2.19´) we shall then get

$$\lambda(1 - 2\log\lambda) - 2T = \sqrt{\frac{\lambda^2 + 4T - T^2 - 1}{3}} \tag{2.22}$$

$$\lambda(1 - 2\log\lambda) - 2T = -\sqrt{\frac{\lambda^2 + 4T - T^2 - 1}{3}} \tag{2.22´}$$

On purpose to shorten the notation, let us denote

$$K(\lambda, T) = \lambda(1 - 2\log\lambda) - 2T, \quad G(\lambda, T) = \sqrt{\frac{\lambda^2 + 4\lambda T - T^2 - 1}{3}}$$

$$\Delta_1 = \left\{(\lambda, T),\ 0 < \lambda \le \frac{1}{\sqrt{e}},\ 0 < T \le \lambda\right\}$$

$$\Delta_2 = \left\{(\lambda, T), \frac{1}{\sqrt{e}} < \lambda < 1,\ 0 < T < T(\lambda)\right\}$$

$$\Delta_3 = \left\{(\lambda, T)\frac{1}{\sqrt{e}} < \lambda < 1, \quad T = T(\lambda)\right\}$$

$$\Delta_4 = \left\{(\lambda, T) \ \frac{1}{\sqrt{e}} < \lambda < 1, \quad T(\lambda) < T \le \lambda\right\}$$

$$\Delta_5 = \left\{(\lambda, T), \ \frac{1}{2} < \lambda < 1, \quad T_1(\lambda) < T \le \lambda\right\}$$

where $T(\lambda) = \frac{1}{2}\lambda(1 - 2\log\lambda)$, while $T_1(\lambda) = 2\lambda - \sqrt{5\lambda^2 - 1}$

After examining the signs of the values of $K(\lambda, T)$ and $G(\lambda, T)$ we acquire the following information:

$$K(\lambda, T) > 0 \Leftrightarrow (\lambda, T) \in (\Delta_1 \cup \Delta_2)$$

$$K(\lambda, T) = 0 \Leftrightarrow (\lambda, T) \in \Delta_3$$

$$K(\lambda, T) < 0 \Leftrightarrow (\lambda, T) \in \Delta_4$$

$$G(\lambda, T) = 0 \Leftrightarrow T = T_1(\lambda), \quad \lambda \in <\tfrac{1}{2}, 1)$$

$$G(\lambda, T) > 0 \Leftrightarrow (\lambda, T) \in \Delta_5$$

By the information and the notation adopted above,it remains to consider two possibilities

(2.23) $\quad K(\lambda, T) = G(\lambda, T), \quad (\lambda, T) \in \Delta_5 \cap (\Delta_1 \cup \Delta_2)$

(2.23′) $\quad K(\lambda, T) = -G(\lambda, T), \quad (\lambda, T) \in \Delta_5 \cap \Delta_4$

In Fig. 1 we shall present domain in which (λ, T) may vary, and in this domain we shall sketch the graphs of the functions

$$T(\lambda), \ T_1(\lambda) \text{ and } T_2(\lambda) \text{ where } T_2(\lambda) = 2\lambda + \sqrt{5\lambda^2 - 1}$$

Denote $k(T) = K(\lambda, T)$ and $g(T) = G(\lambda, T)$, where λ is fixed, $\lambda \in (\frac{1}{2}, 1)$, while T is variable, $T \in (T_1(\lambda), \lambda)$.

After examining the functions $k(T)$ and $g(T)$ we obtain:

I. For each $\frac{1}{2} < \lambda \le \frac{1}{\sqrt{e}}$, equation (2.23) has one solution $T = T_3(\lambda) \in (T_1(\lambda), \lambda)$ if and only if $k(\lambda) \le g(\lambda)$.

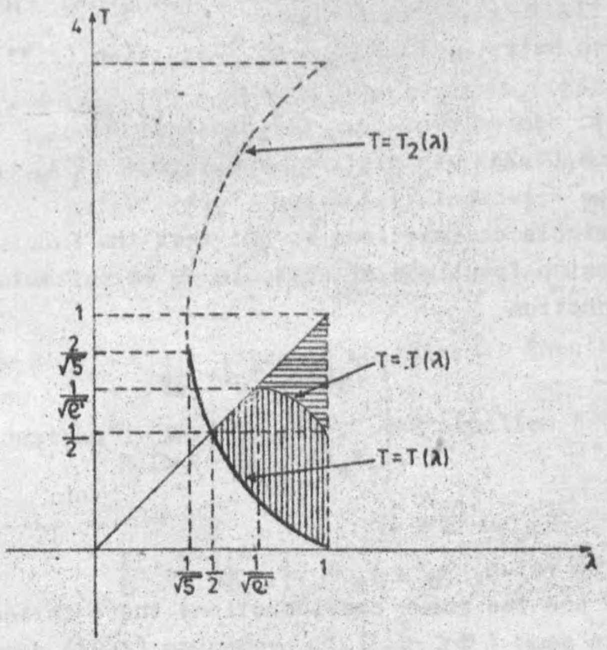

Fig. 1

II. For each $\frac{1}{\sqrt{e}} < \lambda < 1$, equation (2.23) has one solution $T = T_4(\lambda) \in (T_1(\lambda), T(\lambda))$.

III. For each $\frac{1}{\sqrt{e}} < \lambda < 1$, equation (2.23') has one solution $T = T_5(\lambda) \in (T(\lambda), \lambda)$ if and only if $k(\lambda) \le -g(\lambda)$.

IV. For $\frac{1}{2} < \lambda \le \frac{1}{\sqrt{e}}$ equation (2.23') has no solutions belonging to the interval $(T(\lambda), \lambda)$.

After examining the functions $k_1(\lambda) = k(\lambda)$ and $g_1(\lambda) = g(\lambda)$ for $\lambda \in (\frac{1}{2}, \frac{1}{\sqrt{e}}\rangle$ and the functions $k_1(\lambda) = k(\lambda)$ and $g_2(\lambda) = -g(\lambda)$ for $\lambda \in (\frac{1}{\sqrt{e}}, 1)$ we get

<u>Lemma 3.</u> For each $\lambda \in \langle \lambda_0, \frac{1}{\sqrt{e}} \rangle$, equation (2.23) has exactly one solution $T = T_3(\lambda) \in (T_1(\lambda), \lambda)$, where $\lambda_0 \in (\frac{1}{2}, \frac{1}{\sqrt{e}})$ is the only root of the equation

$$- \lambda (1 + 2 \log \lambda) = \sqrt{\frac{4\lambda^2 - 1}{3}}$$

II. For each $\lambda \in \frac{1}{\sqrt{e}}, 1)$, equation (2.23) possesses exactly one solution $T = T_4(\lambda) \in (T_1(\lambda), T(\lambda))$.

III. For each pair $(\lambda, T) \in \Delta_5 \cap \Delta_4$, equation (2.23) possesses no solutions.

It remains to decide whether, for distinct $\lambda \in <\lambda_0, 1)$, the roots of equation (2.23) are distinct and, if $\lambda \in <\lambda_0, 1)$, then in what interval the values of $T_3(\lambda)$ and $T_4(\lambda)$ vary.

After some simple calculations we get that the functions $T_3(\lambda)$, $T_4(\lambda)$ are decreasing functions of variable λ in suitable intervals, i.e. that the function

$$T^*(\lambda) = \begin{cases} T_3(\lambda), & \lambda_0 \le \lambda \le \frac{1}{\sqrt{e}} \\ T_4(\lambda), & \frac{1}{\sqrt{e}} \le \lambda < 1 \end{cases}$$

decreases from the value $\lambda_0 = T_0 \in (\frac{1}{2}, \frac{1}{\sqrt{e}})$ to $\frac{13}{3}$.

From lemma 3 and the above considerations there follows:

Lemma 4. For each $T \in <\frac{3}{13}, T_0>$, equation (2.22) possesses exactly one solution $\lambda = \lambda(T) \in <T_0, 1)$ where $T_0 = \lambda_0 \in (\frac{1}{2}, \frac{1}{\sqrt{e}})$ is the only root of the equation

$$\sqrt{\frac{4\lambda^2 - 1}{3}} = -\lambda(1 + 2\log\lambda)$$

For $T \in (0, 1)$, $T \notin <\frac{3}{13}, T_0>$ equation (2.22) has no solution belonging to the interval $(T, 1)$. If $T = \frac{3}{13}$, then $(\lambda)T = 1$, whereas if $T = T_0$, then $\lambda(T_0) = T_0$; these are limit cases.

For $T \in (0, 1)$ and $\lambda \in (T, 1)$, equation (2.22′) has no solutions.

Corollary 1. For each $M \in <M_0, \frac{13}{3}>$, equation (2.19) possesses exactly one solution β_0 belonging to the interval $(4, 4M)$ where M_0 is the only root of equation (2.10). For $M > 1$, $M \notin <M_0, \frac{13}{3}>$, equation (2.19) has no solutions from the interval $(4, 4M)$.

II. For each $M > 1$, $\beta \in (4, 4M)$ equation (2.19′) possesses no solutions.

We have thus shown

Lemma 5. If in the family $S_R(M)$, $M > 1$, there is an extremal function $w = F^*(z)$ satisfying equation B, then

$$H(F^*) = \frac{2}{3}\left(\frac{\beta_0 - 4}{2M} - \frac{\beta_0}{2M}\log\frac{\beta_0}{4M}\right)\left[1 + \frac{1}{M^2}\left(\frac{\beta_0^2}{8} - \beta_0 + 1\right)\right]$$

where β_0 is the only root of equation (2.19), belonging to the interval $(4, 4M)$, while $M \in (M_0, \frac{13}{3})$ where M_0 is the only root of equation (2.10). Besides, the extremal function $w = F^*(z)$ satisfies (2.14) where

$$C = \frac{\beta}{2M}\log\frac{\beta}{4M} + A_2^* - \frac{\beta}{2M} + \frac{2}{M}$$

For $M \notin (M_0, \frac{13}{3})$, there is no extremal function satisfying equation B.

3. Equation C. In this case, the equation can be written in the form

(2.24)
$$\left(\frac{zw'}{w}\right)^2 \frac{(w + M)^2 (w + \tau)\left(w + \frac{M^2}{\tau}\right)}{M^4 w^2} = \frac{(z + 1)^2 (z + \rho)\left(z + \frac{1}{\rho}\right)}{z^2}$$

where $-\tau (0 < \tau < M)$ is one of the roots of the equation $w^2 + (\beta - 2)Mw + M^2 = 0$, while $-\rho (0 < \rho \leq 1)$ is one of the roots of the equation $z^2 + (\frac{\beta}{M} - 2)z + 1 = 0$.

Note that in the domain $\hat{E} = E - (-1, -\rho\rangle$ there is a single-valued branch $p(z) = \sqrt{\frac{z+\rho}{z+\frac{1}{\rho}}}$. From (2.24) it follows that $-\tau = F^*(-\rho)$ since $F^*(-\rho) \neq 0$. Consequently, in the domain $F^*(\hat{E})$ there is a single-valued branch of the $P(w)$ defined in (2.13). Let us adopt $p(o) = \rho$, $P(o) = \frac{\tau}{M}$.

Since $F^*(o) = 0$, $F^{*\prime}(o) = 1$, equation (2.24) can be represented for $z \in \hat{E}$ in the form

(2.25)
$$\frac{zw'}{w} \frac{(w + M)\left(w + \frac{M^2}{\tau}\right)P(w)}{M^2 w} = \frac{(z + 1)\left(z + \frac{1}{\rho}\right)p(z)}{z}$$

Integrating equation (2.25), we obtain after some simple transformations

$$(2.26)\qquad \frac{(M+\tau)^2}{2M^2\tau}\log\frac{\tau - MP(w)}{\tau + MP(w)} - \frac{(M+\tau)^2}{2M^2\tau}\log\frac{1-P(w)}{1+P(w)} -$$

$$-\frac{M^2-\tau^2}{M}\cdot\frac{P(w)}{M^2P^2(w)-\tau^2} - \frac{M^2-\tau^2}{M^2\tau}\cdot\frac{P(w)}{P^2(w)-1} =$$

$$= \frac{(1+\varrho)^2}{2\varrho}\log\frac{\varrho - p(z)}{\varrho + p(z)} - \frac{(1+\varrho)^2}{2\varrho}\log\frac{(1-p(z)}{1+p(z)} -$$

$$+ (1-\varrho^2)\frac{p(z)}{p^2(z)-\varrho^2} - \frac{1-\varrho^2}{\varrho}\cdot\frac{p(z)}{p^2(z)-1} + C$$

where C is a constant, and $P(w)$, $p(z)$ are the branches of the roots, chosen before, while the branch of the logarithm is so chosen that $\log 1 = 0$.

Since, with $z \to -\varrho$ $w \to -\tau$, therefore, passing to the limit of $z \to -\varrho$ in (2.26), we have

$$C = 0$$

Next, expanding both sides of equation (2.26) in a power series in a neighbourhood of $z = 0$, after comparing the absolute terms and making use of the relations $\frac{M}{\tau} + \frac{\tau}{M} = \beta - 2$, $\varrho + \frac{1}{\varrho} = \frac{\beta}{M} - 2$, we get

$$(2.27)\qquad A_2^* = 2\left(1 - \frac{1}{M}\right)$$

Let us make another use of the expansion of equation (2.26) in a power series in question and compare the coefficients at z. We then obtain

$$(2.28)\qquad \frac{\beta}{2M}\left[A_2^* - \frac{\beta-2}{2M}\right] + \frac{\beta}{2M^2} + A_3^* - A_2^{*2} +$$

$$+ \frac{\beta^2}{8M^2} - \frac{1}{M^2} = \frac{\beta}{2M}\left(1 - \frac{\beta}{2M}\right) + \frac{\beta}{2M} - 1 + \frac{\beta^2}{8M^2}$$

Determining the A_3^* from (1.9, C) and substituting it, along with (2.27), into (2.28), we have

$$\beta = \frac{11M - 13}{2}$$

Solving the inequality $\beta \geq 4M$, we deduce that $M \geq \frac{13}{3}$.

We have thus proved .

Lemma 6. If in the family $S_R(M)$, $M \geq \frac{13}{3}$, there is an extremal function $w = F^*(z)$ satisfying equation C, then $A_2 \cdot A_3 \leq P_2(M) \cdot P_3(M)$ where $P_2(M)$, $P_3(M)$ are defined in the introduction. This estimation is true for $M \geq \frac{13}{3}$. For $M < \frac{13}{3}$ there is no extremal function satisfying equation C.

Consider the function $w = \varphi(z, M)$, holomorphic and univalent in $|z| < 1$, defined by equation (0.5). Since $\varphi(z, M) = z + 2(1 - \frac{1}{M})z^2 + (3 - \frac{8}{M} + \frac{5}{M^2})z^3 + \dots$, therefore $\varphi(z, M) \in S_R(M)$ and $H(\varphi) = H(F^*)$. It can easily be noticed that the function $\varphi(z, M)$ satisfies equation (2.26).

4. Equation D. In this case, the equation can be written in the form

$$(2.29)\qquad \left(\frac{zw'}{w}\right)^2 \frac{(w + M)^2 (w - \tau)(w - \frac{M^2}{\tau})}{M^4 w^2} = \frac{(z + 1)^2 (z - \varrho)(z - \frac{1}{\varrho})}{z^2}$$

where $\tau \in (0, M)$ is one of the roots of the equation $w^2 + M(\beta - 2)w + M^2 = 0$, while $\varrho \in (0, 1)$ is one of the roots of the equation $z^2 + (\frac{\beta}{M} - 2)z + 1 = 0$.

Proceeding similarly as in case C, we have

$$A_2^* = 2(1 - \frac{1}{M}) > 0$$

whence we get a contradiction since, in the case under consideration, $A_2^* < 0$.

So, we have shown

Lemma 7. There is no extremal function with respect to the functional $H(F)$, satisfying equation D.

5. Equation E. In this case, the equation after some transformations takes the form

$$(2.30)\qquad \frac{zw'}{w} \frac{w^2 - M^2}{M^2 w} = \frac{z^2 - 1}{z}$$

Integrating both sides of the above equation, we have

$$\frac{w}{M^2} + \frac{1}{w} = z + \frac{1}{z} + C \tag{2.31}$$

where C is a constant.

Let us expand the left-hand side of (2.31) in a power series in a neighbourhood of $z = 0$ and compare the absolute terms as well as the coefficients at z. We shall then get

$$C = -A_2^* \tag{2.32}$$

$$\frac{1}{M^2} + A_2^{*2} - A_3^* = 1 \tag{2.32'}$$

Determining the A_3^* from (1.9, E) and substituting it into (2.32'), we obtain

$$A_2^{*2} = \frac{1}{3}\left(1 - \frac{1}{M^2}\right)$$

Hence, after taking account of the sign of A_2^* ($A_2^* < 0$), we get

$$A_2^* = -\sqrt{\frac{1}{3}\left(1 - \frac{1}{M^2}\right)}$$

and, thereby, we have determined the $H(F^*)$ in the case considered. By examining equation (2.31) for $C = \sqrt{\frac{1}{3}\left(1 - \frac{1}{M^2}\right)}$, one can easily demonstrate that it possesses a solution belonging to the family $S_R(M)$ only for $M \geq \frac{13}{11}$.

We have thus proved

Lemma 8. If in the family $S_R(M)$, $M > 1$, there is an extremal function $w = F^*(z)$ satisfying equation E, then

$$H(F^*) = \frac{2}{3\sqrt{3}}\sqrt{\left(1 - \frac{1}{M^2}\right)^3}$$

and moreover, this function satisfies the equation

$$\frac{w}{M^2} + \frac{1}{w} = z + \frac{1}{z} + \sqrt{\frac{1}{3}\left(1 - \frac{1}{M^2}\right)}$$

This case can hold only for $M \geq \frac{13}{11}$.

6. Equation F. Let us write that equation in the form

(2.33) $$\left(\frac{zw'}{w}\right)^2 \frac{(w - M)^2 (w + \tau)(w + \frac{M^2}{\tau})}{M^4 w^2} =$$

$$= \frac{(z - 1)^2 (z + \rho)(z + \frac{1}{\rho})}{z^2}$$

Proceeding similarly as in case C, we get

$$A_2^* = -2(1 - \frac{1}{M})$$

$$\beta = \frac{-11M + 13}{2}$$

Let us substitute the above relations into (1.9, F); we shall then obtain $H(F^*) = -P_2(M) \cdot P_3(M)$ where $P_2(M)$, $P_3(M)$ are defined in the introduction.

From the inequality $\beta > 0$ follows that $M < \frac{13}{11}$.

Consequently, we have proved

Lemma 9. If in the family $S_R(M)$, $M > 1$, there is an extremal function satisfying equation F, then

$$H(F^*) = -P_2(M) \cdot P_3(M)$$

and moreover, this function satisfies the equation

(2.34) $$\frac{\beta}{2M} \log \frac{\tau - MP(w)}{\tau + MP(w)} - \frac{\beta}{2M} \log \frac{1 - P(w)}{1 + P(w)} +$$
$$+ \frac{M^2 - \tau^2}{M} \cdot \frac{P(w)}{M^2 P^2(w) - \tau^2} -$$
$$- \frac{M^2 - \tau^2}{M^2 \tau} \frac{P(w)}{P^2(w) - 1} = \frac{\beta}{2M} \log \frac{\rho - p(z)}{\rho + p(z)} -$$
$$- \frac{\beta}{2M} \log \frac{1 - p(z)}{1 + p(z)} + (1 - \rho^2) \frac{p(z)}{p^2(z) - \rho^2} +$$
$$- \frac{1 - \rho^2}{\rho} \cdot \frac{p(z)}{p^2(z) - 1}$$

This case can hold only for $M < \frac{13}{11}$.

Consider the function $w = -\varphi(-z, M)$ where $w = \varphi(z, M)$ is defined by equation (0.5). Since $-\varphi(-z, M) = z - 2(1 - \frac{1}{M})z^2 + (3 - \frac{8}{M} + \frac{5}{M^2})z^3 + \ldots$, therefore $-\varphi(-z, M) \in S_R(M)$ and $H(-\varphi(-z, M)) = H(F^*)$. It is easy to demonstrate that the function $w = -\varphi(-z, M)$ satisfies equation (2.34).

3. The fundamental theorem

In the investigations made so far, we have considered all possible forms of differential-functional equation (1.3). At present, we shall proceed to proving

The fundamental theorem

For any function $w = F(z) = z + A_2z^2 + A_3z^3 + \ldots$ of the family $S_R(M)$, $M > 1$, the following sharp estimation

(3.1)-(3.5)

$$A_2 \cdot A_3 \le \begin{cases} -2(1 - \frac{1}{M})^2(3 - \frac{5}{M}) & \text{if } 1 < M \le \frac{13}{11} \\ \frac{2}{3\sqrt{3}}\sqrt{(1 - \frac{1}{M^2})^3} & \text{if } \frac{13}{11} \le M \le \sqrt{e} \\ \frac{2}{3\sqrt{3}}\sqrt{(1 - \frac{1}{M^2})^3 \frac{\log M}{1 - \log M}} & \text{if } \sqrt{e} \le M \le M_0 \\ \frac{2}{3}(\frac{\beta-4}{2M} - \frac{\beta}{2M}\log\frac{\beta}{4M})\left[1 + \frac{1}{M^2}(\frac{\beta^2}{8} - \beta + 1)\right] & \text{if } M_0 \le M \le \frac{13}{11} \\ 2(1 - \frac{1}{M})^2(3 - \frac{5}{M}) & \text{if } M \ge \frac{13}{3} \end{cases}$$

takes place; where in (3.3), (3.4) M_0 is the only root of the equation $12\log M(1 - \log M) + 1 - M^2 = 0$, while the β occuring in (3.4) is the only root of the equation

$$\frac{\beta - 4}{2M} - \frac{\beta}{2M}\log\frac{\beta}{4M} = \frac{\beta}{4M} + \sqrt{\frac{\beta^2 + 16\beta - 16(M^2 + 1)}{48M}}$$

(belonging to the interval $(4, 4M)$).

P r o o f. In lemmas 2,5, and 6, 8, 9 we have obtained the formulae for the values $H(F^*)$ of functional (1.1), as well as the intervals in which suitable estimations can hold true. Since the intervals obtained are not disjoint, the proof of the theorem will consist in determining the sharp estimations in the common parts of the above-mentioned intervals.

Note first that, in the interval $(1, \frac{13}{11}\rangle$ formulae (3.1) or (3.3) can hold (see lemmas 9, 2). Comparing both values of the functional, we arrive at the conclusion that the estimation valid in this interval is (3.1); the function $w = -\varphi(-z, M)$, where $w = \varphi(z, M)$ is a Pick function defined by equation (0.5), realizes the equality in estimation (3 1) and belongs to the family $S_R(M)$.

Similarly, in the interval $\langle \frac{13}{11}, \sqrt{e}\rangle$ there can hold two cases, i.e. (3.2) or (3.3) (see lemmas 2, 8). Comparing both values of the functional in the interval considered, we infer that the estimation valid is (3.2), equality taking place for the function $w = F^*(z)$ defined by equation (2.30), This function maps the disc E onto the disc $|w| < M$ from which two arcs lying on the real axis and issuing from the points $w_{1,2} = \pm M$ have been removed.

Let us next consider the interval $\langle\sqrt{e}, M_0\rangle$ where M_0 is defined in theorem 1. In this interval, formulae (3.2) or (3.3) can take place. After comparing suitabale values of the functional $H(F)$ in the interval under consideration we deduce that the estimation holding true is (3.3); the equality in estimation (3.3) is realized by the function $w = F^*(z)$ defined by equation (2.9); the proof of the existence of such function of the family $S_R(M)$ can be found in [4].

Let us now take into consideration the interval $\langle M_0, \frac{13}{3}\rangle$. In this interval, estimations (3.4) or (3.2) can hold. Comparing both values of the functional in the interval considered, we infer that (3.4) is valid; the equality in estimation (3.4) is realized by the function $w = F^*(z)$ defined by equation (2.14); the proof of the existence of such function of the family $S_R(M)$ can be found in paper [4].

What is left for us is to consider the interval $\langle \frac{13}{3}, +\infty)$. From lemmas 6 and 8 it follows that, in the interval under consideration, formulae (3.5) or (3.2) can hold true. From the comparison of both values of the functional we conclude that formula (3.5) is valid in this interval; the function realizing the equality in this estimation is $w = \mathcal{P}(z, M)$ defined by equation (0.5) which, as we know, belongs to the family $S_R(M)$. Consequently, the theorem has been proved.

Corrollary 2. For any function $w = F(z) = z + A_2z^2 + A_3z^3 + \ldots$ of the family $S_R(M)$, $M > 1$, the product $A_2 \cdot A_3$ is greater than or equal to suitable values (3.1)-(3.5) multiplied by minus one.

This corollary results from the fact that if $F \in S_R(M)$, then the function $w = -F(-z)$, too, belongs to the family $S_R(M)$.

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OSZACOWANIE FUNKCJONAŁU $A_2 \cdot A_3$ W KLASIE FUNKCJI OGRANICZONYCH, SYMETRYCZNYCH I JEDNORODNYCH

Oznaczmy przez $S_R(M)$, $M > 1$, rodzinę funkcji $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, holomorficznych i jednolistnych w kole $E = \{z : |z| < 1\}$, o rzeczywistych współczynnikach i spełniających warunek $|F(z)| \le M$, $z \in E$.

W pracy uzyskano dokładne oszacowanie funkcjonału $H(F) = A_2 \cdot A_3$ w klasach $S_R(M)$, $M > 1$.