A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
FOLIA OECONOMICA 132, 1993

Halina Klepacz, Elżbieta Żółtowska**

PROPERTIES OF ESTIMATORS OF CES FUNCTION PARAMETERS, OBTAINED BY THE USE QUENOUILLE AND FILTRATION METHODS

Abstract. By numerical experiments the relative efficiency of Quenouille and filtration estimators was established. Assymetry of distribution of estimators and biasedness are the reasons for suggestion of weighting method.

Key words: CES function, Quenouille and filtration methods of estimation, efficiency.

1. INTRODUCTION

The CES function is the most general of the production functions with low substitution flexibility. This is from where most of its advantages come: if does not assume in advance the shaping of substitution flexibility parameter, or the level of production homogenity, it allows direct determination of values of as many as four parameters having good economic interpretation.

Also the analysis conducted by [S a t o (1969)] with the aggregation theory shows that in many cases, aggregated production function can be shown directly in from of the CES function, or applied with positive results in the local approximation of the VES aggregated function. The result is that the CES function is worth considering in terms of economic theory.

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* Lecturers at the Institute of Econometrics and Statistics of the University of Łódź.

Empirical investigations of the production models conducted by many authors [see i.e. K l e p a c z, Ż ó ł t o w s k a (1990), K u d r y c k a (1973), S a t o (1969), Ż ó ł t o w s k a (1989)], show difficulties in obtaining reliable information about, among others, capital series and the level of utilization of the production capacities. It allows us to assume that one of the explanatory variables, namely the value of the fixed assets K in the two factor CES production function of type

$$Y = \alpha[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-\nu/\rho} e^{\varepsilon} \tag{1}$$

where: Y, K, L - are respectively: production, fixed assets and employment,

α, δ, ν, ρ - structural parameters,

ε - random component,

is measured with error. Assuming that the errors η of measurement are random (independent from the random component ε) one can say that

$$\tilde{K} = K \cdot e^{\eta} \tag{2}$$

There are a lot of methods of estimation of the model parameters (1). In the case, when the elasticity of substitution is expected to be close to one, one takes advantage of Kmenta transformation derived from the development of CES function into a power series in the vicinty of $\rho = 0$. After an adequate grouping of its initial terms the model parameters a_0, a_1, a_2, a_3 are estimated from

$$\ln Y = a_0 + a_1(\ln K - \ln L) + a_2(\ln K - \ln L)^2 + a_3 \ln L + o(\rho^2) + \varepsilon \tag{3}$$

where:

$$\begin{cases} a_0 = \ln\alpha \\ a_1 = \nu\delta \\ a_2 = -0.5\nu\delta(1 - \delta)\rho \\ a_3 = \nu \end{cases} \tag{4}$$

and, generally, the influence of component $o(\rho^2)$ on the values of the explanatory variables is omitted. Estimators of parameters α, δ, ν, ρ are determined on the basis of estimators a_0, a_1, a_2, a_3 and relation (4). Let us notice, that due to the non-observability of the value of variable K, model (3) cannot be estimated directly.

2. BASIC ASSUMPTION FOR THE CES FUNCTION PARAMETERS ESTIMATION METHOD ON THE BASIS OF JACKKNIFING

The starting point is model (3) and relation (2) [K l e p a c z (1988), K l e p a c z, Ż ó ł t o w s k a (1987)], from which, by determining lnK, we obtain

$$\ln K = \ln\widetilde{K} - \eta \tag{5}$$

Inserting this dependence into (3), skipping $o(\rho^2)$ and grouping suitably components we obtain, from (3), the formula

$$y = a_0 + a_1\widetilde{X}_1 + a_2\widetilde{X}_2 + a_3X_3 + \varepsilon^* \tag{6}$$

in which:

$$\begin{cases} y = \ln Y \\ \widetilde{X}_1 = \ln\widetilde{K} - \ln L \\ \widetilde{X}_2 = (\ln\widetilde{K} - \ln L)^2 \\ X_3 = \ln L \\ \varepsilon^* = \varepsilon - a_1\varepsilon - 2a_2(\ln K - \ln L)\eta - a_2\eta^2 \end{cases} \tag{6a}$$

This model, however, does not satisfy the assumptions of the least squares method as its random component ε^* does not have zero expectation and variables $\widetilde{X}_1$ and $\widetilde{X}_2$ are random. In paper [K l e p a c z (1988)] it was proposed to determine estimators of parameters for the relation of the first differences instead of estimating parameters of model (6),

$$\Delta y = a_1\Delta\widetilde{X}_1 + a_2\Delta\widetilde{X}_2 + a_3\Delta X_3 + \Delta\varepsilon^* \tag{7}$$

where:

Δy, $\Delta\widetilde{X}_1$, $\Delta\widetilde{X}_2$, ΔX_3 are increments of respective variables. There was also discussed a detailed presentation of the method of determining the estimation of the free term a_0 there, with application of jackknifing. To achieve this estimation, there was applied Quenouille method, producting a series of estimations $\{a_2^i\}$, and also $\{a_1^i\}$ i $\{a_3^i\}$. These fractional results served to construct two types of jackknife estimators:

1) on the basis of a_1^i, a_2^i, a_3^i estimators, B^i, C^i, R^i are calculated by formulas (4), for each of the subsamples in i-th subsample, and then direct jackknife estimators: B^J, C^J, R^J of δ, ν, ρ parameters are built;

2) on the basis of a_1^i, a_2^i, a_3^i estimators, jackknife estimators a_1^J, a_2^J, a_3^J of a_1, a_2, a_3 parameters are built, and then, from formulas (4), indirect jackknife estimators: B^{pJ}, C^{pJ}, R^{pJ} of δ, ν, ρ parameters are determined;

3) on the basis of a_1, a_2, a_3 parameters estimators derived from the whole sample, B, C, R estimators are constructed according to formulas (4).

Properties of these three groups of estimators and estimator A of parameter α are presented on the basis of Monte Carlo experiments in [K l e p a c z (1989)].

3. BASIC ASSUMPTIONS IN THE METHOD OF ESTIMATION OF THE CES FUNCTION PARAMETERS ON THE BASIS OF KALMAN FILTRATION ALGORITHM

In this case the starting point is model (1) and relation (2) from which

$$K = \tilde{K}e^{-\eta},$$

after inserting in (1), leads to relationship

$$Y = \tilde{\alpha}[\tilde{\delta}\ \tilde{K}^{-\rho} + (1 - \tilde{\delta})\ L^{-\rho}]^{-\nu/\rho}\ e^{\varepsilon} \tag{8}$$

where:

$$\tilde{\alpha} = \alpha(\delta(e^{-\rho\eta} - 1) + 1)^{-\nu/\rho},$$

$$\tilde{\delta} = \frac{\delta e^{-\rho\eta}}{\delta e^{-\rho\eta} + (1 - \delta)},$$

are as we see, functions of random variable η.

Having adapted Kmenta's approach, we transformed model (8) in the formula

$$Y = A_1\tilde{X}_1 + A_2\tilde{X}_2 + A_0 + A_3X_3 + o(\rho^2) + \varepsilon \tag{9}$$

analogous to (3), with

$$\begin{cases} A_1 = \nu\ \delta\ e^{-\rho\eta}\ \xi^{-1} \\ A_2 = -0.5\nu\rho\delta(1 - \delta)\xi^{-2} \\ A_0 = \ln\alpha - \nu\rho^{-1}\ln\xi \\ \xi\ = \delta(e^{-\rho\eta} - 1) + 1 \\ A_3 = \nu \end{cases} \tag{10}$$

Just like $\tilde{\alpha}$, $\tilde{\delta}$ in model (8), parameters A_0, A_1, A_2 of model (9) are random variables, being function of η.

As it was shown in study [Ż ó ł t o w s k a (1988)] and [Ż ó ł t o w s k a (1989)], expectations of variables A_0, A_1, A_2 are equal to parameters a_0, a_1, a_2 of model (6), respectively. Hence, dependencies between expectations: $E(A_0)$, $E(A_1)$, $E(A_2)$ and parameters α, δ, ρ of model (1) have form (4).

To estimate model (9) there was applied Kalman filtration algorithm for which the model of the system is as follows

$$\begin{cases} Z(t+1) = Z(t) + \varepsilon_A(t) \\ y(t) = C(t)Z(t) + \varepsilon \\ t = 1, 2, \ldots, n \end{cases} \tag{11}$$

where:

$$Z^T(t) = [A_1(t), A_2(t), A_0(t), A_3] \tag{12}$$

is a vector of realization of random parameters at moment t,

$$C(t) = [\tilde{X}_1(t), \tilde{X}_2(t), 1, X_3(t)] \tag{13}$$

is a vector of realization at moment t of explanatory variables of model (9),

$$\varepsilon_A^T(t) = [\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t), 0] \tag{14}$$

is a vector, the components of which represent disturbances of state vector Z(t) at moment t.

Properties of estimators a, b, c, r of model (1) parameters α, δ, ν, ρ were presented on the basis of Monte Carlo experiments results, in [Ż ó ł t o w s k a (1987)].

4. SIMILARITIES AND DIFFERENCES BETWEEN BOTH METHODS

The starting point for both methods of estimation was model (1), i.e. CES function with multiplicatively introduced random component. In both cases the same model (2) for measurement error was assumed as well as the use, in the process of estimation, of Kmenta transformation. Both solutions lead to linear models with explanatory random variables. Diversified dinal results come from the moment of taking into consideration the occurrence of the measurement error of the explanatory variable. Namely, taking into consideration condition (2) after transformation (3) results in model (6), in which structural parameters are non-random, and random explanatory variables and the non-observable random component are dependent and the expectation of the last one is different form zero. It resulted in the need to estimate the model with the first differences and, in consequence, seeking for a free trem on the basis of the Quenouille method results. The latter was also used. to construct two groups of jackknife estimators: direct and indirect. Considering observable variable $\tilde{K}$ in model (1) results in obtaining a non-linear model with random structural parameters and, after Kmenta transformation, a linear model (9), also with random structural parameters, in which random explanatory variables and component ε are independent. To obtain estimates of performance of the structural parameters there was built a simple model of the system and then the filtration algorithm was applied.

The scheme for construction of models series suitable in the jackknifing and filtration methods are shown in Fig. 1, while a general presentation of both methods was included in items 2 and 3.

5. ANALYSIS OF RESULTS OBTAINED FROM BOTH METHODS ON THE BASIS OF MONTE CARLO EXPERIMENTS

Comparing both ways of estimating production function let us focus our attention on statistical characteristics of model (1) parameters estimators, obtained in the performed numerical experiments. We shall discuss in detail interrelations between respective amòunts for these spaces of samples, in which $R^2 = 0.99$, $RB = 0.99$ with 0.95, where $1 - R^2$ is the share of variance ε in variance Y, and $1 - RB$ is the share of the measurement error variance η in variance $\tilde{K}$.

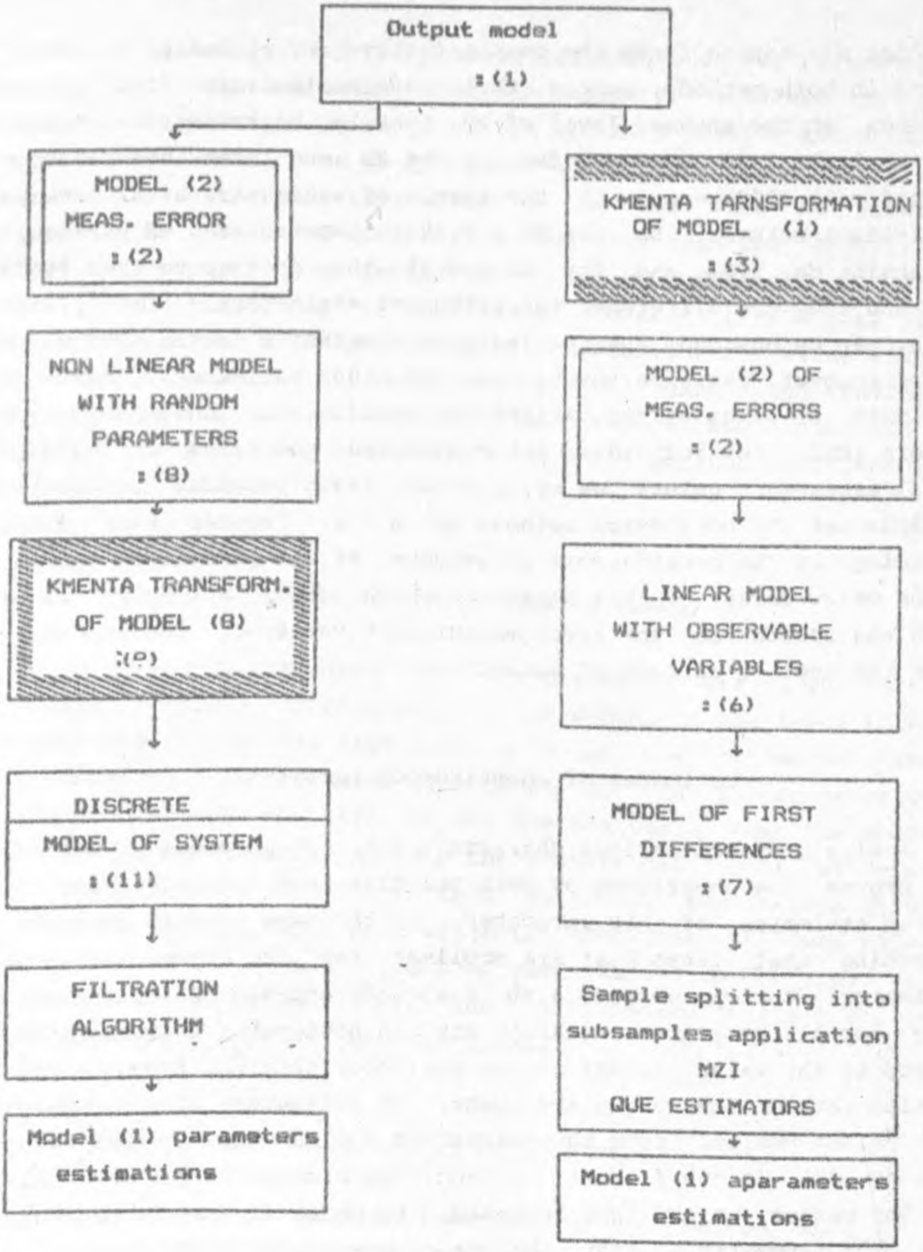

Fig. 1. Construction scheme of models series
for considered methods

Estimates of the parameter of splitting

Let uns notice first the common feature of parameter δ estimations in both methods, namely, their underestimations. Its amount depends on the assumed level of RB, that is, on the error variance for variable $\ln\tilde{K}$. However, for a given RB one does not observe significant differences in the levels of estimators bias between individual methods. So, for RB = 0.99 estimators bias of parameter δ consist ca. 3.3%, and for RB = 0.95 they grow up to from 8% to 11% and they are different for different estimators. The least ones, can be noticed for the indirect jackknife estimator which might suggest its superiority over the other estimators. But the analysis of sizes of the variability coefficients and mean square errors (MSE) for individual estimators does not allow for so univocal assessment unless we restrict our investigations to applications of the considered methods to m = 1. Another fact worth noticing is the considerable percentage of all parameters estimations below value of this parameter which grows strongly along with the growth in the error measurement variance, independently from the applied variant of estimation.

Estimates of substitution parameter

Analysis of statistical characteristics of parameter ρ estimates proves the occurrence of both positive and negative bias of medium estimates of this parameter. At the same time it seems interesting that these bias are smollest for the direct jackknife estimator. One can observe also a significant degree of variability for all estimators although one can notice for decreasing RB, a drop in the value in MSE for an estimator obtained from the filtration method. Analysing the number of estimates of a parameter with values smaller from the parameters one can state that they make ca. 40% in the jackknife method, and almost 60% for the filtration method, which might indicate, respectively, right side and left side asymmetry of distributions of respective estimators.

Estimates of scale parameter

Analising statistical characteristics adequate for parameter α one can state that estimates of this parameter obtained in accordance with the filtration algorithm are more precise and accurate, at the same time we do not observe too significant differences in the accuracy of this estimator in the case of growing values 1 - RB. Bias of estimator a, obtained from the filtration method, are several times lower than those of estimator A. This result is not surprising, if one takes into account how many indirect estimations must be performed to determine its value within the method described in item 2. On the other hand, the considerable variability of estimates of this parameter and higher MSE in estimates of parameter α obtained from the algorithm of filtration are worth noticing.

Estimates of homogeneity parameter

Bias of medium estimates of this parameter are close to the considered variants of both methods and depend on the level of RB. For RB = 0.99 thery are kept with in 1% of the parameter value, drop of RB to 0.95 increases them to 35 of the parameter value. Taking into account the size of MSE one can state that the observed bias is not significant from the statistical point of view. The distribution of number of this parameter estimations shows, at the same time, a right side asymmetry when going to lower values of coefficient RB. In the ligth of characteristics obtained for estimates of this parameter in the performed estimations one can assume that both methods are equivalent.

Average errors of predictor

Average errors of variable Y predictor, determined in a point designated for K and L, can be considered a common measure of estimators accuracy for all parameters. The point was assumed to be determined on the basis of average values of quantities $\tilde{K}$ and L observed with error. Hence, the point was different in each sample of the experiment and depended on the realized values of $\widetilde{K}$. Of course selection of such point is disputable due to the theory of production, although when the predictor is treated as a measure it is not so important. It is certain that, according to the expectations, a drop in the value of RB will result in a absolute increase of the predictors errors. It is characteristic at the same time that in most cases these errors are positive for the applied jackknife method, while negative for the filtration method, preserving, however, order of magnitude. It proves that one can not prefer one type of estimators to other.

Final remarks

The performed numerical experiments do not provide us with arguments which might in conclusion result in the statement that one these two investigated methods is more effective than other, although estimators obtained after their application are not identical. It seems that due different asymmetries of their distributions and different bias it would be reasonable to consider new estimators constructed on the basis of these obtained from the jackknife and filtration methods as their average bias. The problem, however, is that in the choice of bias and can presumably be solved through experiments. One of the possibilities would be the construction of such bias which might minimize the weights of a new estimator. Then, such bias could be designated on the basis of information on the size of weights. Another possibility is the minimization of MSE of individual estimators would be different as it seems. Still another possibility would be such a choice of bias which would minimize predictors averages, it leads, however to a non-linear optimization and, numerically, can be difficult to carry out.

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Halina Klepacz, Elżbieta Żółtowska

ANALIZA WŁASNOŚCI ESTYMATORÓW PARAMETRÓW FUNKCJI CES WG METOD: QUENOUILLE'A I FILTRACJI

Przeprowadzone eksperymenty numeryczne nie pozwalają ustalić, która ze stosowanych metod estymacji (Quenouille'a i filtracji) jest bardziej efektywna, mimo że otrzymane w wyniku ich stosowania estymatory nie są identyczne. Wydaje się, że ze względu na różne asymetrie ich rozkładów i różne wielkości ich obciążeń, dobrze byłoby rozważyć nowe estymatory jako ich średnie ważone.