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Czesław Domański*, Andrzej S. Tomaszewicz**

NORMAL APPROXIMATION OF MULTIPLE RUNS DISTRIBUTIONS

Tests based on the number of runs can be applied to verification of many types of hypotheses, for instance the hypothesis that

- elements of the sample are independent,
- two or more samples are drawn from the same distribution,
- regression model of one or two explanatory variables is linear, etc.

Let us consider a sequence of random variables

$$X_1, X_2, \ldots, X_n \tag{1}$$

which have the same discrete distribution with s values, i.e.

$$P(X_1 = x_j) = p_j, \quad j = 1, \ldots, s, \quad \sum_{j=1}^{s} p_j = 1.$$

We also use the symbols n_j to denote the number of elements of j-th type in (1):

$$n_j = \text{card}\,\{i: X_i = x_j\}.$$

The object of our consideration is the number of runs in the sequence (1):

$$K_n = 1 + \text{card}\,\{i: 2 \le i \le n, \quad X_i \neq X_{i-1}\}.$$

For instance:

- for n = 8, s = 2, using the traditional notation: $x_1 = A$, $x_2 = B$ we can have the sequence

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* Associate Professor at the University of Łódź.
** Associate Professor at the University of Łódź.

A A B A A B B A

when we observe 5 runs,

- for $n = 9$, $s = 3$, $x_3 = C$ - we also have 5 runs in the sequence

A B B C C C A A C

The critical region for the above specified types of hypotheses is based on the distribution of the number of runs in (1) under the assumption that X_i's are independent. This distribution for two kinds of elements was investigated by S t e v e n s (1939), M o o d (1940), W a l d and W o l f o w i t z (1940), S v e d and E i s e n h a r t (1943) and, for three and more kinds of elements - B a r t o n and D a v i d (1957, 1960). Some new results could be found in monographies B a r t o n (1960) and G i b b s o n (1987). They showed some combinatorial formulae for number-of-run distribution. Besides, the convergence to the normal distribution was proved.

Tables of number-of-run distribution for more than two kinds of elements were constructed just for small values of n (for instance, for 3 and 4 kinds of elements - up to $n = 12$ only). It was because the classical formulae are not convenient for computations. For $s = 2$ we published quite general recursive formulae (cf. D o m a ń s k i and T o m a s z e w i c z (1984)) which can be used when the sequence (1) is generated by stationary Markov chain.

The number of runs is a discrete variable. Therefore (excluding exceptional cases) it is not possible to select such a critical value to make the test size exactly equal to fixed significance level α. Commonly the critical values as integer numbers

$$k^L(n, \alpha) = \max \{k: P(K_n \leq k) \leq \alpha\},$$

$$k^R(n, \alpha) = \min \{k: P(K_n \geq k) \leq \alpha\}$$

are accepted (the first of them concerns a left-hand-sided test, the second one - a right-hand-sided test. Hence, the test size is, in general, less than α:

$$P(K_n \leq k^L(n, \alpha)) < \alpha,$$

$$P(K_n \geq k^R(n, \alpha)) < \alpha.$$

For that reason the randomized tests are applied.

The randomized test we used is defined in common way: reject H_0 when

$$K_n \le k^L(n, \alpha) \text{ (or } K_n \ge k^R(n, \alpha)),$$

accept H_0 when

$$K_n > k^L(n, \alpha) + 1 \text{ (or } K_n < k^R(n, \alpha) - 1),$$

reject H_0 with the probability

$$p^L_{rand}(n, \alpha) = \frac{\alpha - P(K_n \le k^L(n, \alpha))}{P(K_n = k^L(n, \alpha) + 1)}, \quad \text{when} \quad K_n = k^L(n, \alpha) + 1$$

(or reject H_0 with the probability

$$p^R_{rand}(n, \alpha) = \frac{\alpha - P(K_n \ge k^R(n, \alpha))}{P(K_n = k^R(n, \alpha) - 1)}, \quad \text{when} \quad K_n = k^R(n, \alpha) - 1).$$

The idea of interpolated quantiles is connected with randomized tests. We define

$$k^L_i(n, \alpha) = k^L(n, \alpha) + p^L_{rand}(n, \alpha),$$
$$k^R_i(n, \alpha) = k^R(n, \alpha) - p^R_{rand}(n, \alpha).$$

Of course, when we know the interpolated quantile we know both the integer quantile and the randomization probability.

Using normal approximation we obtain following estimates of interpolated quantiles

$$k^L_i(n, \alpha) \approx \tilde{k}^L_i(n, \alpha) = \mu_K(n) + \Phi^{-1}(\alpha)\sigma_K(n) - \tfrac{1}{2},$$

$$k^P_i(n, \alpha) \approx \hat{k}^P_i(n, \alpha) = \mu_K(n) - \Phi^{-1}(\alpha)\sigma_K(n) + \tfrac{1}{2}.$$

Φ denotes the inverse of standard normal cumulative distribution function and

$$\mu_K(n) = E(K_n), \quad \sigma_K(n) = \mathrm{var}(K_n).$$

Randomized test based on these quantiles are defined using estimates of the integer quantiles and the randomization probabilities:

$$\hat{k}^L(n, \alpha) = \mathrm{entier}(\hat{k}^L_i(n, \alpha)),$$
$$\hat{p}^L_{rand}(n, \alpha) = \hat{k}^L_i(n, \alpha) - \hat{k}^L(n, \alpha),$$

$$\hat{k}^R(n, \alpha) = -\text{entier}(-\hat{k}_1^R(n, \alpha)),$$

$$\hat{p}_{rand}^R(n, \alpha) = \hat{k}^R(n, \alpha) - \hat{k}_1^R(n, \alpha),$$

Thus, the size of the test based on normal approximation is

$$\alpha^L(n, \alpha) = P(K_n \le \hat{k}^L(n, \alpha)) + p_{rand}^L(n, \alpha)\, P(K_n^L = k(n, \alpha) + 1).$$

$$\alpha^R(n, \alpha) = P(K_n \ge \hat{k}^R(n, \alpha)) + p_{rand}^R(n, \alpha)\, P(K_n^R = k(n, \alpha) - 1).$$

(These formulae follow immediately from the randomized test definition).

As a measure of goodness of approximation we chose the difference between the test size and the assumed significance level α:

$$\delta^L(n, \alpha) = \hat{\alpha}^L(n, \alpha) - \alpha,$$

$$\delta^R(n, \alpha) = \hat{\alpha}^R(n, \alpha) - \alpha.$$

It is hard to deny that the smaller the value of this difference, the better approximation.

We would like to present some evaluation of normal approximation of tests based on number-of-runs distribution in symmetric case for 3 types of elements. That means we assume that

$$s = 3,$$

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

and

$$n_1 = n_2 = n_3$$

are fixed. The recursive formula

$$P(K_n = k \mid n_1, n_2, n_3) = \frac{n_1}{n}\,(P(K_{n-1} = k \mid n_1 - 1, n_2, n_3) +$$

$$+ P(K_{n-1} = k-1 \mid n_2, n_1 - 1, n_3) + P(K_{n-1} = k - 1 \mid n_3,$$

$$n_1 - 1, n_2))$$

was used to compute the distribution. We present some results for the special cases:

$$n_1 = n_2 = n_3 = 1, 2, \ldots, 40 \quad (n = 3, 6, \ldots, 120).$$

Table 1 contains the interpolated quantiles k_i^L, k_i^R, their normal approximates $\hat{k}_i^L$, k_i^R, and test size errors δ^L, δ^R. The dif-

Table 1

Quantiles (k^L, $\hat{k}^L$) and test size errors (α)

n	k^L, $\hat{k}^L$, α	Left-hand sided			Right-hand sided		
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
15	$k_1^L(n, \alpha)$	6.267	7.534	9.077	12.956	14.209	14.990
	$\hat{k}_1^L(n, \alpha)$	6.568	7.720	9.077	12.923	14.280	15.432
	$\alpha^L(n, \alpha)$	0.0052	0.0095	0.0000	-0.0070	0.0037	0.0046
30	$k_1^L(n, \alpha)$	14.416	16.260	18.380	23.653	25.556	26.958
	$\hat{k}_1^L(n, \alpha)$	14.704	16.402	18.403	23.597	25.598	27.296
	$\alpha^L(n, \alpha)$	0.0029	0.0063	0.0025	-0.0068	0.0019	0.0029
60	$k_1^L(n, \alpha)$	31.940	34.460	37.467	44.567	47.338	49.615
	$\hat{k}_1^L(n, \alpha)$	32.150	34.600	37.479	44.521	47.403	49.849
	$\alpha^L(n, \alpha)$	0.0018	0.0038	0.0010	-0.0037	0.0018	0.0020
90	$k_1^L(n, \alpha)$	50.008	53.123	56.778	65.257	68.702	71.552
	$\hat{k}_1^L(n, \alpha)$	50.214	53.227	56.779	65.221	68.773	71.786
	$\alpha^L(n, \alpha)$	0.0015	0.0027	0.0000	-0.0022	0.0019	0.0015
120	$k_1^L(n, \alpha)$	68.330	71.992	76.194	85.837	89.837	93.145
	$\hat{k}_1^L(n, \alpha)$	68.588	72.078	76.191	85.809	89.922	93.419
	$\alpha^L(n, \alpha)$	0.0014	0.0019	0.0002	-0.0016	0.0020	0.0012

Source: The author's calculations.

ferences between the quantiles seem to be not large. But, in our opinion, a much better measure of goodness of approximation is the test size error δ^L and δ^R. In Table 2 values of δ^L δ^R are shown. They are absolute values of δ in some intervals of sample size n:

$$\delta_*^L(n_1, n_2, \alpha) = \max_{n_1 \le n \le n_2} | \delta^L(n, \alpha) |,$$

$$\delta_*^R(n_1, n_2, \alpha) = \max_{n_1 \le n \le n_2} | \delta^R(n, \alpha) |$$

Table 2

Test size errors

n_1-n_2	Left-hand sided			Right-hand sided		
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
15-30	0.0052	0.0095	0.0033	0.0084	0.0048	0.0046
33-45	0.0030	0.0049	0.0014	0.0056	0.0048	0.0029
48-60	0.0022	0.0040	0.0010	0.0039	0.0033	0.0025
63-75	0.0019	0.0037	0.0008	0.0035	0.0030	0.0022
78-90	0.0017	0.0031	0.0009	0.0022	0.0020	0.0019
93-105	0.0015	0.0026	0.0003	0.0024	0.0025	0.0017
108-120	0.0014	0.0026	0.0003	0.0022	0.0020	0.0015

Source: The author's calculations.

Some authors suggest that the normal approximation is good enough even for n = 15. For very rough statistical investigation maybe it is not a big difference whether the test size is 5% or 6%. But in many kinds of statistical analysis (for instance test power investigation) more accuracy is needed. Thus, the error values 1 or 2 pro mille which we observe for sample size 100-120 is not satisfactory.

Therefore we cannot base only on normal approximation of number-of-runs distribution. Even for moderate sample sizes the exact distribution should be applied.

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Czesław Domański, *Andrzej S. Tomaszewicz*

ROZKŁADY DŁUGOŚCI I LICZBY SERII WIELOKROTNYCH

Rozkłady liczby i długości serii dla dwóch rodzajów elementów zostały stosunkowo dobrze poznane. Znacznie mniej natomiast wiadomo o własnościach rozkładów liczby bądź długości serii dla trzech lub więcej elementów.

W artykule prezentujemy niektóre wyniki dotyczące własności testów opartych na seriach złożonych z trzech lub więcej rodzajów elementów, weryfikujących hipotezę o niezależności obserwacji w próbie. Ze względu na to, że rozkłady badanych statystyk są dyskretne, analizowano testy zrandomizowane i kwantyle interpolowane.