A C T A U N I V E R S I T A T I S L O D Z I E N S I S
FOLIA MATHEMATICA 3, 1989

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ON SOME GEOMETRICAL CHARACTERIZATION OF SINGULAR NORMED MEASURES

In this paper there are given two characterizations of singular normed measures. These theorems are used to study singular measures in the product of measurable normed spaces.

Let Y be a (real or complex) vector space and let Z be a subset of Y having at least two points. We say that two distinct points $p_1, p_2 \in Z$ are antipodal in Z (or simply antipodal) if and only if, for each $x_1, x_2 \in Z$ and for every real number t, the equality $t(p_1 - p_2) = x_1 - x_2$ implies $|t| \leq 1$.

Let (X, A) be any measurable space. Then the set Y of all signed measures defined on this space is a real vextor space (addition and multiplication by real numbers are applied in the usual sense). Consider the subset M of Y containing all nonnegative measures μ defined on X and normed by the condition $\mu(X) = 1$. Then one can prove the following.

Theorem 1. Two normed measures $\mu, \nu \in M$ are antipodal if and only if $(\mu - \nu)^*(X) = 2$ where $(\mu - \nu)^*(X)$ denotes the total variation of a signed measure $\mu - \nu$ on X.

P r o o f. Remark first that, for all normed measures μ, ν defined on X, we have $(\mu - \nu)^*(X) \leq 2$. Really, for upper and lower variations of the measure $\lambda = \mu - \nu$, we have the inequalities $\lambda^+ \leq \mu$, $\lambda^- \leq \nu$. As $\lambda^* = \lambda^+ + \lambda^-$, so $\lambda^*(X) \leq 2$.

Suppose now that $(\mu - \nu)^*(X) = 2$ and assume that there exists a real number $t \geq 1$ and two normed measures μ_1, ν_1 for which $t(\mu - \nu) = \mu_1 - \nu_1$. Then we have $(\mu_1 - \nu_1)^*(X) = t(\mu - \nu)^*(X) > 2$. This inequality is impossible, so the measures μ, ν fulfilling the condition $(\mu - \nu)^*(X) = 2$ are antipodal.

Now, suppose we have $(\mu - \nu)^*(X) < 2$. Then $t = \frac{2}{(\mu - \nu)^*(X)} > 1$ and $t(\mu - \nu)^*(X) = 2$. Put $\lambda = t(\mu - \nu)$. Then $\lambda^*(X) = \lambda^+(X) + \lambda^-(X) = 2$. Suppose that $\lambda^+(X) < 1$. So $\lambda^-(X) > 1$. Let B be a measurable set such that, for every measurable set A, we have $\lambda(A) \le 0$ for $A \subset B$ and $\lambda(A) \ge 0$ for $A \subset X \setminus B$. Then $\lambda(X \setminus B) = \lambda^+(X) < 1$ and $\lambda(B) = -\lambda^-(X) < -1$. These inequalities imply $\mu(X) < \nu(X)$, which is impossible. Assuming $\lambda^+(X) > 1$, we obtain a false inequality $\mu(X) > \nu(X)$. So we proved that λ^+, λ^- are normed measures. The equality $\lambda = t(\mu - \nu) = \lambda^+ - \lambda^-$ shows that the measures μ, ν are not antipodal. This completes the proof.

By the above theorem, we obtain the other characterization of antipodal points of M.

Theorem 2. Two normed measures $\mu, \nu \in M$ are antipodal if and only if they are singular.

P r o o f. Suppose first the measures μ, ν are singular. Let A be a measurable subset of X for which $\mu^*(A) = \nu^*(X \setminus A) = 0$. Then $(\mu - \nu)^*(X) \ge \mu(X \setminus A) + \nu(A) = 2$. But, on the other hand, $(\mu - \nu)^*(X) \le 2$, so $(\mu - \nu)^*(X) = 2$ and, consequently, μ, ν are antipodal.

Assume now μ, ν to be antipodal. We have

$$2 = (\mu - \nu)^*(X) = [\mu(X \setminus B) + \nu(B)] - [\mu(B) + \nu(X \setminus B)],$$

when B is as in *Theorem 1*. So $\mu(X \setminus B) + \nu(B) = 2$ and $\mu(B) + \nu(X \setminus B) = 0$ and, consequently, $\mu(B) = \nu(X \setminus B) = 0$.

The theorem just proved gives a simple geometrical characterization of singular normed measures on X. We used them to characterize singular measures on a product space. The following lemmas will be useful.

Lemma 1. Let μ, ν be nonnegative σ-finite measures on (X, A) such that $\nu(X) > 0$ and $\nu \ll \mu$ (ν is absolutely continuous with respect to μ). If $\bar{\mu} = \mu + \nu$, then there exists a measurable real function f such that, for every measurable set A, we have

$$\nu(A) = \int_A f \, d\mu$$

and $0 \le f < 1$ $\bar{\mu}$-almost everywhere on X. Moreover, the set $B = \{x \in X : 0 < f(x) < 1\}$ is measurable and $\bar{\mu}(B) > 0$.

P r o o f. It is clear that $\nu \ll \bar{\mu}$ and $\bar{\mu}$ is σ-finite. By the Lebesgue-Radon-Nikodym theorem, there exist only two finite nonnegative measurable functions f_o, f defined on X such that, for every measurable set A, the equalities

$$\nu(A) = \int_A f_o \, d\mu = \int_A f \, d\bar{\mu}$$

hold. From this we obtain

$$\int_X f(1 + f_o)d\mu = \int_X f_o d\mu.$$

Hence $f_o = f(1 + f_o)$ μ - a.e. on X. So, according to the assumption $\nu \ll \mu$, we have that $0 \le f < 1$ $\bar{\mu}$ - a.e. on X.

Moreover, the equivalence $f(x) = 0 \Leftrightarrow f_o(x) = 0$ is fulfilled μ - a.e. on X and the equality $f = 0$ does not hold μ - a.e. on X. So a set $B = \{x \in X : 0 < f(x) < 1\}$ is measurable and $\bar{\mu}(B) > 0$. This ends the proof.

Lemma 2. If σ-finite measures μ, ν on X are nonnegative and fulfil the conditions $\nu \ll \mu$, $\mu \ne \nu$, then

$$(\mu - \nu)^*(X) < \mu(X) + \nu(X).$$

P r o o f. Put $\bar{\mu} = \mu + \nu$. Let f be a function from the first lemma and let

$$B = \{x \in X : 0 < f(x) < 1\},$$

$$B_k = \{x \in X : \tfrac{1}{k} < f(x) < 1 - \tfrac{1}{k}\}.$$

Then $\bar{\mu}(B) > 0$. Moreover, $B = \bigcup_{k=3}^{\infty} B_k$. So there exists a positive integer $k_o \ge 3$ such that $\mu(B_{k_o}) > 0$.

Let $\{A_1, A_2, \ldots A_n\} \subset A$ be any partition of X. Consider a new partition $\{C_1, C_2, \ldots, C_n\} \subset A$ obtained from the first one in the following way:

$$C_i = \begin{cases} A_i \cap B_{k_o} & i = 1, \ldots, n \\ A_{i-n} \setminus B_{k_o} & i = n+1, \ldots, 2n. \end{cases}$$

It is easy to see that $\bigcup_{i=1}^{n} C_i = B_{k_o}$, $\bigcup_{i=n+1}^{2n} C_i = X \setminus B_{k_o}$,

Moreover,

$$\sum_{i=1}^{n} |(\mu - \nu)(A_i)| = \sum_{i=1}^{n} |(\bar{\mu} - 2\nu)(A_i)| \le \sum_{i=1}^{n} \left|\int_{C_i} (1 - 2f)d\bar{\mu}\right| +$$

$$+ \sum_{i=n+1}^{2n} |(\mu - \nu)(C_i)| \le \int_{B_{k_o}} |1 - 2f|d\bar{\mu} + \bar{\mu}(X \setminus B_{k_o}) \le \mu(X) +$$

$$+ \nu(X) - \frac{2}{k_o}\bar{\mu}(B_{k_o}).$$

These inequalities hold for every partition $\{A_1, A_2, \ldots, A_n\}$, so

$$(\mu - \nu)^*(X) \le \mu(X) + \nu(X) - \frac{2}{k_o}\,\mu(B_{k_o}),$$

which implies the proposition of the lemma.

Theorem 3. Let $\{(X_\gamma, \mathcal{A}_\gamma)\}_{\gamma\in\Gamma}$ be a family of measurable spaces and μ_γ, ν_γ $(\gamma \in \Gamma)$ - normed measures defined on X_γ. Let $\mu = \bigotimes_{\gamma\in\Gamma} \mu_\gamma$, $\nu = \bigotimes_{\gamma\in\Gamma} \nu_\gamma$. If there exists $\gamma_o \in \Gamma$ such that μ_{γ_o}, ν_{γ_o} are singular, then the product measures μ, ν are singular, too.

P r o o f. By our assumption, there exists a set $A_{\gamma_o} \in \mathcal{A}_{\gamma_o}$ such that $\mu_{\gamma_o}(A_{\gamma_o}) = \nu_{\gamma_o}(X_{\gamma_o} \setminus A_{\gamma_o}) = 0$. Put

$$A_\gamma = \begin{cases} X_{\gamma_o} & \gamma \neq \gamma_o \\ A_{\gamma_o} & \gamma = \gamma_o, \end{cases} \qquad B_\gamma = \begin{cases} X_\gamma & \gamma \neq \gamma_o \\ X_{\gamma_o} \setminus A_{\gamma_o} & \gamma = \gamma_o. \end{cases}$$

Then the sets $A = \mathop{\mathbb{P}}_{\gamma\in\Gamma} A_\gamma$, $B = \mathop{\mathbb{P}}_{\gamma\in\Gamma} B_\gamma$ are measurable subsets of the product space $(\mathop{\mathbb{P}}_{\gamma\in\Gamma} X_\gamma, \mathop{\mathbb{P}}_{\gamma\in\Gamma} \mathcal{A}_\gamma)$, such that $\mu(A) = \nu(\mathop{\mathbb{P}}_{\gamma\in\Gamma} X_\gamma \setminus A) = 0$.

If Γ is a finite set, then the above theorem can be reversed.

Theorem 4. The measures $\mu = \bigotimes_{k=1}^{n} \mu_k$, $\nu = \bigotimes_{k=1}^{n} \nu_k$ defined on a product $(\mathop{\mathbb{P}}_{k=1}^{n} X_k, \mathop{\mathbb{P}}_{k=1}^{n} \mathcal{A}_k)$ of measurable spaces and normed by the conditions $\mu_k(X_k) = \nu_k(X_k) = 1$ $(k = 1, 2, \ldots, n)$ are singular if and only if there exists a positive integer k_o $(1 \le k_o \le n)$ such that the measures μ_{k_o}, ν_{k_o} are singular.

P r o o f. It remains to prove that if μ, ν are singular then there exists k_o such that μ_{k_o}, ν_{k_o} are singular.

For simplicity, let $n = 2$. So $\mu = \mu_1 \otimes \mu_2$, $\nu = \nu_1 \otimes \nu_2$, $X = X_1 \times X_2$.

Suppose first that $\nu_k \ll \mu_k$, $k = 1, 2$. Let f_k $(k = 1, 2)$ denote the Lebesgue-Radon-Nikodym derivative $\frac{d\nu_k}{d\mu_k}$. Since μ_k, ν_k are normed measures, therefore no function f_1, f_2 can be μ_k - a.e. equal to zero. This means that

$$(\int_{X_1} f_1^{\frac{1}{2}}\, d\mu_1)\,(\int_{X_2} f_2^{\frac{1}{2}}\, d\mu_2) > 0.$$

By Kakutani´s theorem, we have $\nu \ll \mu$, so μ, ν are not singular.

Now, consider the case when neither the pair μ_1, ν_1 nor μ_2, ν_2 is singular. The Lebesgue decomposition theorem enables us to represent the measures ν_1, ν_2 in the form

$$\nu_k = \nu_k' + \nu_k''$$

when $\nu_k' \ll \mu_k$, $\nu_k'' \perp \mu_k$. Furthermore, $\nu_k'(X_k) > 0$. Hence

$$\nu = (\nu_1' \otimes \nu_2') + (\nu_1' \otimes \nu_2'') + (\nu_1'' \otimes \nu_2') + (\nu_1'' \otimes \nu_2'').$$

Using *lemma 2*, we obtain $(\mu - \nu)^*(X) < 2$. We have proved that then μ, ν are not singular. This ends the proof.

Now, we construct the example that *theorem 3* cannot by reversed if μ, ν are product measures on a product of infinitely many measurable spaces.

Let $X_n = \langle 0, 1\rangle$, let $\mathcal{A}_n$ be the σ-algebra of all Borel subsets of X_n. For each $n \in N$ and for each Borel subset $E \subset \langle 0, 1\rangle$, put

$$\nu_n(E) = \begin{cases} 0 & \frac{1}{n} \notin E \\ 1 & \frac{1}{n} \in E \end{cases}$$

$$\mu_n(E) = \begin{cases} \frac{k}{n^2} & \operatorname{card} E \cap \{\frac{1}{n^2}, \frac{2}{n^2}, \ldots, 1\} = k \\ 0 & E \cap \{\frac{1}{n^2}, \frac{2}{n^2}, \ldots, 1\} = \emptyset. \end{cases}$$

Then $\nu_n \ll \mu_n$. Let $\mu = \bigotimes_{n=1}^{\infty} \mu_n$, $\nu = \bigotimes_{n=1}^{\infty} \nu_n$. For any measurable subset B of the product σ-algebra $\mathop{\mathbb{P}}_{n=1}^{\infty} X_n$, we have

$$\nu(B) = \begin{cases} 1 & (1, \frac{1}{2}, \frac{1}{3}, \ldots) \in B \\ 0 & (1, \frac{1}{2}, \frac{1}{3}, \ldots) \notin B \end{cases}$$

and $\mu(\{(1, \frac{1}{2}, \frac{1}{3}, \ldots)\}) = \prod_{n=i}^{\infty} \mu_n(\{\frac{1}{n}\}) = 0$. So, if we put $A =$

$= \{(1, \frac{1}{2}, \frac{1}{3}, \ldots)\}$, then $\mu(A) = \nu(\mathop{\mathrm{P}}\limits_{n=1}^{\infty} X_n \setminus A) = 0$. This result shows that μ, ν are singular.

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O PEWNEJ CHARAKTERYZACJI MIAR OSOBLIWYCH UNORMOWANYCH

W pracy podaje się charakteryzację miar osobliwych unormowanych (twierdzenie 2). W twierdzeniu 3 formułuje się warunek dostateczny na to, by miary unormowane w iloczynie dowolnej ilości przestrzeni mierzalnych były osobliwe. W twierdzeniu 4 pokazuje się, że w przypadku iloczynów skończonych podany warunek jest również warunkiem dostatecznym. Konstruuje się również przykład na to, że w przypadku iloczynów nieskończonych twierdzenie 3 nie daje się odwrócić.