A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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SOME GENERALIZATIONS OF RESULTS OF CARATHÉODORY AND MILLER

This paper concerns measurable functions. It is shown that in the Miller theorem from [2] the assumption about the set J can be weakened. Moreover, it turns out that the carrying over of the Caratheodory theorem from [1] to the case of functions and sets having the Baire property is impossible.

C. C a r a t h e o d o r y has shown in [1] that there exists a Lebesgue measurable function f, $f: \mathbb{R} \to \mathbb{R}$ such that for each non-empty open interval I and for each set J of positive Lebesgue measure the set $f^{-1}(J) \cap I$ has positive Lebesgue measure. Whereas H. J. M i l l e r has proved in [2] there exists a Lebesgue measurable function g, $g: \mathbb{R} \to \mathbb{R}$ such that for each non-empty open interval I and for each set J of the second category the set $f^{-1}(J) \cap I$ is also of the second category.

On account of above we can put the following questions:

1) if in Miller theorem the assumptions about the set J can be weakened;

2) if in Caratheodory theorem one can replace the measurability in Lebesgue sense by the property of Baire.

The following theorem gives the answer to the first question (we assume CH).

THEOREM 1. There exists a Lebesgue measurable function f, $f: \mathbb{R} \to \mathbb{R}$ such that for every non-empty interval I and for every $y \in \mathbb{R}$ the set $f^{-1}(\{y\}) \cap I$ is of the second category.

The proof of this theorem is based upon the following lemma.

LEMMA 1. There exists a family $\mathcal{C}$ of power $\mathfrak{c}$ ($\mathfrak{c}$ - the cardinal of the continuum) of pairwise disjoint sets and such that they are of the second category on every non-empty open interval and $\bigcup \mathcal{C} = \mathbb{R}$.

P r o o f. Let Ω be the smallest ordinal number of the power $\mathfrak{c}$. Let $\mathcal{A} = \{A;\ A$ is $\mathcal{F}_\sigma$ of the first category$\}$. We arrange the elements of the family $\mathcal{A}$ in the transfinite sequence of type Ω i.e. $\mathcal{A} = \{A_\alpha;\ \alpha < \Omega\}$. Let $\mathcal{D}$ be the family of all non-empty open interval $(a, b) \subset \mathbb{R}$, i.e. $\mathcal{D} = \{I_\delta;\ I_\delta$ - open interval, $\delta < \Omega\}$ and let $\mathcal{P}_\Omega$ be the set of all ordinal numbers $\alpha < \Omega$. Let us consider the set $\mathcal{A} \times \mathcal{D} \times \mathcal{P}_\Omega$ obviously card $(\mathcal{A} \times \mathcal{D} \times \mathcal{P}_\Omega) = \mathfrak{c}$. We arrange the elements of the set $\mathcal{A} \times \mathcal{D} \times \mathcal{P}_\Omega$ in the sequence of type Ω, i.e.

$$\mathcal{A} \times \mathcal{D} \times \mathcal{P}_\Omega = \{E_\beta;\ \beta < \Omega\},$$

where $E_\beta = (A_{\alpha(\beta)}, I_{\delta(\beta)}, \gamma(\beta))$ for some $\alpha(\beta), \delta(\beta), \gamma(\beta) < \Omega$.

Now, we inductively define a certain set of power $\mathfrak{c}$. Let x_0 be any element of the set $I_{\delta(0)} - A_{\alpha(0)}$. Suppose, that we have already chosen $\{x_\nu\}_{\nu<\varepsilon}$, where $\varepsilon < \Omega$ and we choose some $x_\varepsilon \in I_{\delta(\varepsilon)} - (\bigcup_{\nu\leq\varepsilon} A_{\alpha(\nu)} \cup \{x_\nu;\ \nu < \varepsilon\})$. Such element exists, because the set $I_{\delta(\varepsilon)} - (\bigcup_{\nu\leq\varepsilon} A_{\alpha(\nu)} \cup \{x_\nu;\ \nu < \varepsilon\})$ is of the second category. Hence, by the transfinite induction, we appoint the set $\{x_\varepsilon;\ \varepsilon < \Omega\}$.

Let $0 < \eta < \Omega$, $C_\eta = \{x_\varepsilon;\ \gamma(\varepsilon) = \eta\}$ and $C_0 = \mathbb{R} - \bigcup_{0<\eta<\Omega} C_\eta$. It is easy to see that $\{x_\varepsilon;\ \gamma(\varepsilon) = 0\} \subset C_0$. We shall prove that $\mathcal{C} = \{C_\eta;\ \eta < \Omega\}$ is a required family of sets.

All chosen elements x are different and if $x_{\varepsilon'} \in C_{\eta_1}$, $x_{\varepsilon'} \in C_{\eta_2}$ for some ε', η_1, η_2 then from the definition of the sets of type C we have that $\gamma(\varepsilon') = \eta_1$ and $\gamma(\varepsilon') = \eta_2$ from where it follows that $\eta_1 = \eta_2$. Hence the sets C_η are parwise disjoint. Now, we suppose that $C_\eta \cap (a, b)$ is a set of the first category for some $\eta < \Omega$ and for some interval $(a, b) = I_{\delta'}$, hence $C_\eta \cap (a, b)$ is contained in a certain set of the first category and of type $\mathcal{F}_\sigma$ contained in $(a, b) = I_{\delta'}$, i.e. there exists α' such that $C_\eta \cap (a, b) \subset A_{\alpha'}$. However it is impossible, because $(A_{\alpha'}, I_{\delta'}, \eta) \in \mathcal{A} \times \mathcal{B} \times \mathcal{P}_\Omega$ and it lies in the sequence $\{E_\beta\}$ on the spot, for instance, μ, i.e. $\alpha' = \alpha(\mu)$, $\delta' = \delta(\mu)$, $\eta = \gamma(\mu)$. The point x_μ was chosen from the set $I_{\delta(\mu)} - A_{\alpha(\mu)}$, in the other words $x_\mu \in C_\eta \cap I_\delta$, and $x \notin A_{\alpha(\mu)}$. Hence for any η and for any open interval I the set $C_\eta \cap I$ is of the second category.

P r o o f of Theorem 1. Let $\{C_\alpha\}_{\alpha<\Omega}$ be a family of pairwise disjoint sets and such that they are of the second category on every non-empty open interval and $\bigcup_{\alpha<\Omega} C_\alpha = \mathbb{R}$.

Let $\mathbb{R} = M \cup B$ where the set M has Lebesgue measure zero and the set B is of the first category, $M \cap B = \emptyset$ (see [3], p. 4). Let $h\colon \mathbb{R} \xrightarrow[1-1]{} \mathcal{P}_\Omega$. We define the function f

$$f(x) = \begin{cases} t & \text{for } x \in M \cap C_{h(t)}, \\ 0 & \text{for } x \in B. \end{cases}$$

It is easy to see that the function f is measurable in Lebesgue sense, because for every $a \in \mathbb{R}$ $f^{-1}((-\infty, a))$ is contained in the set M or contains the set B. Moreover, for any $y \in \mathbb{R}$

$$f^{-1}(\{y\}) = \begin{cases} M \cap C_{h(y)} & \text{for } y \neq 0, \\ B \cup C_{h(0)} & \text{for } y = 0. \end{cases}$$

If I is a non-empty open interval then it is not difficult to observe that for any $y \in \mathbb{R}$ the set $I \cap M \cap C_{h(y)}$ is of the second category, because $I \cap M \cap C_{h(y)} = [I \cap C_{h(y)}] - [C_{h(y)} \cap \cap (I - M)]$.

The answer to the secong question is negative. The following theorem is true.

THEOREM 2. For every function g having the property of Baire there exist a non-empty open interval I and a set J of the second category having the property of Baire such that the set $g^{-1}(J) \cap \cap I$ is of the first category.

P r o o f. Let g be an arbitrary function which has the property of Baire. There exists a residual set E such that $g|E$ is continuous. Let $x_o \in E$ and let $\varepsilon > 0$ be an arbitrary number.

Let us put $\hat{J} = [g(x_o) - \varepsilon, \ g(x_o) + \varepsilon]$. From the continuity of the function $g|E$ it follows that there exists $\delta > 0$ such that $g([x_o - \delta, \ x_o + \delta] \cap E) \subset \hat{J}$. Let $I = (x_o - \delta, \ x_o + \delta)$ and let $J \subset \mathbb{R}$ be a set of the second category with the Baire property disjoint with $\hat{J}$. It is easy to see that $g^{-1}(J) \cap I \subset \mathbb{R} - E$, hence $g^{-1}(J) \cap I$ is of the first category.

We can prove this theorem in a little more general form.

Let (X, S) be a measurable space, $\mathcal{T} \subset S$, $\mathcal{T}$ - σ-ideal such that $S = \mathcal{G} \vartriangle \mathcal{T}$, $\mathcal{G} \cap \mathcal{T} = \emptyset$ where X is a topological space and $\mathcal{G}$ is the family of open sets in X and S is a σ-algebra. Let Y be a topological space with countable basis.

LEMMA 2. Let $f\colon X \to Y$. The function f is S-measurable if and only if there exists a set $P \in \mathcal{T}$ such that $f|(X - P)$ is continuous.

The proof of this lemma is similar to the proof of suitable property of Baire functions (see for example [3]).

Let us assume that X is the Hausdorff space with countable basis. Then the following theorem is true.

THEOREM 3. For every S-measurable function $f\colon X \to X$ there exist an open set $H \subset X$ and a set $J \in S - \mathcal{T}$ such that $f^{-1}(J) \cap \cap H \in \mathcal{T}$.

P r o o f. Let f be an arbitrary S-measurable function. There exists a set $P \in \mathcal{T}$ such that $f|(X - P)$ is continuous. Take $x_o \notin \notin P$ and let $V \subset X$ be a neighbourhood of $f(x_o)$. From the continuity of the function $f|(X - P)$ it follows that there exists a neighbourhood H of x_o such that $f(H - P) \subset V$. Let $J \subset X, J \in S - \mathcal{T}$ be a set disjoint with $\bar{V}$. It is easy to see that $f^{-1}(J) \cap H \subset P$, it means that $f^{-1}(J) \cap H \in \mathcal{T}$.

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PEWNE UOGÓLNIENIA DOTYCZĄCE WYNIKÓW CARATHEODORY´EGO I MILLERA

Prezentowany artykuł dotyczy funkcji mierzalnych. Pokazano w nim, że w twierdzeniu Millera z pracy [2] można istotnie osłabić założenie dotyczące zbioru J. Ponadto okazało się, że przeniesienie twierdzenia Caratheodory´ego z pracy [1] na przypadek funkcji i zbiorów mających własność Baire´a jest niemożliwe.