Rohan French and Lloyd Humberstone

AN OBSERVATION CONCERNING PORTE'S RULE IN MODAL LOGIC

Abstract

It is well known that no consistent normal modal logic contains (as theorems) both $\Diamond A$ and $\Diamond \neg A$ (for any formula A). Here we observe that this claim can be strengthened to the following: for any formula A, either no consistent normal modal logic contains $\Diamond A$, or else no consistent normal modal logic contains $\Diamond \neg A$.

1. Introduction

We work with the standard language of (propositional) monomodal logic, for convenience taken to have a countable stock of sentence letters, p_i, constants (0-ary connectives) $\top$ and $\bot$ among the Boolean connectives, alongside others together functionally complete (we use $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ below for the Boolean connectives commonly so notated), and for definiteness, the 1-ary connective $\Box$ (with $\Diamond A$ defined as $\neg \Box \neg A$). A *pure* formula is one not containing any sentence letters (such as $\Box \bot \vee \Diamond \Box \bot$). Here modal logics are treated as sets of formulas, and we are interested specifically in normal modal logics, i.e. such sets as contain every truth-functional tautology in the Boolean connectives as well as all formulas of the form $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$, and are closed under Modus Ponens, Uniform Substitution (of arbitrary formulas for sentence letters), and Necessitation (the rule taking us from A to $\Box A$).

Let us recall the status of Porte's rule (from [11]):

$$\frac{\Diamond A \wedge \Diamond \neg A}{\bot}$$

in this setting. Porte observed that (the set of theorems of) S5 was closed under this rule – which appears somewhat differently notated as (R5) on p. 417 of [11] – and others (such as Fagin et al. [4]) have noted that the same goes for all normal modal logics. Since the inconsistent logic is obviously closed under the rule, it suffices to establish that all consistent normal modal logics are, which can be done by noting that every such logic is a sublogic of one of the two Post complete modal logics: that containing $\Box A \leftrightarrow A$ for all A ("the trivial system"), or that containing $\Box A$ for all A ("the Verum system"). Where $\mathsf{S} \oplus X$ denotes the smallest normal modal logic extending S and containing all formulas $A \in X$ and K is, as usual, the smallest normal modal logic, then the latter system can alternatively be described as $\mathsf{K} \oplus \{\Box\bot\}$. Since no formula of the form $\Diamond A \wedge \Diamond\neg A$ belongs to either of these logics, no such formula belongs to any consistent normal modal logic.

Alternatively, one may draw the same conclusion semantically using either algebraic or model-theoretic semantics for modal logic. In the former case (as in Makinson [8]) one considers normal modal expansions of the two-element Boolean algebra,[1] noting that neither can validate any formula of the form $\Diamond A \wedge \Diamond\neg A$, and in the latter via a consideration (as in van Benthem [2], Lemma 2.19, p. 31) of the special status of the two one-point Kripke frames, again neither of them validating a formula of the form in question.

The consistent normal modal logics are thus closed under Porte's rule because the rule is, as one says, *vacuously* admissible: no such logic contains any candidate premiss for an application of the rule. We can put this another way, rewriting the rule as a two-premiss rule, with premisses $\Diamond A$ and $\Diamond\neg A$, and concentrating on K since the existence of *some* consistent normal modal logic containing such and such formulas is equivalent to the consistency of the normal extension of K by those formulas. The familiar point associated with Porte's rule is then:

For all formulas A, $\mathsf{K} \oplus \{\Diamond A, \Diamond\neg A\}$ is inconsistent.   (*)

By an elementary argument given in the following section, we show that a considerable strengthening of this point is available:

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[1] The notion of normality for a modal algebra used here is that defined in Blok and Köhler, p. 943.

For all formulas A, either $\mathsf{K} \oplus \{\Diamond A\}$ *or* $\mathsf{K} \oplus \{\Diamond\neg A\}$, *is inconsistent.*(**)

Thus, reverting to Porte's conjunctive form, the inconsistency of a normal modal logic S containing any formula $\Diamond A \wedge \Diamond\neg A$ can always be 'blamed on' one of the conjuncts and is never due to any interaction between them (or to any interaction between them and whatever else is in S other than what is already in K).

2. Justifying the Stronger Claim

The present section is devoted to establishing the observation (**) above. It is a well known fact that the logic KD ($= \mathsf{K} \oplus \Diamond\top$) contains, for every pure formula B, either B itself or else $\neg B$; we will put this by saying that KD *decides* every pure formula. (See the first part of the proof of Theorem 1 in [15].)

With a view to showing that for any formula A, either $\mathsf{K} \oplus \{\Diamond A\}$ or $\mathsf{K} \oplus \{\Diamond\neg A\}$ is inconsistent, fix on one such A and note that each of these logics extends KD. Let B be any pure substitution instance of A and note that

(1) $\Diamond B \in \mathsf{K} \oplus \{\Diamond A\}$ (2) $\Diamond\neg B \in \mathsf{K} \oplus \{\Diamond\neg A\}$.

Now, since B is a pure formula and KD decides each such formula we have $B \in \mathsf{KD}$ or $\neg B \in \mathsf{KD}$. In the first case, therefore, $\Box B \in \mathsf{KD}$ (by normality) while in the second case we have $\Box\neg B \in \mathsf{KD}$. Since each of $\mathsf{K} \oplus \{\Diamond A\}$, $\mathsf{K} \oplus \{\Diamond\neg A\} \supseteq \mathsf{KD}$, we therefore have either $\Box\neg B \in \mathsf{K} \oplus \{\Diamond A\}$, showing $\mathsf{K} \oplus \{\Diamond A\}$ to be inconsistent in view of (1), or else $\Box B \in \mathsf{K} \oplus \{\Diamond\neg A\}$, showing $\mathsf{K} \oplus \{\Diamond\neg A\}$ to be inconsistent in view of (2). Thus one or other of these extensions of K is inconsistent, as (**) claims.

3. Enriching the Language

It is worth asking what becomes of (*) and (**) in the setting of bimodal logic, i.e., when there are two primitive non-Boolean connectives instead of one. Consider the case of tense logic, in which each of the two operators – we shall write them as $\Box$ and $\Box^{-1}$ here – is 1-ary, and satisfies the normality conditions given in connection with $\Box$ in Section 1, as well as the 'Lemmon

bridging axioms', all formulas of the form $\Diamond\Box^{-1}A \to A$ and $\Diamond^{-1}\Box A \to A$. (Here $\Diamond^{-1}$ abbreviates $\neg\Box^{-1}\neg$; $\Box$, $\Diamond$, $\Box^{-1}$ and $\Diamond^{-1}$ are often written, in a notation due to A. N. Prior, as G, F, H and P respectively.) The basic tense logic K_t is smallest set of formulas in this language satisfying these conditions, and we can ask about the fate of (*) and (**) for K_t rather than monomodal K. Naturally one could focus on $\Diamond^{-1}$ rather than $\Diamond$ in pursuing this question but in view of the symmetrical treatment of the two operators in K_t there is nothing to be gained by doing so. When discussing extensions of a logic S in this language, $\mathsf{S}\oplus X$ denotes the smallest extension of S containing all formulas in X and in which each of $\Box, \Box^{-1}$ satisfies the normality conditions.

As is well known, frames for (i.e., validating every formula in) K_t can be presented with a single accessibility relation, taken as interpreting $\Box$, with $\Box^{-1}$ interpreted by its converse. Such frames reveal a contrast with the monomodal case:

PROPOSITION 3.1. *The claim resulting from* (**) *by putting "K_t" for "K" is false.*

PROOF: We need to supply a formula A for which each of $\mathsf{K} \oplus \{\Diamond A\}$, $\mathsf{K} \oplus \{\Diamond\neg A\}$, is consistent. Take A as $\Box^{-1}\Diamond^{-1}\top$. Thus we can write $\neg A$ as $\Diamond^{-1}\Box^{-1}\bot$. To see that $\mathsf{K}_t \oplus \{\Diamond A\}$ is consistent, note that the one-element reflexive frame – or more explicitly $\langle\{0\}, \{\langle 0,0\rangle\}\rangle$ – is a frame for this logic. In the case of $\mathsf{K}_t \oplus \{\Diamond\neg A\}$ consider, instead, the frame $\langle\{0,1\}, \{\langle 0,1\rangle, \langle 1,1\rangle\}\rangle$. □

The choice of A in the proof just given refutes the tense-logical analogue of (**) but not that of (*), since from $\Diamond A$, i.e., $\Diamond\Box^{-1}\Diamond^{-1}\top$, by one of the bridging axioms (and Modus Ponens) we get $\Diamond^{-1}\top$, from which Necessitation (w.r.t. $\Box^{-1}$ and then, after that, $\Box$) delivers $\Box\Box^{-1}\Diamond\top$, which is equivalent to $\Box A$, making the envisaged extension by both $\{\Diamond A, \Diamond\neg A\}$ inconsistent. But must the same happen, as it does in the monomodal case, for all possible choices of A? That is:

QUESTION. Is (*) correct for K_t in place of K?

A remark spanning pages 712 and 713 in Segerberg [15] (and echoed in the bottom paragraph of p. 135 of [6]) suggests that the answer to this question may be negative, but we have no definite information.

A simpler extension of the language of monomodal logic would involve the addition of a non-Boolean sentential constant. As has recently been observed ([5], [6]) adding such a constant κ makes for considerable changes to standard metatheoretical results familiar from the straight monomodal case, and indeed the multimodal-with-all-operators-normal case. We can use our constant by considering the smallest modal logic in this language in which $\square$ obeys the normality conditions and which also contains, in order to contain a candidate premiss for Porte's rule, the axiom $\Diamond\kappa \wedge \Diamond\neg\kappa$. If the notion of normality is extended in a natural way to other than 1-ary connectives (see Schotch and Jennings [14], p. 271, where the definition given is credited to unpublished work of Goldblatt), this would be counted as a bimodal logic in which $\square$ but not κ was normal. (The latter would require, by the appropriate generalization of Necessitation, that κ itself belong to the logic.) The conclusion, $\bot$, of an application of Porte's rule to this new axiom clearly does not belong to the logic, which contains exactly the formulas (and $\bot$ is not one of them) valid on all frames $\langle W, R, W_0 \rangle$ with $R \subseteq W \times W$ and $W_0 \subseteq W$ satisfies the condition (in which we denote by $R(x)$ the set of $y \in W$ such that Rxy) that for all $x \in W$, $R(x) \cap W_0 \neq \varnothing$ and $R(x) \cap (W \smallsetminus W_0) \neq \varnothing$, truth in a model being defined with the condition that κ is true precisely at elements of W_0. (Validity on a frame is truth throughout W in every model on the frame.) The axiom tells us that it is contingent whether κ, and was used by Anderson in an early version (see note 40 of [1]) of his reduction of deontic logic to alethic modal logic – using κ to record the violation of an obligation so that "it ought to be that A" could be thought of as $\square(\neg A \rightarrow \kappa)$. More recently, Pizzi ([9], [10]) has used a similar idea to define necessity – understood as subsuming any interpretation of the $\square$ operator – in terms of contingency (or noncontingency), even in logics too weak to allow for a definition of $\square$ in terms of $\triangledown$ ("it is contingent whether"). Of course, given the change of primitives, he writes the above axiom as $\triangledown\kappa$ rather than $\Diamond\kappa \wedge \Diamond\neg\kappa$, but shows how to use κ alongside $\triangledown$ in an explicit definition of $\square$ even in these weaker logics. For a review of this project, see [7], esp. Section 4, where further references and a contrast with a different kind of contingency constant (used by Meredith and Prior) may be found.[2] The reason for

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[2]Pizzi [10] is there referred to as 'Relative Contingency and Multimodal Logics,' its working title as of the time of writing. The bottom paragraph of p. 1296 in [7] seems to suggest that Prior [12] had observed that without the addition of a new constant

giving this example here is simply to show that there are naturally arising extensions of the basic monomodal apparatus in which □ remains normal but Porte's rule ceases to be admissible.

References

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[2] Johan van Benthem, **Modal Logic and Classical Logic**, Bibliopolis, Naples 1985.

[3] W. J. Blok and P. Köhler, *Algebraic Semantics for Quasi-Classical Modal Logics*, **Journal of Symbolic Logic** 48 (1983), pp. 941–964.

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[5] Rohan French, *Denumerably Many Post-complete Normal Modal Logics with Propositional Constants*, **Notre Dame Journal of Formal Logic** 53 (2012), pp. 549–556.

[6] Robert Goldblatt and Tomasz Kowalski, *The Power of a Propositional Constant*, **Journal of Philosophical Logic** 43 (2014), pp. 133–152.

[7] Lloyd Humberstone, *Zolin and Pizzi: Defining Necessity from Noncontingency*, **Erkenntnis** 78 (2013), pp. 1275–1302.

[8] David Makinson, *Some Embedding Theorems for Modal Logics*, **Notre Dame Journal of Formal Logic** 12 (1971), pp. 252–254.

[9] Claudio Pizzi, *Necessity and Relative Contingency*, **Studia Logica** 85 (2007), pp. 395–410.

[10] Claudio Pizzi, *Relative Contingency and Bimodality*, **Logica Universalis** 7 (2013), pp. 113–123.

[11] Jean Porte, *The Deducibilities of S5*, **Journal of Philosophical Logic** 10 (1981), pp. 409–422.

[12] A. N. Prior, **Time and Modality**, Oxford University Press, Oxford 1957.

[13] A. N. Prior, Review of various papers by A. R. Anderson, **Journal of Symbolic Logic** 24 (1959), pp. 177–178.

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nothing was provably contingent in the modal logics under consideration there, but this observation was actually made by Prior in line 10 from the base of [13], p. 177. (The reference in that paragraph of [7] to note 4 of the [7] should instead be to note 26 there, which mentions specifically Appendix D of Prior [12], rather than the more pertinent [13].)

[14] P. K. Schotch and R. E. Jennings, *Modal Logic and the Theory of Modal Aggregation*, **Philosophia** 9 (1975), pp. 265–278.

[15] Krister Segerberg, *Post Completeness in Modal Logic*, **Journal of Symbolic Logic** 37 (1972), pp. 711–715.

Department of Philosophy,
Monash University,
Victoria 3800,
Australia