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SOME BERNSTEIN-TYPE CONSTRUCTION AND ITS APPLICATIONS

Sierpiński in 1932 constructed a Bernstein-type set on the real line $\mathbb{R}$, such that each of its translations differs from it by the set of power $< 2^{\aleph_0}$. In the paper, we generalize his result by considering, instead of translations, an arbitrary family of power $\leqslant 2^{\aleph_0}$ which consists of one-to-one functions from $\mathbb{R}$ onto $\mathbb{R}$. Moreover, some corollaries concerning σ-ideals are obtained.

Let $\mathbb{R}$ denote the real line.

Recall (see [2], p. 422) that $B \subsetneq \mathbb{R}$ is a Bernstein set if and only if both B and $\mathbb{R} \setminus B$ intersect each nonempty perfect set.

Note that the following property holds:

Lemma. If a set $E \subsetneq \mathbb{R}$ intersects each nonempty perfect set, then the intersection is of power $2^{\aleph_0}$.

P r o o f. Let P be an arbitrary perfect set such that $P \cap E \neq \emptyset$. Choose a set $C \subsetneq P$ homeomorphic to the Cantor set (see [2], p. 355). There exists a homeomorphism h which maps $C \times C$ onto C (see [2], p. 235). Let $P_t = h(C \times \{t\})$ for $t \in C$. The family $\{P_t : t \in C\}$ consists of $2^{\aleph_0}$ nonempty perfect sets, pairwise disjoint, included in P. Since $P_t \cap E \neq \emptyset$ for all $t \in C$, the assertion is clear.

It is known that a Bernstein set is not Lebesgue measurable and has not the Baire property (see [2], p. 423).

A set is called totally imperfect if and only if it does not contain any nonempty perfect set. Each Bernstein set is totally imperfect.

For $x \in \mathbb{R}$ and $A, D, E \subsetneq \mathbb{R}$, denote

$$A + x = \{ y \in \mathbb{R}: y = a + x \text{ for some } a \in A \}$$

$$D \triangle E = (D \setminus E) \cup (E \setminus D).$$

Let $\mathcal{P}(\mathbb{R})$ be the family of all subsets of $\mathbb{R}$ and let $\mathcal{K} \subseteq \mathcal{P}(\mathbb{R})$ be the family of all sets of power $< 2^{\aleph_0}$.

In [6] S i e r p i ń s k i obtained the following result:

Theorem 0. There is a Bernstein set B such that each set $B \triangle (B + x)$, $x \in \mathbb{R}$, belongs to $\mathcal{K}$.

Using the same methods, we shall exend this construction to a more general case.

Throught the paper, it will be assumed that $\mathcal{F}$ denotes a family of power $\leq 2^{\aleph_0}$ of one-to-one functions which map $\mathbb{R}$ onto $\mathbb{R}$.

Theorem 1. There is a Bernstein set B such that each set $B \triangle f(B)$, $f \in \mathcal{F}$ belongs to $\mathcal{K}$.

P r o o f. Let β be the first ordinal number of power $2^{\aleph_0}$. Let

(1) $$r_0, r_1, \ldots, r_\alpha, \ldots; \alpha < \beta$$

$$P_0, P_1, \ldots, P_\alpha, \ldots; \alpha < \beta$$

$$f_0, f_1, \ldots, f_\alpha, \ldots; \alpha < \beta$$

denote the transfinite sequences: of all real numbers, of all nonempty perfect subsets of $\mathbb{R}$, and of all functions belonging to $\mathcal{F}$, respectively (if $\mathcal{F}$ is of power $< 2^{\aleph_0}$, we repeat one of the functions from $\mathcal{F}$ in the third of the above sequences sufficiently many times). Choose x_0 as the first term of (1) which belongs to P_0, and y_0 as the first term of (1) which belongs to $P_0 \setminus \{x_0\}$. Assume that $0 < \alpha < \beta$ and the elements x_γ, y_γ, $\gamma < \alpha$, have been already defined. Denote by $\mathcal{F}_\alpha$ the set of all functions of the form

$$f_{\alpha_0}^{k_0} \circ f_{\alpha_1}^{k_1} \circ \ldots \circ f_{\alpha_n}^{k_n}$$

where $n = 0, 1, 2, \ldots$; $\alpha_i < \alpha$, $k_i = \pm 1$ for $i = 0, 1, \ldots, n$.

Observe that $\mathcal{F}_\alpha$ is of power $< 2^{\aleph_0}$. Let

$$S_\alpha = \{ f(y_\gamma) : \gamma \leq \alpha, \ f \in \mathcal{F}_\alpha \}.$$

Choose x_α as the first term of (1) which belongs to $P_\alpha \setminus S_\alpha$. Let

$$T_\alpha = \{ f(x_\gamma) : \gamma \leq \alpha, \ f \in \mathcal{F}_\alpha \}.$$

Choose y_α as the first term of (1) which belongs to $P_\alpha \setminus T_\alpha$. Put

$$B = \bigcup_{\alpha<\beta} T_\alpha, \quad Y = \{ y_\alpha : \alpha < \beta \}.$$

We shall prove that $B \cap Y \neq \emptyset$. Suppose that $z \in B \cap Y$. Since $z \in B$, there are $\alpha < \beta$, $\gamma \leq \alpha$, $f \in \mathcal{F}_\alpha$ such that

(2) $$z = f(x_\gamma).$$

Since $z \in Y$, there is $\xi < \beta$ such that

(3) $$z = y_\xi.$$

If $\gamma \leq \xi$, then, in virtue of (2) and the definition of T_ξ, we have $z \in T_\xi$. Since $y_\xi \notin T_\xi$, therefore $z \neq y_\xi$. This contradicts (3). If $\gamma > \xi$, then, by the definition of x_γ, we have $x_\gamma \notin S_\gamma$. But $f^{-1}(y_\xi) \in S_\gamma$, so $x_\gamma \neq f^{-1}(y_\xi)$. This contradicts (2) and (3). Thus $B \cap Y = \emptyset$ and since $x_\alpha \in P_\alpha \cap B$, $y_\alpha \in P_\alpha \cap Y$ for each $\alpha < \beta$, therefore $P_\alpha \cap B \neq \emptyset \neq P_\alpha \setminus B$ for each $\alpha<\beta$. Hence B is a Bernstein set. Now, let $f_\alpha, \alpha<\beta$, be an arbitrary function of $\mathcal{F}$. It is easy to verify that, for all $\gamma, \alpha \leq \gamma < \beta$, we have $f_\alpha(T_\gamma) = T_\gamma$. Consequently

$$B \triangle f(B) \subseteq \bigcup_{\gamma<\alpha} (T_\gamma \cup f_\alpha(T_\gamma))$$

Observe that $T_\gamma, f_\alpha(T_\gamma) \in \mathcal{K}$ for $\gamma<\alpha$. It is known that a union of $< 2^{\aleph_0}$ sets of power $< 2^{\aleph_0}$ is of power $< 2^{\aleph_0}$ (it is a consequence of the König theorem, see [3], pp. 198-199). Therefore the above inclusion implies $B \triangle f(B) \in \mathcal{K}$. The proof has been completed.

Remark. The following stronger assertion results from the proof: $B \triangle f(B)$ belongs to $\mathcal{K}$ for each f from the group generated by $\mathcal{F}$ and the operation of superposition.

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PEWNA KONSTRUKCJA TYPU BERNSTEINA I JEJ ZASTOSOWANIA

Uogólniono konstrukcję Sierpińskiego z 1932 r. podzbioru typu Bernsteina prostej rzeczywistej $\mathbb{R}$, który różni się od swojego obrazu w dowolnej translacji o zbiór mocy $< 2^{\aleph_0}$. Zamiast rodziny translacji rozważa się dowolną rodzinę mocy $\leq 2^{\aleph_0}$, złożoną z funkcji przekształcających wzajemnie jednoznacznie $\mathbb{R}$ na $\mathbb{R}$. Uzyskano kilka wniosków dotyczących σ-ideałów.