ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 216, 2008

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BAYESIAN ESTIMATION OF BONUS MALUS COEFFICIENTS IN CR AUTOMOBILE LIABILITY INSURANCE

ABSTRACT. The basis of insurance activity is proper premium estimation. The gross premium is the net premium enlarged by a security loading and costs of insurance activity. In the paper individual net premium is calculated by means of three methods: the expected value method, the variance method and the zero utility method. Subsequently, by means of Bayesian estimators, the bonus-malus coefficients for the premiums calculated by the three methods mentioned above were estimated and compared. The research was performed for different parameters of the number of damages distribution.
Key words: bonus-malus system, automobile liability insurance, Bayes estimators.

I. INTRODUCTION

In CR automobile insurance the classification of insured to the tariff classes is done on the basis of prior factors (observable risk factors such as, for example, car type and production year, motor capacity, driver's sex and age) and posterior factors (driver's damage history). That is why the CR premium is calculated in two stages. The first stage is to calculate the basic premium on the basis of prior factors, the second stage is the posterior tariffication (see Lemaire 1995).

The paper is focused on the second stage called the bonus-malus system. The term bonus-malus refers to the methods of determining individual premiums taking into account a driver's number of damages from the past.

Every bonus-malus system must have an established starting class to which all insured with clear damage history will be assigned, the vector of basic premium and the principles of classes changing.

Annual net premium is determined as the product of the basic premium for a given tariff class (prior tariffication) and the coefficient constituting the estimated percentage rate of the premium.

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In the paper, additional rises and reductions characteristic for particular insurers, are not considered.

In wealth insurance the gross premium is calculated as the sum of three components: the net premium, security loading and insurance activity costs. Let us skip the third component. In this way the gross premium is the net premium plus the security loading.

In automobile liability insurance we assume that the number of damages in homogenous portfolio is a random variable following the Poisson distribution i. e.

$$P(K=k)=e^{-\lambda}\frac{\lambda^{k}}{k!}, \quad (k=0, 1, 2, ...), \tag{1}$$

where λ is the parameter of damage rate.

If the portfolio is not homogenous (compare Hossack et al.1999) then the parameter Λ of damage rate is a random variable with parameters α and β and the density function

$$f(\lambda)=\frac{\beta^{\alpha}e^{-\beta\lambda}\lambda^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha>0, \beta>0, \lambda>0. \tag{2}$$

Then the number of damages in the portfolio follows the negative binominal distribution of the form

$$P(K=k)=\binom{q+k-1}{k}p^{q}(1-p)^{k}, \quad (k=0, 1, 2, ...) \tag{3}$$

with parameters p and q, where

$$q=\alpha \quad \text{and} \quad p=\beta/(1+\beta). \tag{4}$$

The estimators of parameters α and β derived with the moments of method (see Domański et al. 2000) have the form

$$\beta=\frac{\bar{k}}{S_{k}^{2}-\bar{k}}, \quad \alpha=\frac{\bar{k}^{2}}{S_{k}^{2}-\bar{k}}, \tag{5}$$

where $\bar{k}$ is the mean number of claims in the portfolio, S_k^2 stands for the variance of the number of damages.

In the bonus-malus insurance system the prior premium is determined and then one takes into account the individual risk parameter called the premium rate (see Lemaire 1995). In the paper a method using Bayesian estimators was applied to determine the individual risk parameters (see Domański et al. 2000).

Let K_j be random variable denoting the number of damages in year j for a given policy; $(k_1, k_2,\ldots,k_t)$ the vector of observations of numbers of damages for t years for a given policy; $\lambda_{t+1}(k_1,k_2,...,k_t)$ unknown claim parameter in year t+1 for the policy described by the vector of observations $(k_1, k_2,\ldots,k_t)$.

The unknown parameter $\lambda_{t+1}(k_1,k_2,...,k_t)$ may be estimated by means of the Bayesian estimator from the vector of observations $(k_1, k_2,\ldots,k_t)$.

Let as assume that the distribution of the number of damages in the portfolio is negative binominal. The parameter Λ of damage rate follows the prior gamma distributions with parameters α and β.

Thus, the posterior distribution of parameter Λ is the gamma distribution with parameters $\hat{\alpha} = \alpha + k$ and $\hat{\beta} = \beta + t$.

The Bayesian estimator of parameter Λ has the form

$$\lambda_{t+1}(k_1,...,k_t) = \frac{\hat{\alpha}}{\hat{\beta}} = \frac{\alpha + k}{\beta + t}. \tag{6}$$

The parameters α and β are determined from formula (5).

In CR automobile liability insurance the individual net premium in period t+1 is equal to

$$P_{t+1}(k_1,...,k_t) = (EX)\cdot(E\Lambda)\cdot b_{t+1}(k_1,...,k_t), \tag{7}$$

where $P_{t+1}(k_1,...,k_t)$ is the individual net premium in period t+1, (EX) is the expectation of single damage, $(E\Lambda)$ is the expected number of damages, $b_{t+1}(k_1,...,k_t)$ is the rate of the premium estimated.

Let us assume that (EX)=1 and $(E\Lambda) = \frac{\alpha}{\beta}$. Then, the equation (7) has the form

$$P_{t+1}(k_1,...,k_t) = \frac{\alpha}{\beta}\cdot b_{t+1}(k_1,...,k_t). \tag{8}$$

Hence, a driver who after t years reported k damages, should pay the rate of the premium estimated equal to

$$b_{t+1}(k_1,...,k_t)=\frac{\beta}{\alpha}P_{t+1}(k_1,...,k_t)\cdot 100\%. \tag{9}$$

II. THE EXPECTED VALUE PRINCIPLE

The simplest rule of the premium calculation in automobile liability insurance is the expected value principle (cf. Hossack et al.1999). According to this rule, the estimated individual net premium enlarged by the security loading θ is equal to

$$P_{t+1}(k_1,...,k_t)=(1+\theta)\lambda_{t+1}(k_1,...,k_t)=(1+\theta)\frac{\alpha+k}{\beta+t}. \tag{10}$$

From formula (9) and (10) it follows that a driver, who after t years reported k damages in the year t+1 should pay the premium equal to

$$b_{t+1}(k_1,...,k_t)=(1+\theta)\frac{\beta(\alpha+k)}{\alpha(\beta+t)}\cdot 100\% \tag{11}$$

III. THE VARIANCE PRINCIPLE

According to this rule the estimated individual net premium enlarged by the security loading θ is equal to

$$\begin{aligned}P_{t+1}(k_1,...,k_t)&=(1+\theta)E_\lambda[\lambda|k_1,...k_t]+\theta Var_\lambda[\lambda|k_1,...k_t]=\\&=(1+\theta)\frac{\alpha+k}{\beta+t}+\theta\frac{\alpha+k}{(\beta+t)^2}\end{aligned} \tag{12}$$

the estimated rate of the net premium for a driver who after t years reported k damages is equal to

$$b_{t+1}(k_1,...,k_t)=\frac{\beta}{\alpha}\left[(1+\theta)\frac{\alpha+k}{\beta+t}+\theta\frac{\alpha+k}{(\beta+t)^2}\right]. \tag{13}$$

IV. THE ZERO UTILITY PRINCIPLE

Let us apply the zero utility rule to the estimation of the individual net premium.

The zero utility rule is based on the assumption that the expected income of the insurer, when risk X is insured for price P, is equal to the utility of the starting insurer's reserve w i.e.

$$u(w-P)=Eu(w-X). \tag{14}$$

Let function $u(w)$ be the expected utility function of the form

$$u(w)=\frac{1}{c}\left(1-e^{-cw}\right), \tag{15}$$

where $c>0$ is a risk aversion defining parameter. If the utility function is given by formula (15) the net premium P is equal to

$$P=\frac{\alpha}{c}\left|\ln\left(1-\frac{e^{c}-1}{\beta}\right)\right| \quad dla \quad \beta>e^{c}-1. \tag{16}$$

Taking into account that $\hat{\alpha}=\alpha+k$ and $\hat{\beta}=\beta+t$, the individual net premium estimated according to the utility rule is equal to

$$P_{t+1}(k_1,\ldots,k_t)=\frac{\alpha+k}{c}\left|\ln\left(1-\frac{e^{c}-1}{t+\beta}\right)\right|. \tag{17}$$

As the purpose of the investigation is the estimation of the percentage of the basic premium a driver who after t years reported k damages should pay, from equations (9) and (17) it follows that the estimated premium rate in bonus-malus system is equal to

$$b_{t+1}(k_1,\ldots,k_t)=\frac{\beta}{\alpha}\frac{\alpha+k}{c}\left|\ln\left(1-\frac{e^{c}-1}{t+\beta}\right)\right|\cdot 100\%. \tag{18}$$

V. APPLICATIONS

In the paper the impact of the method of estimating the individual net premium on the estimated bonus-malus coefficients is investigated. The influence of the parameters of the number of damages distribution on the estimation of premium rate was also assessed.

Table 1

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution ($\bar{k}$ =0,8, S_k^2=0,86, p=0,93, q=10,67, θ=0,25, c=0,25)

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	93	93	93	102	102	102	110	111	111	119	119	119
2	87	87	87	95	95	95	103	103	103	111	111	112
3	82	82	82	89	89	89	97	97	97	105	105	105
4	77	77	77	84	84	84	91	91	92	99	98	99
5	73	73	73	80	79	80	86	86	87	93	93	93

Source: own investigations.

Table 2

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution ($\bar{k}$ =1,4, S_k^2=1,51, p=0,93, q=17,82, θ=0,25, c=0,25)

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	93	93	93	98	98	98	103	103	103	108	109	108
2	86	87	87	91	91	91	96	96	96	101	101	101
3	81	81	81	85	85	86	90	90	90	95	95	95
4	76	76	76	80	80	81	85	85	85	89	89	89
5	72	72	72	76	76	76	80	80	80	84	84	84

Source: own investigations.

Table 3

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution

$(\bar{k}=2, S_k^2=2{,}16, p=0{,}93, q=25, \theta=0{,}25, c=0{,}25)$

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	93	93	93	96	97	96	100	100	100	104	104	104
2	86	86	86	90	90	90	93	93	93	97	97	97
3	81	81	81	84	84	84	87	87	87	90	90	91
4	76	76	76	79	79	79	82	82	82	85	85	85
5	71	71	72	74	74	75	77	77	77	80	80	80

Source: own investigations.

Table 4

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution

$(\bar{k}=0{,}8, S_k^2=1{,}24, p=0{,}65, q=1{,}45, \theta=0{,}25, c=0{,}25)$

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	65	62	66	109	105	112	154	148	157	198	191	203
2	48	45	49	81	76	84	113	107	118	146	138	152
3	38	35	39	64	60	67	90	84	94	116	109	121
4	31	29	33	53	49	56	74	69	78	96	89	101
5	27	25	28	45	42	48	63	59	67	82	76	86

Source: own investigations.

Table 5

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution

$(\bar{k}=1{,}4, S_k^2=2{,}18, p=0{,}65, q=2{,}51, \theta=0{,}25, c=0{,}25)$

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	64	62	66	90	87	92	115	111	118	141	136	144
2	47	45	49	66	63	69	85	80	88	104	98	108
3	37	35	39	52	49	55	67	63	70	82	77	86
4	31	29	32	43	40	45	56	52	58	68	63	71
5	26	24	28	37	34	39	47	44	50	58	54	61

Source: own investigations.

Table 6

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution

($\bar{k}$ =2, S_k^2 =3,14, p=0,65, q=3,51, θ=0,25, c=0,25)

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	64	61	65	82	79	84	100	97	102	118	114	121
2	47	44	48	60	57	62	73	70	76	87	82	90
3	37	35	38	47	44	49	58	54	60	68	64	71
4	30	28	32	39	36	41	48	45	50	57	53	59
5	26	24	27	33	31	35	41	38	43	48	45	51

Source: own investigations.

Table 7

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution

($\bar{k}$ =0,8, S_k^2 =1,78, p=0,45, q=0,65, θ=0,25, c=0,25)

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	45	40	49	114	102	124	184	164	199	253	226	274
2	29	25	32	74	64	82	119	102	132	163	141	181
3	21	18	24	54	46	61	88	74	98	121	102	135
4	17	14	19	43	36	49	69	58	78	96	80	108
5	14	12	16	36	30	41	57	48	65	79	66	90

Source: own investigations.

Table 8

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution

($\bar{k}$ =1,4, S_k^2 =3,1, p=0,45, q=1,15, θ=0,25, c=0,25)

t \ k	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	45	40	49	84	75	91	123	110	134	163	145	176
2	29	25	32	54	47	60	80	69	88	105	90	116
3	21	18	24	40	34	45	59	50	66	77	66	87
4	17	14	19	32	27	36	47	39	53	61	51	69
5	14	12	16	26	22	30	39	32	44	51	42	58

Source: own investigations.

Table 9

The rate of estimated premium determined with respect to three rules(*wo*- the expected value rule, *w* – the variance rule, *zu* – the zero utility rule). The parameters of the damage number distribution ($\bar{k}$ =2, S_k^2 =4,4, p=0,45, q=1,67, θ=0,25, c=0,25)

t \ *k*	0			1			2			3 and more		
	wo	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*	*wo*	*w*	*zu*
0	100	100	100									
1	45	40	49	73	65	78	100	89	108	127	113	137
2	29	25	33	47	40	52	64	55	72	82	71	91
3	22	18	24	35	29	39	48	40	53	61	51	68
4	17	14	19	27	23	31	38	32	43	48	40	54
5	14	12	16	23	19	26	31	26	36	40	33	45

Source: own investigations.

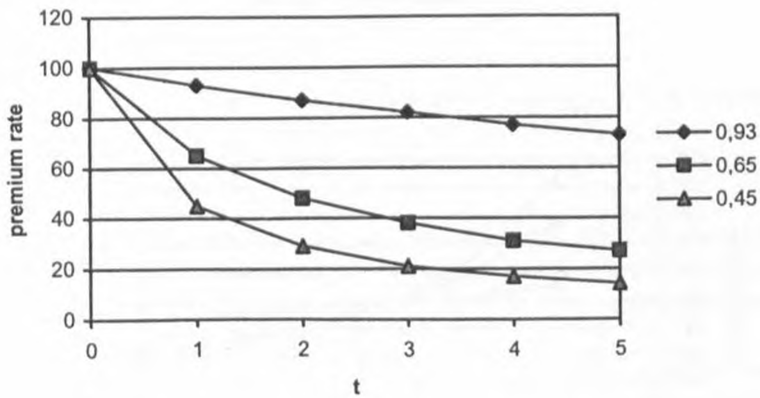

Figure 1. The comparison of premiums estimated by means of the expected value rule for k=0 on the basis of the data from tables 1, 4, 7.
Source: own investigations.

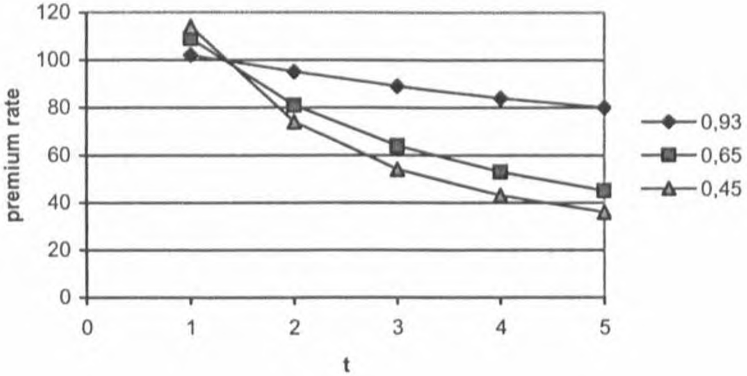

Figure 2. The comparison of premiums estimated by means of the expected value rule for $k=1$ on the basis of the data from tables 1,4,7.
Source: own investigations.

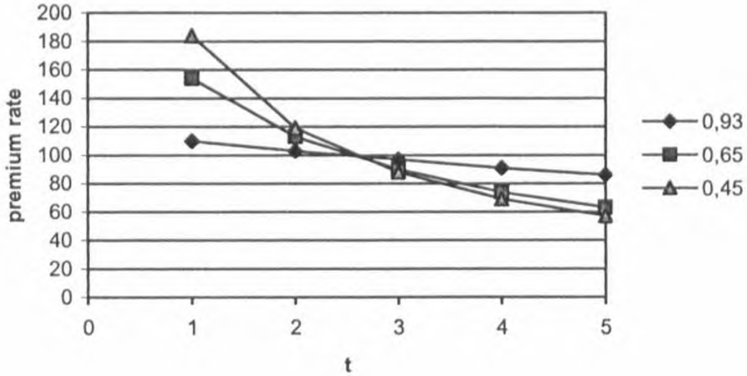

Figure 3. The comparison of premiums estimated by means of the expected value rule for $k=2$ on the basis of the data from tables 1,4,7.
Source: own investigations.

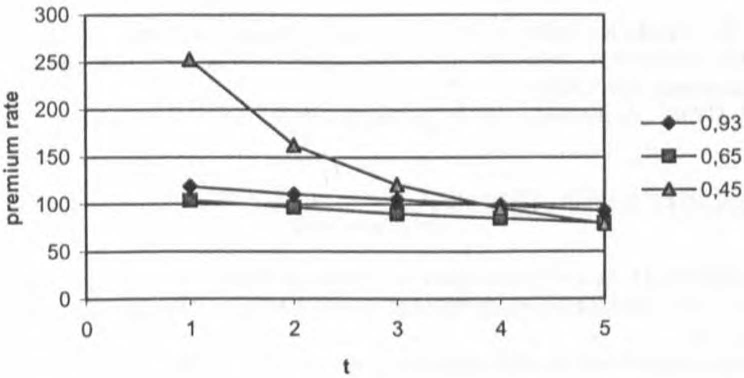

Figure 4. The comparison of premiums estimated by means of the expected value rule for $k \geq 3$ on the basis of the data from tables 1,4,7.
Source: own investigations.

The research carried out proves that for small dispersion (with respect to the mean) of the portfolio damage number the methods of expected value, variance and zero utility do not differ significantly as far as the estimated net premium is concerned.

The higher the dispersion of the portfolio damage number, the higher the differences between the premium rates estimated with the three above mentioned methods. The estimated premium rates are highest for the expected value methods and lowest for the variance method.

Figures 1-4 depict the premium rates estimated with the expected value rule with respect to insurance year t, damage number k and parameter p of the damage number distribution. If a driver inflicts no damage, every year he pays lower rate of the net premium. The premium rates are lowest in portfolios with the smallest values of the parameter p of the damage number distribution (compare figure 1). The higher the number of damages inflicted by a driver the higher the premium paid the following insurance years. For the damage number equal to or greater than 3 the premiums will be highest in portfolios in which the parameter p of the damage number distribution is smallest (compare figure 4).

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ESTYMACJA BAYESOWSKA WSPÓŁCZYNNIKÓW BONUS-MALUS W UBEZPIECZENIACH KOMUNIKACYJNYCH OC

Podstawą działalności ubezpieczeniowej jest prawidłowe szacowanie składek ubezpieczeniowych. Składka brutto jest to składka netto powiększona o dodatek bezpieczeństwa oraz koszty działalności ubezpieczeniowej. W pracy indywidualne składki netto wyznaczano trzema metodami: metodą wartości oczekiwanej, metodą wariancji oraz metodą zerowej użyteczności. Następnie oszacowano za pomocą estymatorów bayesowskich i porównano współczynniki bonus-malus dla składek wyznaczanych trzema wymienionymi metodami. Badania przeprowadzono dla różnych parametrów rozkładu liczby szkód.