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THE ŁOJASIEWICZ EXPONENT VIA THE VALUATIVE HAMBURGER-NOETHER PROCESS

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ABSTRACT. Let k be an algebraically closed field of any characteristic. We apply the Hamburger-Noether process of successive quadratic transformations to show the equivalence of two definitions of the Łojasiewicz exponent $\mathfrak{L}(\mathfrak{a})$ of an ideal $\mathfrak{a} \subset k[[x, y]]$.

1. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic. Let Ξ denote the set of pairs of formal power series $\varphi \in k[[t]]^2$ such that $\varphi \neq \mathbf{0}$ and $\varphi(0) = \mathbf{0}$. We call the elements of Ξ *parametrizations*. We say that a parametrization φ is a *parametrization of* a formal power series $f \in k[[x, y]]$ if $f \circ \varphi = 0$. For $\varphi = (\varphi_1, \dots, \varphi_n) \in k[[t]]^n$ we put $\text{ord } \varphi := \min_j \text{ord } \varphi_j$, where $\text{ord } \varphi_j$ stands for the order of the power series φ_j . Let $\mathfrak{a} \subset k[[x, y]]$ be an ideal. We consider the *Łojasiewicz exponent of \mathfrak{a}* defined by the formula

$$(1.1) \quad \mathfrak{L}(\mathfrak{a}) := \sup_{\varphi \in \Xi} \left(\inf_{f \in \mathfrak{a}} \frac{\text{ord } f \circ \varphi}{\text{ord } \varphi} \right).$$

Such concept was introduced and studied by many authors in different contexts. Lejeune-Jalabert and Teissier [10] observed that, in the case of several complex variables, $\mathfrak{L}(\mathfrak{a})$ is the optimal exponent $r > 0$ in the Łojasiewicz inequality

$$\exists_{C, \varepsilon > 0} \forall_{\|x\| < \varepsilon} \max_j |f_j(x)| \geq C \|x\|^r,$$

where (f_1, \dots, f_k) is an arbitrary set of generators of \mathfrak{a} . Moreover, they proved that, with the help of the notion of integral closure of an ideal, the number $\mathfrak{L}(\mathfrak{a})$ may be seen algebraically. This is what we generalize below (see Theorem 1) partly

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answering [3, Question 2]. D'Angelo [6] introduced $\mathfrak{L}(\mathfrak{a})$ independently, as an order of contact of \mathfrak{a} . He showed that this invariant plays an important role in complex function theory in domains in \mathbb{C}^n .

There has been some interest in understanding the nature of the curves that 'compute' $\mathfrak{L}(\mathfrak{a})$. In fact, the supremum in (1.1) may be replaced by maximum. A more exact result in this direction says that if $\mathfrak{a} = (f_1, \dots, f_m)k[[x, y]]$ is an (x, y) -primary ideal, then there exists a parametrization φ of $f_1 \times \dots \times f_m$ such that

$$\mathfrak{L}(\mathfrak{a}) = \inf_{f \in \mathfrak{a}} \frac{\text{ord } f \circ \varphi}{\text{ord } \varphi}.$$

For holomorphic ideals, this was proved by Chądzyński and Krasieński [5], and independently by McNeal and Némethi [12]. The case of ideals in $k[[x, y]]$, where k is as above, is due to the authors [3]. De Felipe, García Barroso, Gwoździejewicz and Płoski [7] gave a shorter proof of this result; moreover, they answered [3, Question 1], by showing that $\mathfrak{L}(\mathfrak{a})$ is always a Farey number, i. e. a rational number of the form $N + b/a$, where N, a, b are integers such that $0 < b < a < N$.

2. METHODS AND RESULTS

Once and for all we agree that all the rings considered in the paper are commutative with unity. Let $\bar{\mathfrak{a}}$ denote the integral closure of an ideal \mathfrak{a} (see Section 4). Our main result is

Theorem 1. *Let $\mathfrak{a} \subset k[[x, y]]$ be an ideal. Then*

$$(2.1) \quad \mathfrak{L}(\mathfrak{a}) = \inf \left\{ \frac{p}{q} : (x, y)^p k[[x, y]] \subset \bar{\mathfrak{a}}^q \right\}.$$

The general idea of the proof is the following. It is easy to see, that the right hand side of (2.1) is equal to

$$\sup_{\nu} \frac{\nu(\mathfrak{a})}{\nu((x, y)k[[x, y]])},$$

where ν runs through the set of all rank one discrete valuations with center $(x, y)k[[x, y]]$. This is a consequence of the well-known valuative criterion of integral dependence (see Theorem 5). On the other hand, there is a correspondance between valuations of the field $k(C)$ and parametrizations centered at points of a given irreducible curve C (see [14, Chapter V §10]). A mathematician's basic instinct, then, lead us to believe that the same reasoning could be repeated for parametrizations in place of valuations. For this we need a version of criterion of integral dependence which is based on parametrizations (well-known in the complex analytic setting). This is where the Hamburger-Noether process comes in. Namely, if (R, \mathfrak{m}) is a local regular two-dimensional domain, then using Abhyankar theorem (Theorem 15) we may find for any given valuation ν with center \mathfrak{m} a sequence of quadratic transformations of R producing rings and their associated valuations which, respectively, approximate the valuation ring of ν and ν itself.

The aforementioned valuations, given by the process, are in fact expressible in a quite explicit form even in the case $R = k[[x_1, \dots, x_n]]$ (see Lemmas 19 and 20); however, the unique feature of Abhyankar theorem is the ‘approximation phenomenon’, which for non-divisorial valuations only holds in the two-dimensional case (cf. Example 18). Altogether, the above observations plus the usual valuative criterion of integral dependence allows us to prove a parametric version of the criterion over $k[[x, y]]$.

The structure of the paper is as follows. Sections 3 and 4 are of introductory nature. In Section 5 we give detailed description of the concept of the quadratic transformation of a local regular domain. This notion was developed and used by Zariski and Abhyankar in the 50’s in the framework of valuation theory and the resolution of singularities problem. A sequence of successive quadratic transformations starting from a local regular domain containing an algebraically closed field leads to an inductive construction called the Hamburger-Noether process. This is described in Section 6. In this setting Hamburger-Noether process may be considered as a generalization of a classical construction of the normalization of a plane algebroid curve (see [4, 13]) to the case of valuations [8]. Finally, in Sections 7 and 8 we prove the aforementioned parametric criterion of integral dependence and as a result obtain Theorem 1.

3. VALUATIONS

An integral domain V is called a *valuation ring* if every element x of its field of fractions K satisfies

$$x \notin V \implies 1/x \in V.$$

We say that V is a *valuation ring of K* . The set of ideals of a valuation ring V is totally ordered by inclusion. In particular, V is a local ring. In general, this ring need not be Noetherian, nevertheless its finitely generated ideals are necessarily principal.

A *valuation* of a field K is a group homomorphism $\nu: K^* \rightarrow \Gamma$, where Γ is a totally ordered abelian group (written additively), such that for all $x, y \in K^*$, if $x + y \neq 0$ then

$$\nu(x + y) \geq \min\{\nu(x), \nu(y)\}.$$

Occasionally, when convenient, we will extend ν to K setting $\nu(0) := +\infty$. The image of ν is called the *value group* of ν and is denoted Γ_ν . Set

$$R_\nu := \{x \in K : x = 0 \text{ or } \nu(x) \geq 0\},$$

$$\mathfrak{m}_\nu := \{x \in K : x = 0 \text{ or } \nu(x) > 0\}.$$

Then R_ν is a valuation ring of K and \mathfrak{m}_ν is its maximal ideal.

Let Γ be an ordered abelian group. A subgroup $\Gamma' \subset \Gamma$ is called *isolated* if the relations $0 \leq \alpha \leq \beta$, $\alpha \in \Gamma$, $\beta \in \Gamma'$ imply $\alpha \in \Gamma'$. The set of isolated subgroups of Γ is totally ordered by inclusion. The number of proper isolated subgroups of Γ is called the *rank* of Γ , and written $\text{rk } \Gamma$. If ν is a valuation of a field K , then we say

that ν is of rank $\text{rk } \nu := \text{rk } \Gamma_\nu$. It is well known that the rank of ν is equal to the Krull dimension of R_ν [2, VI.4.5 Proposition 5].

If V is a valuation ring of K , then there exists a valuation ν of K such that $V = R_\nu$. If ν_1, ν_2 are valuations of K then $R_{\nu_1} = R_{\nu_2}$ if and only if there exists an order-preserving group isomorphism $\varphi: \Gamma_{\nu_1} \rightarrow \Gamma_{\nu_2}$ satisfying $\nu_2 = \varphi \circ \nu_1$. In such a case we say that valuations ν_1 and ν_2 are *equivalent*.

Let R be an integral domain with field of fractions K . The valuation ν of K is said to be *centered* on R if $R \subset R_\nu$. In this case the prime ideal $\mathfrak{p} = \mathfrak{m}_\nu \cap R$ is called the *center* of ν on R . Quite generally, if $A \subset B$ is a ring extension, \mathfrak{q} is a prime ideal of B and $\mathfrak{p} = \mathfrak{q} \cap A$ then we have a natural monomorphism $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$. Consequently, the residue field of \mathfrak{p} , that is the field of fractions of A/\mathfrak{p} , may be considered as a subfield of the residue field of \mathfrak{q} . In this setting we have the following important dimension inequality due to I. S. Cohen. We write below $\text{tr. deg}_A B$ for the transcendence degree of the field of fractions of B over that of A , where $A \subset B$ is an extension of integral domains.

Theorem 2 ([11, Theorem 15.5]). *Let A be a Noetherian integral domain, and B an extension ring of A which is an integral domain. Let \mathfrak{q} be a prime ideal of B and $\mathfrak{p} = \mathfrak{q} \cap A$; then we have*

$$\text{ht } \mathfrak{q} + \text{tr. deg}_{A/\mathfrak{p}} B/\mathfrak{q} \leq \text{ht } \mathfrak{p} + \text{tr. deg}_A B.$$

In what follows we will be interested in the case where (R, \mathfrak{m}, k) is a local Noetherian domain with residue field k and ν is a valuation with center \mathfrak{m} on R . We set $\text{tr. deg}_k \nu := \text{tr. deg}_k R_\nu/\mathfrak{m}_\nu$. Directly from the above theorem we get:

Proposition 3. *Let (R, \mathfrak{m}, k) be a local Noetherian domain and let ν be a valuation with center \mathfrak{m} on R . Then*

$$\text{rk } \nu + \text{tr. deg}_k \nu \leq \dim R.$$

In particular, $\text{tr. deg}_k \nu \leq \dim R - 1$.

Definition 4. Let (R, \mathfrak{m}, k) be a local Noetherian domain and let ν be a valuation with center \mathfrak{m} on R . If $\text{tr. deg}_k \nu = \dim R - 1$ then we say that ν is *divisorial* with respect to R (or is a *prime divisor* for R).

4. INTEGRAL CLOSURE OF IDEALS

Let \mathfrak{a} be an ideal in a ring R . We say that an element $x \in R$ is *integral over* \mathfrak{a} if there exist $N \geq 1$ and $a_1 \in \mathfrak{a}, a_2 \in \mathfrak{a}^2, \dots, a_N \in \mathfrak{a}^N$ such that

$$x^N + a_1 x^{N-1} + \dots + a_N = 0.$$

The set of elements of R that are integral over \mathfrak{a} is called the *integral closure* of \mathfrak{a} and is denoted $\bar{\mathfrak{a}}$. It turns out that the integral closure of an ideal is always an ideal.

Next theorem is the celebrated valuative criterion of integral dependence.

Theorem 5 ([9, Proposition 6.8.4]). *Let \mathfrak{a} be an ideal in an integral Noetherian domain R . Let \mathcal{V} be the set of all discrete valuation rings V of rank one between R and its field of fractions for which the maximal ideal of V contracts to a maximal ideal of R . Then*

$$\bar{\mathfrak{a}} = \bigcap_{V \in \mathcal{V}} \mathfrak{a}V \cap R.$$

5. QUADRATIC TRANSFORMATION OF A RING

Definition 6. Let (R, \mathfrak{m}) be a local regular domain and let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Set $S = R \left[\frac{\mathfrak{m}}{x} \right]$ and let \mathfrak{p} be a prime ideal in S containing x . Then the ring $S_{\mathfrak{p}}$ is called a (first) quadratic transform of R . If ν is a valuation with center \mathfrak{m} on R and $xR_{\nu} = \mathfrak{m}R_{\nu}$ then $S_{\mathfrak{p}}$, where $\mathfrak{p} := R \cap \mathfrak{m}_{\nu}$, is called a (first) quadratic transform of R along ν .

Remark 7. Keep the notations from the above definition. Then $xS = \mathfrak{m}S$ and for any $k \in \mathbb{N}$, $x^k S \cap R = \mathfrak{m}^k S \cap R = \mathfrak{m}^k$. Indeed, the equalities $xS = \mathfrak{m}S$, $x^k S = \mathfrak{m}^k S$ and the inclusion $\mathfrak{m}^k \subset \mathfrak{m}^k S \cap R$ are clear. Take $r \in \mathfrak{m}^k S \cap R$. Then there exist $l \geq 0$ and $a_j \in \mathfrak{m}^{k+j}$, $j = 0, \dots, l$, such that

$$a_0 + \frac{a_1}{x} + \dots + \frac{a_l}{x^l} = r.$$

Thus $x^l r \in \mathfrak{m}^{k+l}$. On the other hand, (R, \mathfrak{m}) is a local regular domain, hence the associated graded ring $\text{gr}_{\mathfrak{m}} R$ is an integral domain (as isomorphic to the ring of polynomials $\frac{R}{\mathfrak{m}}[Y_1, \dots, Y_n]$). We have $(x^l + \mathfrak{m}^{l+1}) \cdot (r + \mathfrak{m}^k) = x^l r + \mathfrak{m}^{k+l}$, which is zero in $\text{gr}_{\mathfrak{m}} R$. Consequently, since $x^l \notin \mathfrak{m}^{l+1}$ we must have $r \in \mathfrak{m}^k$.

Remark 8. It is clear from the definition, that if (T, \mathfrak{n}) is a quadratic transformation of (R, \mathfrak{m}) along ν then ν has center \mathfrak{n} on T .

Proposition 9. *Let (R, \mathfrak{m}) be a local regular domain of dimension $n > 1$. Set x_1, \dots, x_n as the generators of \mathfrak{m} . Let $R[Y]$, where $Y = (Y_2, \dots, Y_n)$, be a polynomial ring in $n - 1$ variables over R . If $\varphi: R[Y] \rightarrow S := R \left[\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1} \right]$ is an R -homomorphism given by $\varphi(Y_j) := x_j/x_1$, $j = 2, \dots, n$, then $\ker \varphi = (x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n) R[Y]$.*

Proof. Take $f \in \ker \varphi$. Using successive divisions with remainder we may write f in the form

$$f(Y) = A_2 \cdot \left(Y_2 - \frac{x_2}{x_1} \right) + \dots + A_n \cdot \left(Y_n - \frac{x_n}{x_1} \right) + B,$$

where $A_2, \dots, A_n \in S[Y]$, $B \in R \left[\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1} \right]$. We must have $B = 0$, since

$f \in \ker \varphi$. There exists N such that

$$(5.1) \quad x_1^N f(Y) = A_2' \cdot (x_1 Y_2 - x_2) + \cdots + A_n' \cdot (x_1 Y_n - x_n),$$

where $A_2', \dots, A_n' \in R[Y]$.

Now, observe that $R[[Y]]$ is a regular local ring of dimension $2n - 1$ and $x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n, x_1, Y_2, \dots, Y_n$ is its regular system of parameters [11, Theorems 15.4, 19.5]. Thus $R[[Y]] / (x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n)$ is a regular local domain and, consequently $(x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n) R[[Y]]$ is a prime ideal. Thus $(x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n) R[Y]$ is also prime. Moreover, this ideal does not contain x_1 since x_1, \dots, x_n minimally generates \mathfrak{m} . This and (5.1) gives $f \in (x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n) R[Y]$. \square

Proposition 10. *Under the notations from Proposition 9 we have:*

- 1) S is regular,
- 2) if $\mathfrak{p} \subset S$ is a prime ideal containing x_1 then $S_{\mathfrak{p}}$ is a regular local ring and

$$\text{tr. deg}_{R/\mathfrak{m}} S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = \dim R - \dim S_{\mathfrak{p}},$$
- 3) if $\mathfrak{p} = \mathfrak{m}_{\nu} \cap S$, where ν is a valuation with center \mathfrak{m} on R such that $\nu(x_1) \leq \nu(x_j)$, $j = 2, \dots, n$, then

$$\text{tr. deg}_{R/\mathfrak{m}} \nu - \text{tr. deg}_{S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}} \nu = \dim R - \dim S_{\mathfrak{p}}.$$

Proof. Let $\mathfrak{p} \subset S$ be a prime ideal. We have $R \subset S \subset R_{x_1}$, so $R_{x_1} = S_{x_1}$. Thus, if $x_1 \notin \mathfrak{p}$ then

$$S_{\mathfrak{p}} = (S_{x_1})_{\mathfrak{p}S_{x_1}} = (R_{x_1})_{\mathfrak{p}R_{x_1}} = R_{\mathfrak{p}},$$

hence $S_{\mathfrak{p}}$ is a regular local ring.

Now, assume that $x_1 \in \mathfrak{p}$. Let $R[Y]$, $Y = (Y_2, \dots, Y_n)$, be a polynomial ring. Put $\mathfrak{b} := (x_1 Y_2 - x_2, \dots, x_1 Y_n - x_n) R[Y]$. We have $S \simeq R[Y]/\mathfrak{b}$ by Proposition 9. Let $\mathfrak{p}^* := \mathfrak{p}/x_1 S$, $S^* := S/x_1 S$. Since $\mathfrak{b} \subset \mathfrak{m}R[Y]$ and $x_1 S = \mathfrak{m}S$,

$$(5.2) \quad S^* = \frac{S}{\mathfrak{m}S} \simeq \frac{R[Y]}{\mathfrak{m}R[Y]} \simeq \frac{R}{\mathfrak{m}}[Y].$$

The ring S^* is regular, as a ring of polynomials over a field, thus there exist $y_2, \dots, y_{k+1} \in S$, such that $\mathfrak{p}^* S_{\mathfrak{p}^*}^* = (y_2, \dots, y_{k+1}) S_{\mathfrak{p}^*}^*$ and $\text{ht } \mathfrak{p}^* = k$. Moreover

$$\dim S_{\mathfrak{p}} = \text{ht } \mathfrak{p}S_{\mathfrak{p}} = \text{ht } \mathfrak{p}S = \text{ht } \mathfrak{p}^* + 1 = k + 1$$

and $\mathfrak{p}S_{\mathfrak{p}} = (x_1, y_2, \dots, y_{k+1}) S_{\mathfrak{p}}$. Consequently, $S_{\mathfrak{p}}$ is a regular local ring. This proves 1).

Using the identifications (5.2), we have

$$\begin{aligned} \text{tr. deg}_{R/\mathfrak{m}} \frac{S_{\mathfrak{p}}}{\mathfrak{p}S_{\mathfrak{p}}} &= \text{tr. deg}_{R/\mathfrak{m}} \left(\frac{\frac{R}{\mathfrak{m}}[Y]}{\mathfrak{p}^* \frac{R}{\mathfrak{m}}[Y]} \right)_0 = \dim \frac{\frac{R}{\mathfrak{m}}[Y]}{\mathfrak{p}^* \frac{R}{\mathfrak{m}}[Y]} \\ &= \dim \frac{R}{\mathfrak{m}}[Y] - \text{ht } \mathfrak{p}^* = n - 1 - k = \dim R - \dim S_{\mathfrak{p}}. \end{aligned}$$

This gives 2).

Since $R/\mathfrak{m} \subset S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \subset R_{\nu}/\mathfrak{m}_{\nu}$, the proof of 3) follows from 2) and from the equality

$$\text{tr. deg}_{R/\mathfrak{m}} R_{\nu}/\mathfrak{m}_{\nu} = \text{tr. deg}_{S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}} R_{\nu}/\mathfrak{m}_{\nu} + \text{tr. deg}_{R/\mathfrak{m}} S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}.$$

□

Lemma 11. *Let (T, \mathfrak{n}) be a quadratic transformation of R . Then*

- 1) $\mathfrak{n}^k \cap R = \mathfrak{m}^k$ for any $k \in \mathbb{N}$,
- 2) if $xT = \mathfrak{m}T$ for some $x \in R$, then $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $T = S_{\mathfrak{p}}$, where $S := R \left[\frac{\mathfrak{m}}{x} \right]$ and $\mathfrak{p} := S \cap \mathfrak{n}$.

Proof. By the definition of the quadratic transformation there exist $x' \in \mathfrak{m} \setminus \mathfrak{m}^2$ and a prime ideal \mathfrak{p}' in $S' := R \left[\frac{\mathfrak{m}}{x'} \right]$ such that $x' \in \mathfrak{p}'$, $T = S'_{\mathfrak{p}'}$, $\mathfrak{n} = \mathfrak{p}'T$.

We have

$$\mathfrak{m}^k \supset \mathfrak{n}^k \cap R = (\mathfrak{n}^k \cap S') \cap R \supset \mathfrak{p}'^k \cap R \supset x'^k S' \cap R = \mathfrak{m}^k S' \cap R = \mathfrak{m}^k.$$

This gives the first assertion.

For the proof of the second one, observe that $x \in \mathfrak{m}T \cap R \subset \mathfrak{n} \cap R = \mathfrak{m}$. Moreover, if $x \in \mathfrak{m}^2$, then $\mathfrak{m} = xT \cap R \subset \mathfrak{m}^2 T \cap R \subset \mathfrak{n}^2 \cap R = \mathfrak{m}^2$, which is a contradiction. Thus $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.

Set $S := R \left[\frac{\mathfrak{m}}{x} \right]$. Since $xT = \mathfrak{m}T = x'T$, the element x/x' is invertible in T . Hence $S \subset T$. Let $\mathfrak{p} := \mathfrak{n} \cap S$. Clearly $S_{\mathfrak{p}} \subset T$. On the other hand, the localizations $S'_{\frac{x'}{x}}$ and $S'_{\frac{x}{x'}}$ are equal; denote them by Q . Since $\mathfrak{p}'Q = \mathfrak{n} \cap Q$ and $\mathfrak{p}Q \subset \mathfrak{n} \cap Q$,

$$T = S'_{\mathfrak{p}'} = Q_{\mathfrak{p}'Q} = Q_{\mathfrak{n} \cap Q} \subset Q_{\mathfrak{p}Q} = S_{\mathfrak{p}}.$$

□

Definition 12. Let (R, \mathfrak{m}) be a local regular domain and let $f \in R$, $f \neq 0$. Then we write $\text{ord}_R f$ for the greatest $l \geq 0$ such that $f \in \mathfrak{m}^l$. As usually, we also put $\text{ord}_R 0 := +\infty$. We will call ord_R the *order function* on R . Moreover, for an ideal $\mathfrak{a} \subset R$ we put $\text{ord}_R \mathfrak{a} := \min_{f \in \mathfrak{a}} \text{ord}_R f$.

Corollary 13. *Let (R, \mathfrak{m}) be a local regular domain. Then the order function ord_R is a valuation of the field of fractions of R . Moreover, if $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, $S := R \left[\frac{\mathfrak{m}}{x} \right]$ and $\mathfrak{p} := xS$, then $T := S_{\mathfrak{p}}$ is a valuation ring of the order function on R .*

Proof. Since as in the proof of Proposition 10, S/xS is isomorphic with the ring of polynomials with coefficients in R/\mathfrak{m} , the ideal xS is prime and $\text{ht } xS = 1$. Thus, again by Proposition 10, T is a local regular one-dimensional domain. Hence it is a discrete valuation ring of rank one with valuation given by ord_T . By Lemma 11, $\mathfrak{n}^r \cap R = \mathfrak{m}^r$, so $(\mathfrak{n}^r \setminus \mathfrak{n}^{r+1}) \cap R = \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$ and we get that ord_T restricted to R is equal to ord_R . Consequently, ord_R extends to a valuation of the field of fractions of R with valuation ring equal to T . □

From Proposition 10 we infer that the quadratic transformation $S_{\mathfrak{p}}$ of R is again a regular local domain. If $\text{ht } \mathfrak{p} > 1$ then $\dim S_{\mathfrak{p}} > 1$, thus we may set $R' = S_{\mathfrak{p}}$ and consider a quadratic transformation of R' . This leads to an inductive process, where at each step we must choose the ‘center’ of the next quadratic transformation. This process is finite exactly when at some point as the ‘center’ we take a height one prime ideal. In this case we end up with a discrete valuation ring of rank one.

In what follows we will be interested in the situation in which the above process is driven by a certain valuation ν with center \mathfrak{m} on R . Here, at each step as the next ‘center’ we take the ideal $R_i \cap \mathfrak{m}_{\nu}$. As a result we get a sequence (finite or not) of quadratic transformations along ν :

$$(5.3) \quad R = R_0 \subset R_1 \subset \cdots \subset R_{\nu}.$$

Remark 14. Actually, the sequence 5.3 is uniquely determined by the valuation ν . To see this it is enough to check that a local quadratic transformation (T, \mathfrak{n}) of (R, \mathfrak{m}) along ν is unique. Let $x, x' \in \mathfrak{m} \setminus \mathfrak{m}^2$ be such that $xR_{\nu} = \mathfrak{m}R_{\nu} = x'R_{\nu}$. Set $S := R \left[\frac{\mathfrak{m}}{x} \right]$, $\mathfrak{p} := \mathfrak{m}_{\nu} \cap S$, $T := S_{\mathfrak{p}}$ and similarly $S' := R \left[\frac{\mathfrak{m}}{x'} \right]$, $\mathfrak{p}' := \mathfrak{m}_{\nu} \cap S'$, $T' := S'_{\mathfrak{p}'}$. Since $x'/x \in S \setminus \mathfrak{p}$, x'/x is invertible in T . Hence $x'T = xT = \mathfrak{n}$ and $S' \subset T$, where we set $\mathfrak{n} := \mathfrak{p}T$. Moreover, $\mathfrak{n} \cap S' = (\mathfrak{m}_{\nu} \cap T) \cap S' = \mathfrak{m}_{\nu} \cap S' = \mathfrak{p}'$. Thus $T = T'$ by Lemma 11.

Theorem 15 ([1, Proposition 3, Lemma 12]). *The sequence (5.3) is finite if and only if ν is a divisorial valuation with respect to R . In this case there exists $m \geq 1$ such that*

$$R = R_0 \subset R_1 \subset \cdots \subset R_{m-1} \subset R_m = R_{\nu}.$$

Moreover, if $\dim R = 2$ and the sequence (5.3) is infinite, then

$$R_{\nu} = \bigcup_i R_i \quad \text{and} \quad \mathfrak{m}_{\nu} = \bigcup_i \mathfrak{m}_i,$$

where \mathfrak{m}_i stands for the maximal ideal of R_i .

Lemma 16. *Let (R, \mathfrak{m}) be a two-dimensional local regular domain and let ν be a valuation with center \mathfrak{m} on R . Assume that (5.3) is a sequence of quadratic transformations along ν . Let $F \subset R_{\nu} \setminus \{0\}$ be a finite set and let $h \in R_{\nu} \setminus \{0\}$ be such that for every $f \in F$ we have $f/h \in \mathfrak{m}_{\nu}$. Then there exists $i \geq 0$ such that $\dim R_i = 2$ and $\min_{f \in F} \text{ord}_{R_i} f > \text{ord}_{R_i} h$.*

Proof. By Theorem 15 there exists i such that $f/h \in \mathfrak{m}_i$ for any $f \in F$. Hence $\min_{f \in F} \text{ord}_{R_i} f > \text{ord}_{R_i} h$. Thus, we get the assertion if $\dim R_i = 2$. So, assume that $\dim R_i = 1$. This means that the sequence (5.3) is necessarily finite and $R_i = R_{\nu}$ is a valuation ring of $\text{ord}_{R_{i-1}}$. It follows that $\text{ord}_{R_{i-1}} = \text{ord}_{R_i}$. Since $\dim R_{i-1} = 2$, we get the assertion. \square

6. HAMBURGER-NOETHER EXPANSION

Let (R, \mathfrak{m}) be an n -dimensional local regular domain, $n > 1$. We will assume in this section that there exists an algebraically closed field $k \subset R$ such that $k \rightarrow R/\mathfrak{m}$ is an isomorphism.

Lemma 17. *Let (T, \mathfrak{n}) be a quadratic transformation of R . Then the following conditions are equivalent:*

1. $\dim T = n$,
2. $\text{tr. deg}_{R/\mathfrak{m}} T/\mathfrak{n} = 0$,
3. the natural homomorphism $k \rightarrow T/\mathfrak{n}$ is an isomorphism,
4. for every regular system of parameters x_1, \dots, x_n of R there exist $j \in \{1, \dots, n\}$ and $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n \in k$ such that

$$\frac{x_1}{x_j} - a_1, \dots, \frac{x_{j-1}}{x_j} - a_{j-1}, x_j, \frac{x_{j+1}}{x_j} - a_{j+1}, \dots, \frac{x_n}{x_j} - a_n$$

is a regular system of parameters of T .

Proof. 1. \implies 2. Follows from Proposition 10.

2. \implies 3. By the assumptions the field R/\mathfrak{m} is algebraically closed and the field extension $k = R/\mathfrak{m} \subset T/\mathfrak{n}$ is algebraic. Hence, the last inclusion is in fact equality. Consequently, the field $k \subset T$ is isomorphic with the residue field of T .

3. \implies 4. The ideal $\mathfrak{m}T$ is principal, hence without loss of generality we may assume that $\mathfrak{m}T = x_1T$. Choose $a_i \in k$ as the image of x_i/x_1 in T/\mathfrak{n} . Put $S := R \left[\frac{\mathfrak{m}}{x_1} \right]$, $\mathfrak{p} := \mathfrak{n} \cap S$. Then by Lemma 11 we have $T = S_{\mathfrak{p}}$, $\mathfrak{n} = \mathfrak{p}T$. Every $f \in S$ may be written in the form

$$f = f_0 + A \left(\frac{x_2}{x_1} - a_2, \dots, \frac{x_n}{x_1} - a_n \right),$$

where $f_0 \in R$ and $A \in R[Y_2, \dots, Y_n]$ is a polynomial without constant term. We have $f \in \mathfrak{n}$ if and only if $f_0 \in \mathfrak{m}$, hence

$$\mathfrak{p} = \left(x_1, \frac{x_2}{x_1} - a_2, \dots, \frac{x_n}{x_1} - a_n \right) R \left[\frac{\mathfrak{m}}{x_1} \right].$$

Thus

$$\frac{\mathfrak{p}}{x_1 S} \simeq (Y_2 - a_2, \dots, Y_n - a_n) \frac{R}{\mathfrak{m}} [Y],$$

by Proposition 9. Consequently $\dim T = \dim S_{\mathfrak{p}} = \text{ht } \mathfrak{p} = n$.

4. \implies 1. Obvious. □

Example 18. Set

$$\nu(x) := (0, 0, 1),$$

$$\nu(y) := (0, 1, 0),$$

$$\nu(z) := (1, 0, 0)$$

and for any $f \in \mathbb{k}[[x, y, z]] \setminus \{0\}$ put as $\nu(f)$ the lexicographic minimum of

$$\{a\nu(x) + b\nu(y) + c\nu(z) : (a, b, c) \in \text{supp } f\},$$

where $\text{supp } f$ denotes the set of $(a, b, c) \in \mathbb{Z}^3$ such that the monomial $x^a y^b z^c$ appears in the expansion of f with non-zero coefficient. It is easy to see that ν extends to a valuation with center $(x, y, z) \mathbb{k}[[x, y, z]]$. The value group Γ_ν is equal to \mathbb{Z}^3 with lexicographical ordering. Let

$$\mathbb{k}[[x, y, z]] =: R_0 \subset R_1 \subset \cdots \subset R_\nu$$

be the sequence of successive quadratic transformations of $\mathbb{k}[[x, y, z]]$ along ν . Observe that $\nu(z/y) = (1, -1, 0) > \mathbf{0}$, hence $z/y \in R_\nu$. Nevertheless, we claim that $z/y \notin \bigcup_{i=0}^{\infty} R_i$. Indeed, set $S := R_0[y/x, z/x]$ and notice that, since $\nu(x) < \nu(y) < \nu(z)$, we have

$$\mathfrak{p} := \mathfrak{m}_\nu \cap S = \left(x, \frac{y}{x}, \frac{z}{x}\right) S$$

is a maximal ideal in S . Thus $R_1 = (R_0)_\mathfrak{p}$ and $x_1 := x$, $y_1 := y/x$, $z_1 := z/x$ is the regular system of parameters in R_1 , where again $\nu(x_1) < \nu(y_1) < \nu(z_1)$. Obviously $z/y = z_1/y_1 \notin R_1$ and in the same way $z/y \notin R_2$ and so on. This proves that the second statement in the Theorem 15 does not hold in the multidimensional case.

Lemma 19. *Let (T, \mathfrak{n}) be an n -dimensional local regular domain such that there exists a sequence*

$$(6.1) \quad R = R_0 \subset R_1 \subset \cdots \subset R_m = T,$$

where for each $i = 1, \dots, m$, R_i is a quadratic transformation of R_{i-1} . Set x_1, \dots, x_n as the generators of \mathfrak{m} . Then there exists a regular system of parameters y_1, \dots, y_n of T and polynomials $A_1, \dots, A_n \in \mathbb{k}[Y_1, \dots, Y_n]$ such that $x_j = A_j(y_1, \dots, y_n)$, $j = 1, \dots, n$.

Proof. Induction with respect to m . The case $m = 0$ is trivial. Assume that the assertion is true for some $m - 1 \geq 0$. Consider the sequence (6.1). By Proposition 10 we have $\dim R_0 \geq \dim R_1 \geq \cdots \geq \dim R_m$. Thus, for each $i = 0, \dots, m$, $\dim R_i = n$. By the induction hypothesis there exist a regular system of parameters y'_1, \dots, y'_n of R_{m-1} and polynomials $A'_1, \dots, A'_n \in \mathbb{k}[Y_1, \dots, Y_n]$ such that $x_j = A'_j(y'_1, \dots, y'_n)$, $j = 1, \dots, n$. On the other hand, by Lemma 17, there exist j_0 , a regular system of parameters y_1, \dots, y_n of R_m and $a_1, \dots, a_{j_0-1}, a_{j_0+1}, \dots, a_n \in \mathbb{k}$

such that

$$\begin{aligned}
 y'_1 &= y_{j_0} (y_1 + a_1), \\
 &\vdots \\
 y'_{j_0-1} &= y_{j_0} (y_{j_0-1} + a_{j_0-1}), \\
 y'_{j_0} &= y_{j_0}, \\
 y'_{j_0+1} &= y_{j_0} (y_{j_0+1} + a_{j_0+1}), \\
 &\vdots \\
 y'_n &= y_{j_0} (y_n + a_n).
 \end{aligned}$$

Now, according to the above equalities we may easily define polynomials A_1, \dots, A_n . \square

Let $R := k[[x_1, \dots, x_n]]$ be the ring of formal power series and let $f \in R \setminus \{0\}$. We will write in f for the *initial form* of f , which is the lowest degree non-zero homogeneous form in the expansion of f . Clearly, $\text{ord}_R f$ is equal to the degree of the initial form of f . For the ring of formal power series R as above we will often write $\text{ord}_{(x_1, \dots, x_n)}$ instead of ord_R .

Lemma 20. *Let $R := k[[x_1, \dots, x_n]]$ be a ring of formal power series. Let (T, \mathfrak{n}) be an n -dimensional local regular domain between R and field of fractions of R . Assume that there exists a regular system of parameters y_1, \dots, y_n of T and polynomials $A_1, \dots, A_n \in k[Y_1, \dots, Y_n]$ such that $x_j = A_j(y_1, \dots, y_n)$, $j = 1, \dots, n$. Then for every non-zero $f \in R$ we have*

$$\text{ord}_T f = \text{ord}_{(Y_1, \dots, Y_n)} f(A_1(Y_1, \dots, Y_n), \dots, A_n(Y_1, \dots, Y_n)).$$

Proof. Set $\Phi := (A_1, \dots, A_n)$. Take $f \in R$, $f \neq 0$.

First, assume that f is a polynomial. We have

$$f(x_1, \dots, x_n) = f(\Phi(Y_1, \dots, Y_n))|_{Y_1=y_1, \dots, Y_n=y_n}.$$

Thus $f(\Phi(Y_1, \dots, Y_n))$ is a non-zero polynomial. Let $P := \text{in } f(\Phi(Y_1, \dots, Y_n))$. Since y_1, \dots, y_n is a regular system of parameters of T ,

$$\text{ord}_T f = \text{ord}_T P(y_1, \dots, y_n) = \deg P = \text{ord}_{(Y_1, \dots, Y_n)} f(\Phi(Y_1, \dots, Y_n)),$$

which gives the assertion in this case.

If f is an arbitrary non-zero power series then, cutting the tail in the power series expansion of f , we find a polynomial $\tilde{f} \in R$ such that $\text{ord}_T f = \text{ord}_T \tilde{f}$ and $\text{ord}_{(Y_1, \dots, Y_n)} f = \text{ord}_{(Y_1, \dots, Y_n)} \tilde{f}$. By the case considered above we have $\text{ord}_T \tilde{f} = \text{ord}_{(Y_1, \dots, Y_n)} \tilde{f}$. \square

7. PARAMETRIC CRITERION OF INTEGRAL DEPENDENCE

Let $R = k[[x, y]]$, $\Delta = k[[t]]$ be the rings of formal power series over an algebraically closed field k . Let \mathfrak{m} and \mathfrak{d} be the maximal ideals of R and Δ respectively. For any $\varphi \in \mathfrak{d} \times \mathfrak{d}$ we have a natural local k -homomorphism $\varphi^*: R \rightarrow \Delta$ given by the substitution.

Theorem 21. *Let \mathfrak{a} be an ideal in R and let $h \in R$. Then h is integral over \mathfrak{a} if and only if $\varphi^*h \in \varphi^*\mathfrak{a}$ for any $\varphi \in \mathfrak{d} \times \mathfrak{d}$.*

Proof. Assume that h is integral over \mathfrak{a} . There exist an integer N and the elements $a_j \in \mathfrak{a}^j$, $j = 1, \dots, N$, such that

$$h^N + a_1 h^{N-1} + \dots + a_N = 0.$$

Take parametrization $\varphi \in \mathfrak{d}^2$. Let $r := \text{ord}_R \mathfrak{a}$. Then

$$\text{Nrd}_\Delta \varphi^* h \geq \min_j (rj + (N - j) \text{ord}_\Delta \varphi^* h).$$

This gives $\text{ord}_\Delta \varphi^* h \geq r$, hence $\varphi^*h \in \varphi^*\mathfrak{a}$.

Assume now, that h is not integral over \mathfrak{a} . Since the case $\mathfrak{a} = 0$ is clear, in what follows we will assume that $\mathfrak{a} \neq 0$. By the valuative criterion of integral dependence (Theorem 5) there exists a valuation ν with center \mathfrak{m} on R such that $h \notin \mathfrak{a}R_\nu$. Consider the sequence of successive quadratic transformations of R along ν :

$$R = R_0 \subset R_1 \subset \dots \subset R_\nu.$$

Denote by \mathfrak{m}_i the only maximal ideal of R_i , $i \geq 0$. Let $F \subset R \setminus \{0\}$ be any finite set of generators of \mathfrak{a} . Then $f/h \in \mathfrak{m}_\nu$ for any $f \in F$. Hence, by Lemma 16 there exists $i \geq 0$ such that $\dim R_i = 2$ and $\min_{f \in F} \text{ord}_{R_i} f > \text{ord}_{R_i} h$. By Lemmas 19 and 20, there exist polynomials $A, B \in k[X, Y]$ such that for any $g \in R$, $\text{ord}_{R_i} g = \text{ord}_{(X, Y)} g(A(X, Y), B(X, Y))$. Set $P_g(X, Y) := \text{in}_g(A(X, Y), B(X, Y))$ for $g \in R$. Then $\deg P_g = \text{ord}_{R_i} g$. Let $(a, b) \in k^2$ be such that $P_h(a, b) \neq 0$ and $P_f(a, b) \neq 0$ for $f \in F$. Put $\varphi := (A(at, bt), B(at, bt))$. Clearly $\text{ord}_\Delta \varphi^* h = \deg P_h$ and $\text{ord}_\Delta \varphi^* f = \deg P_f$ for $f \in F$. Hence $\text{ord}_\Delta \varphi^* h < \min_{f \in F} \text{ord}_\Delta \varphi^* f = \min_{f \in \mathfrak{a}} \text{ord}_\Delta \varphi^* f$, so $\varphi^*h \notin \varphi^*\mathfrak{a}$. \square

Example 22. Let $R = k[[x, y]]$, where k is an algebraically closed field. Consider $\mathfrak{a} := (x^2 + y^3, x^3)$, $h := y^4$, $f := x^2 + y^3$. Let $\varphi := (t^3, -t^2) \in \mathfrak{d} \times \mathfrak{d}$. Notice that $\varphi^*f = 0$. Now, for any $g \in R \setminus \{0\}$ we define $\nu(g) := (k, \text{ord}_\Delta \varphi^*g')$, where $g = f^k g'$ and $\text{gcd}(f, g') = 1$. It is easy to check that ν extends to a valuation with center $(x, y)R$ on R . We will find the Hamburger-Noether expansion along ν . Using this we will show that h is not integral over \mathfrak{a} .

First step.: We have $\nu(x) = (0, 3)$, $\nu(y) = (0, 2)$, so we put $x_1 := \frac{x}{y}$, $y_1 := y$.

Second step.: Now $\nu(x_1) = (0, 1)$, $\nu(y_1) = (0, 2)$, so let $x_2 := x_1$, $y_2 := \frac{y_1}{x_1}$.

Continuing in the above manner we get

Recursive formula for x_i, y_i	Valuation	x_i, y_i in terms of x, y	x, y in terms of x_i, y_i
$x_1 := \frac{x}{y},$ $y_1 := y$	$\nu(x_1) = (0, 1),$ $\nu(y_1) = (0, 2)$	$x_1 = \frac{x}{y},$ $y_1 = y$	$x = x_1 y_1,$ $y = y_1$
$x_2 := x_1,$ $y_2 := \frac{y_1}{x_1}$	$\nu(x_2) = (0, 1),$ $\nu(y_2) = (0, 1)$	$x_2 = \frac{x}{y},$ $y_2 = \frac{y^2}{x}$	$x = x_2^2 y_2,$ $y = x_2 y_2$
$x_3 := x_2,$ $y_3 := \frac{y_2}{x_2} + 1$	$\nu(x_3) = (0, 1),$ $\nu(y_3) = (1, -6)$	$x_3 = \frac{x}{y},$ $y_3 = \frac{y^3 + x^2}{x^2}$	$x = x_3^3 (y_3 - 1),$ $y = x_3^2 (y_3 - 1)$
$x_4 := x_3,$ $y_4 := \frac{y_3}{x_3}$	$\nu(x_4) = (0, 1),$ $\nu(y_4) = (1, -7)$	$x_4 = \frac{x}{y},$ $y_4 = \frac{(y^3 + x^2)y}{x^3}$	$x = x_4^3 (x_4 y_4 - 1),$ $y = x_4^2 (x_4 y_4 - 1)$
$x_5 := x_4,$ $y_5 := \frac{y_4}{x_4}$	$\nu(x_5) = (0, 1),$ $\nu(y_5) = (1, -8)$	$x_5 = \frac{x}{y},$ $y_5 = \frac{(y^3 + x^2)y^2}{x^4}$	$x = x_5^3 (x_5^2 y_5 - 1),$ $y = x_5^2 (x_5^2 y_5 - 1)$
$x_6 := x_5,$ $y_6 := \frac{y_5}{x_5}$	$\nu(x_6) = (0, 1),$ $\nu(y_6) = (1, -9)$	$x_6 = \frac{x}{y},$ $y_6 = \frac{(y^3 + x^2)y^3}{x^5}$	$x = x_6^3 (x_6^3 y_6 - 1),$ $y = x_6^2 (x_6^3 y_6 - 1)$
\vdots	\vdots	\vdots	\vdots
$x_i := x_{i-1},$ $y_i := \frac{y_{i-1}}{x_{i-1}}$	$\nu(x_i) = (0, 1),$ $\nu(y_i) = (1, -i-3)$	$x_i = \frac{x}{y},$ $y_i = \frac{(y^3 + x^2)y^{i-3}}{x^{i-1}}$	$x = x_i^3 (x_i^{i-3} y_i - 1),$ $y = x_i^2 (x_i^{i-3} y_i - 1)$

Successive steps of the Hamburger-Noether algorithm.

Hence, $\mathfrak{a}R_i = (x_i^6(x_i^{i-3}y_i - 1)^2 + x_i^6(x_i^{i-3}y_i - 1)^3, x_i^9(x_i^{i-3}y_i - 1)^3)R_i = x_i^9R_i$ and $hR_i = x_i^8R_i$ for $i \geq 6$. Thus $h \notin \bigcup_{i \geq 6} \mathfrak{a}R_i = \mathfrak{a}R_\nu$. Observe also that $y^5 \in \bar{\mathfrak{a}} \setminus \mathfrak{a}$.

8. THE MAIN RESULT

We keep the notations from the previous section. In particular $R = k[[x, y]]$, k is algebraically closed and for an ideal $\mathfrak{a} \subset R$ we have

$$\mathfrak{L}(\mathfrak{a}) = \sup_{\mathfrak{o} \neq \varphi \in \mathfrak{o} \times \mathfrak{o}} \left(\inf_{f \in \mathfrak{a}} \frac{\text{ord}_\Delta \varphi^* f}{\text{ord}_\Delta \varphi^* (x, y) R} \right) = \sup_{\mathfrak{o} \neq \varphi \in \mathfrak{o} \times \mathfrak{o}} \frac{\text{ord}_\Delta \varphi^* \mathfrak{a}}{\text{ord}_\Delta \varphi^* (x, y) R}.$$

Recall that we want to prove the following

Theorem 1. *Let $\mathfrak{a} \subset R$ be an ideal. Then*

$$(8.1) \quad \mathfrak{L}(\mathfrak{a}) = \inf \left\{ \frac{p}{q} : (x, y)^p R \subset \overline{\mathfrak{a}^q} \right\}.$$

Proof. The cases $\mathfrak{a} = R$ or $\mathfrak{a} = 0$ are trivial. Assume that \mathfrak{a} is a proper ideal and $\text{ht } \mathfrak{a} = 1$. Then, clearly, the right hand side of (8.1) is equal to ∞ . Let $\mathfrak{p} \subset R$ be

a height one prime ideal such that $\mathfrak{a} \subset \mathfrak{p}$. By [13, Appendix C] there exists $f \in R$ such that $\mathfrak{p} = fR$. Hence, one can find $\varphi \in \mathfrak{d} \times \mathfrak{d}$ such that $\varphi^*f = 0$ [13, Theorem 2.1]. Consequently $\mathfrak{L}(\mathfrak{a}) = \infty$.

Now, assume that $\text{ht } \mathfrak{a} = 2$, so that \mathfrak{a} is (x, y) R -primary.

‘ \leq ’ Fix any $p > 0$, $q > 0$ such that $(x, y)^p R \subset \overline{\mathfrak{a}^q}$. Take $\varphi \in \mathfrak{d} \times \mathfrak{d}$. Without loss of generality we may assume that $\text{ord}_\Delta \varphi^*x \leq \text{ord}_\Delta \varphi^*y$. Since $x^p \in \overline{\mathfrak{a}^q}$, Theorem 21 asserts that $\text{ord}_\Delta \varphi^*x^p \geq \text{ord}_\Delta \varphi^*\mathfrak{a}^q$. This easily gives

$$\frac{p}{q} \geq \frac{\text{ord}_\Delta \varphi^*\mathfrak{a}}{\text{ord}_\Delta \varphi^*x} = \frac{\text{ord}_\Delta \varphi^*\mathfrak{a}}{\text{ord}_\Delta \varphi^*(x, y)R}.$$

Hence $p/q \geq \mathfrak{L}(\mathfrak{a})$ and consequently we get the desired inequality.

‘ \geq ’ Take any $p > 0$, $q > 0$ such that $p/q \geq \mathfrak{L}(\mathfrak{a})$. Then, for every $\varphi \in \mathfrak{d} \times \mathfrak{d}$, $\varphi \neq \mathbf{0}$, we have

$$\frac{p}{q} \geq \frac{\text{ord}_\Delta \varphi^*\mathfrak{a}}{\text{ord}_\Delta \varphi^*(x, y)R}$$

or, what amounts to the same thing, $\text{ord}_\Delta \varphi^*(x, y)^p R \geq \text{ord}_\Delta \varphi^*\mathfrak{a}^q$. Hence, for any $h \in (x, y)^p R$ we have $\text{ord}_\Delta \varphi^*h \geq \text{ord}_\Delta \varphi^*\mathfrak{a}^q$. Thus, $(x, y)^p R \subset \overline{\mathfrak{a}^q}$, by Theorem 21. As a result, we get the inequality ‘ \geq ’ in (8.1). \square

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