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INTRODUCING A TOPOLOGY BY AN OPERATION OF A SET–THEORETIC BOUNDARY

A definition of a topology by a boundary operation is considered.

Let X be an arbitrary set. It is known that in the set X one can define a topology uniquely when the closure operation is given ([1], p. 36).

In this paper, we shall give a way of introducing a topology in the set X by means of an operation $\mathrm{Fr} : 2^X \to 2^X$ satisfying the conditions:

(F1) $$\mathrm{Fr}(\emptyset) = \emptyset,$$

(F2) $$\mathrm{Fr}(A) = \mathrm{Fr}(X \setminus A),$$

(F3) $$\mathrm{Fr}(A \cup B) \subset \mathrm{Fr}(A) \cup \mathrm{Fr}(B),$$

(F4) $$\mathrm{Fr}(\mathrm{Fr}(A)) \subset \mathrm{Fr}(A),$$

(F5) $$A \subset B \Rightarrow A \cup \mathrm{Fr}(A) \subset B \cup \mathrm{Fr}(B).$$

Theorem 1. *Let X be an arbitrary set, and $\mathrm{Fr} : 2^X \to 2^X$ an operation satisfying conditions (F1)-(F5). Putting*

(1) $$\overline{A} = A \cup \mathrm{Fr}\, A,$$

we obtain the closure operation satisfying the conditions:

$$\text{(C1)} \qquad \overline{\emptyset} = \emptyset,$$

$$\text{(C2)} \qquad A \subset \overline{A},$$

$$\text{(C3)} \qquad \overline{A \cup B} = \overline{A} \cup \overline{B},$$

$$\text{(C4)} \qquad \overline{(\overline{A})} = \overline{A}.$$

In the topological space so obtained, the set-theoretic boundary of the set A is $\operatorname{Fr} A$.

Proof. Equality (C1) follows immediately from (1) and (F1), whereas (C2) - from (1). Making use of (F3), we get

$$\overline{A \cup B} = A \cup B \cup \operatorname{Fr}(A \cup B) \subset A \cup B \cup \operatorname{Fr}(A) \cup \operatorname{Fr}(B) = \overline{A} \cup \overline{B}.$$

On the other hand

$$\overline{A} \cup \overline{B} = A \cup \operatorname{Fr}(A) \cup B \cup \operatorname{Fr}(B) \subset A \cup B \cup \operatorname{Fr}(A \cup B) = \overline{A \cup B}$$

because $A \subset A \cup B$, $B \subset A \cup B$, thus one can use (F5). So, (C3) holds.

The inclusion $\overline{A} \subset \overline{(\overline{A})}$ follows directly from (C2), and $\overline{(\overline{A})} = \overline{(A \cup \operatorname{Fr}(A)} = A \cup \operatorname{Fr}(A) \cup \operatorname{Fr}(A \cup \operatorname{Fr}(A)) \subset A \cup \operatorname{Fr}(A) \cup \operatorname{Fr}(\operatorname{Fr}(A)) \subset A \cup \operatorname{Fr}(A) = \overline{A}$ on the basis of (1), (F3) and (F4).

Now, denote by $\operatorname{fr}(A)$ the boundary of the set A in the topological space X, i.e.

$$\operatorname{fr}(A) = \overline{A} \cap \overline{X \setminus A}.$$

So, for any $A \subset X$, we have

$$\begin{aligned} \operatorname{fr}(A) &= (A \cup \operatorname{Fr}(A)) \cap ((X \setminus A) \cup \operatorname{Fr}(X \setminus A)) \\ &= (A \cup \operatorname{Fr}(A)) \cap ((X \setminus A) \cup \operatorname{Fr}(A)) \\ &= (A \cap \operatorname{Fr}(A)) \cup (\operatorname{Fr}(A) \cap (X \setminus A)) \cup \operatorname{Fr}(A) \\ &= \operatorname{Fr}(A) \end{aligned}$$

Consequently, the operations Fr and fr are identical.

Conditions (F1) - (F5) are independent.

Example 1. Let X be an arbitrary nonempty set and $\mathrm{Fr}(A) = X$ for any $A \subset X$. The operation thus defined satisfies conditions (F2) - (F5) and, of course, does not satisfy (F1).

Example 2. In any nonempty set X let us put $\mathrm{Fr}(A) = A$ for each $A \subset X$. It is easy to verify that conditions (F1), (F3), (F4) and (F5) are satisfied, while (F2) is not.

Example 3. Let $X = \{0,1,2\}$. Define the operation Fr in the set X as follows :

$$\begin{aligned} \mathrm{Fr}(\emptyset) &= \mathrm{Fr}(X) = \emptyset, \\ \mathrm{Fr}(\{0\}) &= \mathrm{Fr}(\{1,2,\}) = \{0\}, \\ \mathrm{Fr}(\{1\}) &= \mathrm{Fr}(\{0,2\}) = \{1\}, \\ \mathrm{Fr}(\{2\}) &= \mathrm{Fr}(\{0,1\}) = \{2\}. \end{aligned}$$

The operation so defined satisfies conditions (F1), (F2), (F4) and (F5), and does not satisfy (F3), for we have

$$\mathrm{Fr}(\{0,1)\}) \not\subset \mathrm{Fr}(\{0\}) \cup \mathrm{Fr}(\{1\}).$$

Example 4. Again, let $X = \{0,1,2\}$. Put

$$\begin{aligned} \mathrm{Fr}(\emptyset) &= \mathrm{Fr}(X) = \emptyset, \\ \mathrm{Fr}(\{0\}) &= \mathrm{Fr}(\{1,2,\}) = \{0,1\}, \\ \mathrm{Fr}(\{1\}) &= \mathrm{Fr}(\{0,2\}) = \{0,1,2\}, \\ \mathrm{Fr}(\{2\}) &= \mathrm{Fr}(\{0,1\}) = \{1,2\}. \end{aligned}$$

The operation Fr thus defined satisfies conditions (F1) - (F3) and (F5), and does not satisfy (F4). Indeed,

$$\mathrm{Fr}(\{0\}) = \{0,1\} \text{ whereas } \mathrm{Fr}(\mathrm{Fr}(\{0\})) = \{1,2\}.$$

Example 5. In $X = \{0,1,2\}$, put

$$\begin{aligned} \mathrm{Fr}(\emptyset) &= \mathrm{Fr}(X) = \emptyset, \\ \mathrm{Fr}(\{0\}) &= \mathrm{Fr}(\{1,2,\}) = \{1,2\}, \\ \mathrm{Fr}(\{1\}) &= \mathrm{Fr}(\{0,2\}) = \{0,2\}, \\ \mathrm{Fr}(\{2\}) &= \mathrm{Fr}(\{0,1\}) = \{0,1\}. \end{aligned}$$

This operation satisfies conditions (F1) - (F4) and does not satisfy (F5) since $\{0\} \subset \{0,2\}$, but $\{0\} \cup \mathrm{Fr}(\{0\}) = X$ and

$$\{0,2\} \cup \mathrm{Fr}(\{0,2\}) = \{0,2\}.$$

Conditions (F1) - (F5) characterizing the boudaris of sets may be replaced by the equivalent system of three conditions. Namely, the following theorem holds :

Theorem 2. *Let X be an arbitrary set. Each operation $\mathrm{Fr} : 2^X \to 2^X$ satisfies conditions (F1) - (F5) if and only if it satisfies the system of conditions*

(f1) $$\mathrm{Fr}(\emptyset) = \emptyset,$$

(f2) $$\mathrm{Fr}(A \cap B) \subset A \cup \mathrm{Fr}(A),$$

(f3) $$\mathrm{Fr}(A \cup B) \cup \mathrm{Fr}(X \setminus A) \cup \mathrm{Fr}(\mathrm{Fr}(A)) \subset \mathrm{Fr}(A) \cup \mathrm{Fr}(B),$$

where $A, B \subset X$.

Proof. "$\Rightarrow$" is immediate.

It is easily seen that conditions (F1) and (F3) follow at once from (f1) and (f3). Putting $B = \emptyset$ in (f3) and making use of (f1), we get (F4). Substituting $B = A$ in (f3), we have

$$\mathrm{Fr}(A) \cup \mathrm{Fr}(X \setminus A) \cup \mathrm{Fr}(\mathrm{Fr}(A)) \subset \mathrm{Fr}(A),$$

thus

(2) $$\mathrm{Fr}(X \setminus A) \subset \mathrm{Fr}(A).$$

Similarly, replacing in (f3) the sets A and B by the set $X \setminus A$, we obtain

$$\mathrm{Fr}(X \setminus A) \cup \mathrm{Fr}(A) \cup \mathrm{Fr}(\mathrm{Fr}(X \setminus A)) \subset \mathrm{Fr}(X \setminus A)$$

and, consequently, $\mathrm{Fr}(A) \subset \mathrm{Fr}(X \setminus A)$, which, together with (2) gives (F2).

If $A \subset B$, then $A \cap B = A$ and, by (f2),

$$\mathrm{Fr}(A \cap B) = \mathrm{Fr}(A) \subset B \cup \mathrm{fr}(B).$$

Hence $A \cup \mathrm{Fr}(A) \subset B \cup \mathrm{Fr}(B)$ and, therefore, implication (F5) is true.

Examples 1, 5 and 3 allow us to find that each of conditions (f1), (f2), f3 is independent of the remaining ones.

References

[1] R. Engelking, *Topologia ogólna*, PWN, Warszawa, 1976.

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WPROWADZENIE TOPOLOGII PRZEZ OPERACJĘ BRZEGU

W pracy rozważa się topologię wprowadzoną przez operację brzegu.

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