A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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# ON THE SYMMETRIC CONTINUITY

S. M a r c u s proved in [4] that, for any set $E \in G_\delta$, there exists a function $f: R \to R$ for which $SC_f = E$ where $SC_f$ denotes the set of all points of symmetric continuity of the function f. Next, C. L. B e l n a in [1] that, for any function $f: R \to R$, the set $SC_f \cap D_f$ is of interior measure zero, where $D_f$ denotes the set of points of discontinuity of the function f.

In the present paper, some necessary conditions (Theorem 2) and sufficient ones (Theorem 3) are given in order that a given set be the set of points of symmetric continuity for some function $f: R \to R$. Moreover, from Theorem 4 and our example it follows that there exists a set which is not the set of all points of symmetric continuity for any function $f: R \to R$. The example is, at the same time, an example of a function $f: R \to R$ for which $SC_f$ is a non-measurable set. The existence of such a function was proved, with the continuum hypothesis applied, by P. E r d ö s in [2].

DEFINITION 1. The symmetric oscillation of a function at a point is given by

$$S_{osc}\, f(x_o) = \overline{\lim_{h\to 0}}\, |f(x_o + h) - f(x_o - h)|.$$

The following theorem is self-evident:

THEOREM 1. If a set E is such that $E = SC_f$ for some $f: R \to R$ then $E = \bigcap_{p\in N} E_p$ where $E_p = \{x \in R: S_{osc}\, f(x) < \frac{1}{p}\}$.

DEFINITION 2. We say that a set A is a weak section of symmetry if there exists a decreasing sequence of sets $\{A_p\}_{p\in N}$ such that

(i) $A = \bigcap_{p\in N} A_p$,

(ii) $(x_o \in A_p) \Longrightarrow \{\exists\delta > 0 \ \ \forall |h| \in (0, \delta) \ \ \forall r < p \ \exists\theta \in \{0,1\}$ $[((x_o + h) \in A_{r+\theta}) \Longleftrightarrow ((x_o - h) \in A_{r+\theta})]\}$.

DEFINITION 3. If there exists a decreasing sequence of sets $\{A_p\}_{p\in N}$ such that

(i) $A = \bigcap_{p\in N} A_p$,

(ii) $(x_o \in A_p) \Longrightarrow \{\exists\delta > 0 \ \ \forall |h| \in (0, \delta) \ \ \forall r < p \ [((x_o + h) \in A_r) \Longleftrightarrow ((x_o - h) \in A_r)]\}$,

then the set A is said to be a section of symmetry.

THEOREM 2. If $f: R \to R$, then the set $E = SC_f$ is a weak section of symmetry.

P r o o f. Using theorem 1, it is enough to prove that the sequence of sets $\{E_k\}_{k\in N}$ where $E_k = \{x \in R: S_{osc} f(x) < \frac{1}{9^k}\}$ has property (ii) from Definition 2, since the monotonicity of the sequence $\{E_k\}_{k\in N}$ and property (i) are obvious. From the monotonicity of the sequence $\{E_k\}$ and the negation of condition (ii) we have that there exist a point $x_o$ and a number $j \in N$ such that $x_o \in E_j$ and, for any number $\delta > 0$, there exists h, $|h| \in (0, \delta)$, and an index $k < j$ such that

$$x_o + h \in E_{k+1} \wedge x_o - h \notin E_k \tag{1}$$

From (1) and Definition 1 and the way the sets $E_k$ are defined we have

$$\lim_{|t|\to 0} \sup |f(x_o + h + t) - f(x_o + h - t)| < \frac{1}{9^{k+1}} \tag{2}$$

$$\lim_{|t|\to 0} \sup |f(x_o - h + t) - f(x_o - h - t)| \geq \frac{1}{9^k}. \tag{3}$$

Since $x_o \in E_j$, we have

$$\lim_{|u|\to 0} \sup |f(x_o + u) - f(x_o - u)| < \frac{1}{9^j} \tag{4}$$

So, there exists $\delta_1 > 0$ such that

$$|f(x_o + u) - f(x_o - u)| < \frac{1}{9^j} \quad \text{for} \quad u \in (0, \delta_1) \tag{5}$$

Let now $\delta_2 = \frac{\delta_1}{2}$. Then there exists $h \in (0, \delta_2)$ such that, for k, conditions (2) and (3) are satisfied. So, there exists $t_o$ such that $|t_o| \in (0, \delta_2)$ and

$$|f(x_o + h + t_o) - f(x_o + h - t_o)| < \frac{1}{9^{k+1}} \tag{6}$$

$$|f(x_o - h + t_o) - f(x_o - h - t_o)| > \frac{8}{9} \cdot \frac{1}{9^k} \tag{7}$$

Note that $|h + t_o| \in (0, \delta_1)$ and $|h - t_o| \in (0, \delta_1)$. Consequently, from (5) we have

$$|f(x_o + h + t_o) - f(x_o - h - t_o)| < \frac{1}{9^j} \tag{8}$$

and

$$|f(x_o + h - t_o) - f(x_o - h + t_o)| < \frac{1}{9^j} \tag{9}$$

From conditions (6), (7), (8) and (9) we get

$$\begin{aligned} \frac{8}{9} \cdot \frac{1}{9^k} &< |f(x_o - h + t_o) - f(x_o - h - t_o)| = \\ &= |f(x_o - h + t_o) - f(x_o + h - t_o) + \\ &+ (f(x_o + h - t_o) - f(x_o + h + t_o)) + \\ &+ (f(x_o + h + t_o) - f(x_o - h - t_o))| < \\ &< \frac{1}{9^j} + \frac{1}{9^{k+1}} + \frac{1}{9^j} \end{aligned} \tag{10}$$

Thus we have

$$\frac{8}{9} \cdot \frac{1}{9^k} < \frac{1}{9^j} + \frac{1}{9^{k+1}} \tag{11}$$

which is impossible because of the fact that $k < j$.

Consequently, the sequence $\{E_k\}_{k \in N}$ satisfies condition (ii) from Definition 2. This ends the proof of the theorem.

If a set $E \subset R$ is a section of symmetry, let us denote

$$\hat{E} = \{x \in R:\ \forall p \in N\ \ \exists \delta_x > 0\ \ \forall |h| \in (0, \delta_x)\ \ \forall r < p$$
$$[((x + h) \in E_r) \Leftrightarrow ((x - h) \in E_r)]\}.$$

REMARK. If $E$ is a dense set and a section of symmetry, then the interior measure of the set $\hat{E} \setminus E$ is equal to zero.

P r o o f. Let the sequence $\{E_p\}_{p\in N}$ satisfy the conditions of Definition 3. Put

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in E \\ \frac{1}{p} & \text{for} \quad x \in E_p \setminus E_{p+1}. \end{cases}$$

Then $\hat{E} = SC_f$ (see the proof of Theorem 3) and $R \setminus E \subset D_f$ where $D_f$ denotes the set of all points of discontinuity of the function f.

Making use of the result of C. L. B e l n a in [1] stating that the set $SC_f \cap D_f$ has the interior measure equal to zero and from the fact that $\hat{E} \setminus E = \hat{E} \cap (R \setminus E) \subset \hat{E} \cap D_f$, we obtain that the set $\hat{E} \setminus E$ has the interior measure equal to zero.

THEOREM 3. If $E$ is a section of symmetry and the set $\hat{E} \setminus E \in F_\sigma$, then there exists a function $f: R \to R$ such that $E = SC_f$.

P r o o f. Let the sequence $\{E_p\}_{p\in N}$ satisfy the conditions of Definition 3 and $E_1 = R$. Then

$$R = E \cup \bigcup_{p\in N} (E_p \setminus E_{p+1}), \quad (E_p \setminus E_{p+1}) \cap (E_s \setminus E_{s+1}) = \emptyset$$

for $s \neq p$.

Define the function

$$\phi(x) = \begin{cases} 0 & \text{for} \quad x \in E, \\ \frac{1}{p} & \text{for} \quad x \in E_p \setminus E_{p+1}. \end{cases}$$

If $x_o \in \hat{E}$, then, for any $\varepsilon > 0$ and any number $p_o \in N$ such that $\frac{1}{p_o} < \varepsilon$, there exists $\delta > 0$ such that, for each $h$ such that $|h| \in (0, \delta)$, there is

$$|\phi(x_o + h) - \phi(x_o - h)| < \varepsilon,$$

whence we get

$$\hat{E} \subset SC_{\phi} \tag{12}$$

If now $x_o \in R \setminus \hat{E}$, then $\exists p_o \in N \quad \forall \delta > 0 \quad \exists |h| \in (0, \delta) \vee r < p_o$ $[(x_o + h) \in E_r \wedge (x_o - h) \notin E_r]$. Consequently, we have

$$|\phi(x_o + h) - \phi(x_o - h)| = |\frac{1}{k} - \frac{1}{s}|, \quad s < r < p_o, \quad k \geqq r.$$

Then we obtain

$$|\phi(x_o + h) - \phi(x_o - h)| = \frac{1}{s} - \frac{1}{k} \geqq \frac{1}{s} - \frac{1}{s+1} \geqq \frac{1}{p_o(p_o + 1)}.$$

Hence we infer that

$$x_o \notin SC_{\phi} \tag{13}$$

From (13) and (12) we have

$$\hat{E} = SC_{\phi} \tag{14}$$

The set $H = SC_{\phi} \setminus E \in F_{\sigma}$. Then

$$R \setminus H \in G_{\delta}.$$

From the theorem in paper [4] by S. M a r c u s it follows that there exists a function $\psi: R \to R$ such that

$$R \setminus H = SC_{\psi}.$$

Put $f = \phi + \psi$. If $x_o \in E$, then $x_o \in SC_{\phi} \wedge x_o \in SC_{\psi}$, and so, $x_o \in SC_f$. That is, $E \subset SC_f$. Whereas if $x_o \in R \setminus E$, then $x_o \in$ $\in H \vee x_o \notin SC_{\phi}$. If $x_o \in H$, then $x_o \notin SC_{\psi}$, and since $x_o \in SC_{\phi}$, therefore $x_o \notin SC_f$. Whereas if $x_o \notin SC_{\phi}$, then $x_o \in SC_{\psi}$, thus also $x_o \notin SC_f$. Consequently, we have proved that $E = SC_f$, which completes the proof of the theorem.

E x a m p l e. There exist a non-measurable set E and a function $f: R \to R$, such that $E = SC_f$.

Let H be a (Hamel) basis for the space R over the field of rational numbers, such that $1 \in H$. Every real number $x$ has a unique representation of the form

$$x = \sum_{h \in H} x_h \cdot h \tag{15}$$

where $x_h \neq 0$ only for a finite number of coefficients $h \in H$,

$x_h \in Q$. Let $E = \{x \in R: x_1 = 0\}$. From papers [3] and [5] it follows that E is a dense set with empty interior in R and that it is a non-measurable linear subspace of the space R over the field Q. We consider the characteristic function of the set E:

$$f(x) = \begin{cases} 1 & \text{for } x \in E, \\ 0 & \text{for } x \notin E. \end{cases}$$

We now prove that $E = SC_f$. Let $x_o \in E$. Then from the assumption that E is a linear space we have

$$f(x_o + h_n) - f(x_o - h_n) = 0 \tag{16}$$

for any sequence $\{h_n\}_{n\in N}$ converging to zero. It follows from (16) that

$$\lim_{n\to\infty} (f(x_o + h_n) - f(x_o - h_n)) = 0.$$

Thus

$$E \subset SC_f. \tag{17}$$

Now, let $x_o \notin E$. Since $\bar{E} = R$, there exists a sequence $\{x_n\}_{n\in N}$ such that $x_n \in E$ for each $n \in N$ and such that $\lim_{n\to\infty} x_n = x_o$. Let $h_n = x_n - x_o$. Then $x_o + h_n = x_n \in E$, while $x_o - h_n = 2x_o - x_n \notin E$. Otherwise, if $2x_o - x_n \in E$, then $(2x_o - x_n) + x_n = 2x_o \in E$, so that $x_o \in E$, and this contradicts the choice of the point $x_o$. Therefore we have shown that there exists a sequence of real numbers converging to zero, such that

$$f(x_o + h_n) - f(x_o - h_n) = 0 + 1 = 1 \tag{18}$$

for any n, which means that $x_o \notin SC_f$. From this and from (17) we have $E = SC_f$.

THEOREM 4. If the set $G \subset R$ is a linear space over the field Q of the second Baire category in R, and $\overline{R \setminus G} = R$, then the set $G' = R \setminus G$ is not a weak section of symmetry.

P r o o f. Let us assume that $G' = R \setminus G$ is a weak section of symmetry. Then there exists a monotone decreasing sequence of sets $\{G_p\}_{p\in N}$ fulfilling the conditions

(a) $G' = \bigcap_{p\in N} G_p$.

(b) $(x_o \in G') \Longrightarrow \{\forall p \in N\ \exists \delta > 0\ \ \forall h \in (0, \delta)\ \ \exists \Theta \in \{0, 1\}$ $[((x_o - h) \in G_{p+\Theta} \wedge (x_o + h) \in G_{p+\Theta}) \vee ((x_o - h) \notin G_{p+\Theta} \wedge (x_o + h) \notin G_{p+\Theta})]\}$.

Let

$$G_p = H_p \cup G' \tag{19}$$

and

$$R_p = G \setminus H_p, \qquad p = 1, 2, \ldots \tag{20}$$

Using (20), we have

$$G \supset \bigcup_{p \in N} R_p \tag{21}$$

Now, let $x \in G$. Then from (20), for any $p \in N$, $x \in H_p$ or $x \in R_p$. If, for each $p \in N$, $x \in H_p$, then, by (19), $x \in G_p$ for any $p \in N$. Hence from (a) we have that $x \in G$, which contradicts the choice of $x$. Thus there exists $p \in N$ such that $x \in R_p$. Therefore $x \in \bigcup_{p \in N} R_p$. Thus we have obtained that $\bigcup_{p \in N} R_p \supset G$, which, together with (21), gives

$$G = \bigcup_{p \in N} R_p \tag{22}$$

Since $G$ is of the second category, thus it follows from (22) that there exists $p_o \in N$ such that $R_{p_o}$ is of the second category in $R$. So, there exists an interval $(a, b)$ for which $(a\ b) \subset \overline{R}_{p_o}$. Now, let

$$x_o \in (a, b) \cap G' \tag{23}$$

Then there exists a sequence of points $\{w_n\}_{n \in N}$, $\lim_{n \to \infty} w_n = x_o$, $w_n > x_o$ and $w_n \in R_{p_o}$ for $n \in N$. From (20) we have that $w_n \in G$ for $n = 1, 2, \ldots$ Thus $w_n = (x_o + h_o) \notin (G' \cup H_{p_o}) = G_{p_o}$ for $n = 1, 2, \ldots$ From condition (b) it follows that for sufficiently large $n > n_o$, we have $(x_o - h_n) = (2x_o - w_n) \notin G_{p_o+1}$. Because of (a), we have that, for $n > n_o$, $(2x_o - w_n) \notin G'$ and hence

$$(2x_o - w_n) \in G. \quad (24)$$

Since $w_n \in G$, G is a linear space over the field Q and, because of (24), we have

$$x_o = \frac{1}{2}\left[(2x_o - w_n) + w_n\right] \in G. \quad (25)$$

Condition (25) contradicts (23). This contradiction completes the proof of the theorem.

Theorems 2 and 3 give a partial characterization of the set $SC_f$ for a function f: R → R. Our example shows that the set $SC_f$ may even be non-measurable. Moreover, let us notice that the set E from the example is a linear space over the field Q of rational numbers, fulfilling the hypothesis of Theorem 4. Thus R \ E is not the set of points of symmetry continuity for any real function of a real variable f.

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O ZBIORZE SYMETRYCZNEJ CIĄGŁOŚCI

W artykule podane są pewne warunki konieczne oraz pewne warunki dostateczne na to, by zbiór był zbiorem wszystkich punktów symetrycznej ciągłości funkcji $f: R \to R$. Ponadto dowodzi się, że istnieją zbiory nie będące zbiorami punktów symetrycznej ciągłości dla żadnej funkcji $f: R \to R$.