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STOCKS AND FLOWS IN THE COINTEGRATION CONTEXT

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1. Introduction

Cointegration analysis for many years (Granger 1981, Engle, Granger 1987) belongs to the canon of econometrics. Also in Polish literature, the subject has been present for almost three decades (Blangiewicz, Charemza 1990 and in multi-dimensional terms Welfe 1994). Until now cointegration has been perceived in two aspects. Firstly, as a remedy for the problems of spurious regression and the potential non-consistency of the structural parameter estimator. Secondly, it was treated somewhat utilitary as a convenient tool for analyzing long-run equilibrium and related adjustment mechanisms. Often, especially multidimensional cointegration analysis, it discouraged numerically difficult estimation methods. The aim of the paper is to show cointegration relationships in the most elementary context of basic economic categories: stocks and flows. In addition, it is worth realizing that with cointegration the economist meets at every step, often without realizing it.

We limit the considerations to the variables integrated of order at most second and to the integer order of integration (thus excluding processes generated by ARFIMA (Hamilton 1994, Koop and others 1997). Seasonal integration will not be considered in this paper. Structural change in the mechanisms of processes generating economic categories (for more information: Gosińska 2015) is not considered. While the literature on processes generating economic variables in the period of system transformation in Poland is rich (for example: Osińska, Romański 1995, Piłatowska 2003), the cointegration analysis referring to structural change or the period after its occurrence is still very poor with non-typical observations resulting from the global crisis.

The paper consists of the following parts. In the second part, stocks, flows and increments of these flows (for which the name accelerants are proposed) are considered in a one-dimensional context, ie integration analysis. The appropriate examples are given. Elements of the nominal-to-real analysis transition are introduced. The third part is devoted to the same economic categories in the context of a multidimensional cointegration analysis. Long-run equilibrium relationships between stocks, flows understood as a idiopathic category or first differences of stocks have been shown. Separately considered are dependencies of flow and stock cointegration, not necessarily identical to previous categories. Mechanisms of achieving medium- and long-run equilibrium are considered. In the fourth part, a specific type of equilibrium relations was discussed separately - the so-called polynomial cointegration. Its non-typical nature, in relation to other cointegration relations, consists in the fact that it is a two-step - hence, it simulates the mechanisms of achieving equilibrium in the market economy. The fifth part is devoted to the analysis of the mechanisms of precipitation of economic systems from long- and medium-run equilibrium. Various types of shocks are considered, in particular stock shocks, flow and flow increments, and consistently: shocks affecting these groups of variables. The context of instruments and goals of economic activity was referred to. The last part concludes.

2. Stocks and flows in the integration analysis context. Accelerants

I(3) processes are very rare in the economy, it can be assumed that in stable economies they do not occur at all, and in crisis situations they occur sporadically. Juselius (2004) and Burke and Hunter (2005, p.159) point out the possibility of such processes occurring.

Considering that the stock is a cumulative flow (and thus the flow can be considered as an increase in the stock in a selected unit of time) it can be expected that stock categories should be integrated one order higher than the flows connected to them. Thus, from the definition of integration order (see Engle, Granger 1987) it follows that for a flow, one needs to use less than one differential filter to achieve variable stationarity.

The consequence of the exclusion of potential processes I(3) from the considerations is the conclusion that the variables with the highest order of integration I(2) encountered in "normal" economies should be stocks, not flows. Juselius (2006) also suggests that these stocks are defined in nominal rather than real terms. From the perspective of economic interpretation, it is easy to explain. Consider, for example, capital treated as cumulative net investments. In nominal terms, one can find a reasonable economic interpretation of the case that capital is I(2). This means that net investments (in current prices) are generated by the random walk process (classic I(1) process). Thus, investment increases (in current prices) are a purely random process (not necessarily white noise). The long memory inherent in investments results from the nature of the investment cycle: the vast majority of commenced investment plans are trying to wind up, possible fluctuations in investment expenditures in a given year are purely random. In this case, with respect to the capital calculated in constant prices (volume), analogical reasoning also does not provide grounds for ruling out that real capital is I(2), thus the process generating net investments has a long memory. This is even clearer when one considers that in the I(2) domain the random walk process I(1) is not a stochastic trend (and therefore not long-run), but a stochastic cycle (Juselius 2004, Majsterek 2008). Then the commencement of the investment will be treated as a stochastic impulse I(1) that may be identified with the beginning of the investment cycle. The stochastic trend I(2) has a long-run effect on nominal and real capital, although in the last case the horizon of this impact (especially in conditions of high inflation) is significantly shorter. It is worth noting that if prices are I(2), it is likely that real capital will become the I(1) process, while if I(1), then real capital will remain the I(2) process.

This is due to the following reasons. If the prices are I(2), then they may (but do not have to) relate with the capital integrated in the same order in the so-called homogeneous cointegration (with a long-term coefficient of one). By noting it in the form of logarithms, nominal capital $cp \sim I(2)$, prices $p \sim I(2)$, real capital $c = (cp - p) \sim I(1)$. However, if nominal capital is not cointegrated with prices or this relationship is not HECM (a homogeneous long-run relationship), then real capital will be still integrated in the second order. The same is true when prices are I(1), so inflation is stationary. From the economic point of view the meaning is as follows. It should first of all be stressed that one should not confuse a higher order of integration of nominal processes (generating prices) with higher inflation, and even more so with hyperinflation. Prices I(1) mean fixed inflation (in a special case this may be "long-run inflation" of hyperinflationary size), while prices of I(2) is equivalent with the case of economy susceptible to accelerating inflation, as well as disinflation. A stationary inflation flow means that deflation of capital does not change its susceptibility to long-run shocks.

In a very similar case, the nominal savings I(2) mean the case when the shock affecting the net income is of a long-run nature (increase or decrease in wages or other personal income). This example shows the differences and similarities in economic interpretation of I(1) and I(2) shocks. The shock affecting flows is not always long-lasting. This differs in the salary increase from, for example, the premium, which can be treated as I(1) shock as a savings stock (assuming that the entity manages rationally and does not spend an additional flow of money mindlessly in the short-run), but increases the income flow in the case of bonuses temporary character. The savings stock in the case of I(2) changes faster (which does not have to be equivalent to the fact that it changes more strongly), because it is cumulated by a changed flow of income. In contrast to the explosive process, the change in the savings stock is stable, not increasing/decreasing (see Haldrup 1999).

In the I(1) domain only stocks may be non-stationary. In the case discussed above, the savings stock will be I(1) if the shock affecting the net income will be temporary (said bonus, lottery win, drop). In the described case, in contrast to I(2) savings, the savings stock can only integrate with prices if the latter are I(1) and not I(2), so when inflation is stationary. With these assumptions, shocks affecting the volume of savings have only a short-run impact. This condition allows to see in the new context a long-standing dispute between followers of the Keynesian and neoclassical schools regarding the neutrality of money in the long run. Persistent impact on the real economy through long-run monetary shocks (prices are at least I(1)) is possible when inflation is non-stationary or cointegration of nominal categories with prices is not homogeneous. The case of a heterogeneous cointegration relationship is in turn difficult to explain from both the mathematical and economic side (wider discussion: Grabowski, Welfe 2011). Some economic schools, such as the real business cycle school (RBC), generally exclude such a possibility, because it would mean the impact of money on real categories (the stochastic trend caused by monetary factors does not end as opposed to short-run shock).

Prices I(1) mean an economy susceptible to shocks that may lead to inflation (price shocks), but not to hyperinflation, as inflationary processes do not tend to get worse. This does not mean, however, that hyperinflation and stationary inflation (I(1) prices) may not co-exist. From an economic point of view, the situation of inflation stabilization at a high level is dangerous (up to such a level in the past, I(2) prices triggered a process of inflation intensification).

An example of prices generated by the classic I(1) process are the prices of shares on the efficient market. According to the market efficiency hypothesis (MEH) formulated by Fama (1970), prices on such a market should come from random walk.

$$p_t = p_{t-1} + \zeta_t . \tag{1}$$

Some authors wrongly understand that the necessary condition for the efficiency of the capital market is the stationarity of the process that generates share prices, while the essence of market efficiency is the invariance of return rates. The market's efficiency with a sluggishness, the marasmus of the market, implied by the stationarity of stock prices is obviously confused. If the process generating equity prices was stationary, it would mean that the market is not susceptible to any innovations, and the latter have only a short-run impact. Profits on the stock market could only be achieved by happy speculators. In turn, the trend-stationarity of the stock price would mean economically non acceptable "perpetual motion" of this market. The situation of generating share prices by the random walk process means, however, that changes in share prices are purely random, and therefore according to the MEH hypothesis, these changes might not be predicted and no analysis of historical rates creates comparative advantages on the market. The non efficient market is the market with I(2) prices, since their changes are characterized by long memory and competent analysis of historic data, eg based on the GARCH models (cf.: Brzeszczyński, Welfe 2007, Brzeszczyński, Kelm 2002) allows forecasting turning points, and thus achieve extraordinary profits. Many empirical studies confirm the assumption that the Polish capital market is becoming more and more efficient (see, for example, Goczek, Kania-Morales 2015). The question is whether it is a spontaneous "efficiency" of this market, or is it the result of the fact that overall prices in the end of the transformation process and the associated disinflation begin to take I(1), while formerly I(2) (Kelm, Majsterek 2006).

An interesting, though difficult to interpret economic case is the situation when stocks are generated by stationary or trend-oriented processes. Two most important conclusions can be made regarding such a hypothetical economy. First, it means that shocks affecting stocks are only of short-run significance. This is a difficult result to accept, especially with regard to nominally defined categories. It does not have to be connected with the economic slowdown if the nominal categories change according to the deterministic development tendency. Paradoxically, for some nominal categories (eg for the money supply), the stationarity around the non-linear, "weakening" tendency of development) may be more similar to the I(2) process than the I(1) or I(1) process around the deterministic trend. It can also mean positive effect of effective control over a given economic category (development tendency "imposes" the decision-maker, and unforeseen shocks have only temporary impact.

An even more interesting interpretation is the second implication. Stationary stocks mean that the corresponding flows are integrated to a negative order. Very little space in literature has been devoted to such processes. As suggested by Hamilton (1994), the series integrated in the negative order should, after removing the trend and average, cross the abscissa more frequently than in the case of the series I(0).

One could perceive the random component I(-1) as a cointegrating relation between variables I(0), although in the light of the Engle and Granger (1987) definition of cointegration it is a certain over-interpretation. First of all, in the case of I(0) only short-run shocks are considered, thus losing (apparently) the sense of long-run equilibrium almost always (except for the medium-run cointegration considered in the case of I(2)) related to cointegration dependencies. From the purely literal side, the definition of cointegration implicitly implies $d \geq b > 0$.

It seems, however, that as nowadays the definition of cointegration generalizes to the case where not all variables are integrated just in order d (the last number determines the highest of the orders of cointegration considered), the same can be applied to the fundamental condition $b > 0$.

The CI(0,1) relation in a short period would mean the relation between two stationary/trendostationary stock categories (it is impossible to imagine a similar relation for flows). Shocks deviating on them would have only a temporary effect, any changes could only result from a deterioration of the developmental tendency. From the properties of process I(-1) given by Hamilton (1994) it can be argued that in the long-run shocks from such a cointegrating dependence would be "hypersensitive", i.e. the error correction mechanism would act in a manner similar to fading oscillations $-1 < \alpha_1 < 0$ (hence $-1 < \alpha_1 < 0$):

$$\Delta y_t = \alpha_1 ECT_{t-1} + ST_t + \xi_t , \tag{2}$$

where:

ECT - equilibrium error from CI(1,0),

ST - short-term factor influencing explained variable.

The consequences of using the model with the random component MA (1) are as follows. The first is lack of long-run equilibrium. Therefore, testing cointegration does not make sense at all. In the case of I(2), medium-run cointegration, which is CI(1,1), would have some sense, and it occurs between the first increments, if additionally the random component is MA (it does not have to be the same if the parameters are estimated and not derived from the model transformation on levels), cointegration tests with high probability will be inconclusive.

It is very difficult to find economic examples of stationary/trend-stationary stock categories, and thus category I(-1). It seems that such cases take place in 'borderline' situations. If, for example, the economy is approaching a situation where the public debt (stock) exceeds the thresholds set by the basic law or legal acts of a slightly lower rank, any shocks affecting the state finances will be particularly controlled, so shocks (contingencies) will have a short-run impact. From the deficit (flow) side, exceeding the public debt means a statutory requirement to achieve a balance budget surplus the following year. This means a radical correction mechanism, so one can assume that the deficit has the characteristics of the process generated by I(-1) during this period.

In summary, it can be seen that for flows in current prices the best interpretable result is I(1), for nominal stocks - I(2), while for real flows - I(1) or stationarity. It can also be noted that useful, and in the economy and statistics applied, the stock-flow dichotomy turns out to be insufficient in cointegration analysis. Here, because there is no third name for flow increment (eg disinflation), which category is also a flow (it is described by a series of periods, not moments), but its integration properties are significantly different. For the purpose of this paper, this group of variables will be referred to as accelerants.

In view of the commonly known low power of unit root test (this applies to both stationary tests and integration tests), the test indications should be treated skeptically, when suggesting conclusions contrary to the expectations based on the above observations (Majsterek 2014). In particular, this is an argument for not limiting research interest to the classic integration analysis, which has been the standard first step in all empirical studies containing one- or multi-dimensional cointegration analysis. It can be suggested, both following Juselius (2004), (2006) and in the ghost of the "from general to specific" deduction, starting the analysis of each integrated system containing stock variables (in particular nominal) from I(2) domain, and then possible simplification of the system within the so-called “I(2) in I(1)”, if the presence of I(2) trends is not confirmed in the cointegration analysis.

3. Stocks and flows in the integration analysis context

Known considerations regarding cointegration analysis in the presence of stochastic shocks of at most I(1) (I(1) domain and alternatively: presence of long-run shocks I(2)) should be investigated in the context of stocks and flows and the different role of these categories in the case of equilibrium (cointegration), as well as the precipitation of the system from the equilibrium (operation of stochastic trends and stochastic cycles).

Consider the forces in the economy that drive the system towards equilibrium. Initially, let's confine ourselves to I(1) domain. In this case, the only possible type of cointegration is CI(1,1). This means cointegration: between variables I(1), stationary (the random component from the long-run relationship is I(0)) and hence static (the equilibrium is achieved "immediately") and timeless as well as long-run. Up to now in the literature (Johansen 1994, Majsterek 2003) it was emphasized that it was a relationship between the levels of variables (and not between their first increments). In the light of the considerations regarding cointegration in the case of I(2), the latter property of CI(1,1) cointegration requires a comment. In the I(1) domain cointegration in this way should be stock-oriented, which should be understood as the mutual adaptation of stocks to the equilibrium relation. The flows do not need such adaptation because the shocks acting in the case I(1) are too weak to cause their disequilibrium. Flows in I(1) analysis are most often treated as stationary (Majsterek 2014).

By analogy to the considerations of exogeneity with respect to the parameters of interest (Engle, Hendry, Richard 1983), it seems that from an economic point of view, the concept of I(1) domain should be more specific to system I(1). The difference is that it is explicitly assumed that I(2) shocks may exist in the system, but their impact does not have a significant impact on their functioning, so they can be neglected. In addition, it is advisable to distinguish between stock increments and flows. In the first case, the flow is a secondary category, interest if not in the main one, at least to a large extent focused on the associated stock. The so understood flows in I(1) case can only be stationary and previous considerations do not need to be supplemented.

In the second case, the emphasis is on the fact that the flow is an autonomous economic category, and its cumulative stock not necessarily may have economic significance in a given system. In this case, it can be admitted that the system I(1) may also include cointegration relationships. For example, in a static model of absolute consumption hypothesis, the dependence of the consumption flow on the current income flow is assumed. Both income and consumption are characterized by inertia, so they can be treated as I(1) or almost I(1) (near integrated). In contrast to Friedman's permanent income model, stock categories (cumulative income) are not important here, so the adjustment is of a single-level nature. In larger I(1) systems, flow and stock categories can coexist. For example, the CI(1,1) system of wage and price coupling (Welfe, Majsterek 2002) includes: flows: nominal wages, labour productivity and stocks (CPI measured prices in the form of a fixed-base index, non-wage costs index in the form of a fixed-base index).

Another important feature of the I(1) system that is a direct consequence of the property discussed earlier is the static (single-stage) adaptive reaction. In this system, a dichotomy is sufficient: a short and a long period. Deviations from the equilibrium are short-run and only in this period the stocks/flows of the cointegrated variables are not adjusted to each other.

Thus, it is possible to distinguish stock and flow cointegration as well as stock-flow cointegration and all these types of cointegration also occur in the case of I(1). At the same time, it is necessary to distinguish stock and flow cointegration from cointegration of stock categories and cointegration of flow categories, this distinction being important in the case of I(2).

In the case of I(2) the situation is much more complicated (more about the variables I(2): Johansen 1992, Paruolo 1996, in the Polish literature the first analyzes I(2) were conducted by Kufel 2001). In this case, cointegration does not have to be either a stationary dependency (deviations may be I(0)), it may refer not only to the output variable levels, but also to their first increments. Equilibrium is not static in the sense that it is not achieved immediately. In view of the complexity of adjustment procedures, it is necessary to distinguish, in addition to the long and short, also the mid period. Shocks affecting variables are interpreted as both instantaneous (I(0) shocks), long-run (the character of stochastic trends have only I(2) shocks in I(2) systems) and as stochastic cycles (I(1)).

At this stage of the analysis, it also becomes necessary to mention the cointegration of stocks categories and cointegration of flows categories.

In the I(2) domain the cointegration between flow categories is still CI(1,1), and thus it is simple, timeless, stationary. The main difference of this cointegration (it is both flow cointegration and cointegration of flow categories) with the cointegration of CI(1,1) in the I(1) domain consists in the fact that in the model with variables I(2) such a cointegration relationship rather medium-run and long-run nature. It is not necessarily this cointegration relationship that needs to be consolidated in the long run. It should also be noted that not all flow categories must be I(1) (it is allowed that some are stationary or stationary around the deterministic trend), while not all I(1) flows have to be cointegrated (see Table 1).

In I(2) systems, cointegration between flow categories is not the only type of flow cointegration. This is due to the fact that in the case of I(2) flow is not always an original spontaneous category, but also (or even above all) an increase in the stock system plays a significant role. In this context, there may be cointegration relations between stock categories that have the character of flow cointegration. The economic interpretation seems clear. Comparison of tables 1 and 2 leads to the conclusion that this type of cointegration is R_1 $\mathbf{B}_1^T\mathbf{Y}_{t-1}$ dependencies between levels of variables (stock variables, because flow are rather not I(2)). Let us consider a model:

$$\Delta^2\mathbf{Y}_t = \Pi\mathbf{Y}_{t-1} + \Gamma\Delta\mathbf{Y}_{t-1} + \sum_{s=1}^{S-2}\Psi_s\Delta^2\mathbf{Y}_{t-s} + \Sigma_t \,, \quad (3)$$

where:

$\mathbf{Y}_{t-s}$ - $M \times T$ observation matrix on all variables used in the model lagged by $t-s$ periods, values of $\mathbf{y}_t$ are assumed as non-random and predetermined for $t<0$;

$\Pi = \mathbf{AB}^T$ - $M \times M$ matrix being a combination of baseline cointegrating vectors;

$\mathbf{A}$ - $M \times R$ adjustment matrix (alternatively called weights matrix),

$\mathbf{B}$ - $M \times R$ baseline cointegrating vectors matrix.

$\varepsilon_{(m)t}$ - random errors vector in period t (disturbances are IID and normally distributed).

$\Gamma = \sum_{s=1}^{S-1}\Gamma_s - \mathbf{I}$ - $M \times M$ mean lag matrix.

CI(2,1) cointegration is therefore apparently a dependence between the levels of variables because the CI(2,1) dependencies are the projections of the baseline cointegration relationships $\mathbf{B}^T\mathbf{Y}_{t-1}$ into CI(2,1) subspace, whereas formula (3) shows that the relationship includes levels.

Indeed this relationship occurs between levels (and therefore stocks), but they cointegrate more slowly than in the case of I(1). In the latter case, the mid period do not occurs explicitly, it can be equated with long (equilibrium relations are already visible in the medium-run and perpetuate, so they can be identified with long-run ones). The CI(2,1) is a two-step relationship. Directly (and therefore in the medium or relatively long run) the adjustment to the equilibrium takes place only between the flows of variable increments. So it is specific a flow cointegration, which can be called an accelerant cointegration. It is only in a very long period that the stocks, and thus the levels of the starting variable, begin to achieve mutual equilibrium. Cointegration CI(2,1) should be treated as the first stage (introduction) of polynomial cointegration (Juselius 2006 identifies these concepts by additionally introducing the term: multicointegration). A good economic example of this type of cointegration (ie CI(2,1) are convergence processes. A poorer country can relatively easily enter the path of development allowing it to catch up with economic and technological backlogs compared to leaders, however, it takes many years for living levels (stocks) have become comparable.

Cointegration of stock categories in the I(1) domain can be identified with stock cointegration. In the case of I(2) the problem is much more complicated. Cointegration relations between stock categories include the CI(2,1) cointegration discussed earlier, which is of the stock-increase nature, as well as the direct cointegration relations CI(2,2), which can be described with a good approximation as stock cintegration. It should be emphasized, however, that CI(2,2) is both stock and flow cointegration, and each flow relation must be preceded by the achievement of the equilibrium of the flow (more in part 4, see Figure 1). This means that in the long (but not very long) period, the increments of these stock categories (ie flows), as well as (subsequently) the stocks themselves are adapting to each other. This is the whole strength of this type of cointegration. The directness of a cointegration relationship is based on the fact that the adaptation of stocks takes place in such a rapid period after cointegration of increments (flows), that from an economic point of view it is not necessary to distinguish the long and medium period (achieving the "final" equilibrium of "stocks already in this second horizon). Deviations from this equilibrium occur only in the short-run. It seems that in the market economy such mechanisms of achieving an almost immediate equilibrium should not occur frequently. Dependences of the CI(2,2) type were given, for example, by Kębłowski, Majsterek, Welfe (2008) in the model of wage and price coupling, which also includes the impact of the fiscal system. This result can be explained by the fact that the data largely covered the disinflation period. On the one hand, this means that prices had I(2) properties, and inflation processes had a long memory (disinflation shock had a long-lasting impact). On the other hand, anti-inflation policy was one of the priorities of both monetary authorities and governments, which meant strong price control. It should be assumed that stationarity (because such are random components in CI(2,2)) deviations of prices from the equilibrium trajectory generated by variables appearing in the model was largely associated with this. Moreover, this is indirect evidence that the cost model of price formation (enriched with fiscal factors) more effectively explained the price-generating processes than the model based on the Fisher exchange equation (Kelm, Majsterek 2006), where the deviations integrated in the first order were also identified.

Table 1. Relationships in the I(2) model

<table>
<tr><th rowspan="2">integration orde of resulting variables</th><th colspan="3">Simple relationships</th><th rowspan="2">Complex relationship</th></tr>
<tr><th>long-run</th><th>medium-run</th><th>short-run</th></tr>
<tr><td rowspan="2">I(0)</td><td colspan="2">R_0
$\mathbf{B}_0^T \mathbf{Y}_{t-1}$ dependencies</td><td rowspan="2">M dependecies
$\Psi_i \Delta^2 \mathbf{Y}_{t-i}$</td><td rowspan="2">$R_1 = P_2$
polynomial cointegration dependencies</td></tr>
<tr><td>do not occur</td><td>P_1
$\mathbf{B}_{1\perp}^T \Delta \mathbf{Y}_{t-1}$
dependencies</td></tr>
<tr><td>I(1)</td><td>R_1 $\mathbf{B}_1^T \mathbf{Y}_{t-1}$
dependencies</td><td>P_2 $\mathbf{B}_{2\perp}^T \Delta \mathbf{Y}_{t-1}$
dependencies</td><td>do not occur</td><td>do not occur</td></tr>
</table>

Source: Own study

Colours description: relationships between flows, relationships between flows and stocks increments, relationships between stocks, relationships between accelerants (non-cointegrating)

It should also be noted that the relationship of long-run equilibrium between stock categories should not be treated in the traditional way as a balance of demand and supply, even if one of the variables I(2) can be perceived as aggregated supply and the other as aggregate demand. This stock balance can be achieved in an even longer period or not be achieved at all. The state of long-run equilibrium should be understood as the dynamic state to which the system is heading after each precipitation of it from this position (Welfe 1991). Therefore, the stock equilibrium in the cointegration analysis may mean permanent imbalance (eg equilibrium in the sense of Kornai 1980, excess demand in the centrally planned economy, natural rate of unemployment in the market economy).

Table 2. Relationships in the I(2) model

<table>
<tr><td rowspan="2">integration orde of resulting variables</td><td colspan="3">Simple relationships</td><td rowspan="2">Complex relationship</td></tr>
<tr><td>long-run</td><td>medium-run</td><td>short-run</td></tr>
<tr><td rowspan="2">I(0)</td><td colspan="2">R_0
$\mathbf{B}_0^T \mathbf{Y}_{t-1}$ dependencies</td><td rowspan="2">$M \quad \Psi_i \Delta^2 \mathbf{Y}_{t-i}$
dependencies</td><td rowspan="2">$R_1 = P_2$
polynomial cointegration dependencies</td></tr>
<tr><td>do not occur</td><td>$P_1 \quad \mathbf{B}_{1\perp}^T \Delta \mathbf{Y}_{t-1}$
dependencies</td></tr>
<tr><td>I(1)</td><td>$R_1 \quad \mathbf{B}_1^T \mathbf{Y}_{t-1}$
dependencies</td><td>$P_2 \quad \mathbf{B}_{2\perp}^T \Delta \mathbf{Y}_{t-1}$
dependencies</td><td>do not occur</td><td>do not occur</td></tr>
</table>

Source: Own study

Colours description: flow relationships, flow relationships perpetuating to stock relationships, stock relationships, accelerant relationships (non-cointegrating)

It can be noticed that at the stage of simple cointegration analysis, the dichotomy of the stock - the flow is sufficient, because the flow increments are assumed to be stationary, so they do not need to enter into cointegration relationships. A synthetic review of stock, flows and stock dependencies as well as between flows, stocks and stock increments from the point of view of different types of cointegration dependencies in the I(2) system are presented in tables 3 and 4.

Table 3. Cointegrating relationships in I(2) models

Cointegration Type	Long-run relationships		Medium-run relationships
	static (simple)	dynamic (polynomial)	
CI(2,2)	R_0 dependencies $\mathbf{B}_0^T \mathbf{Y}_{t-1}$	do not occur	do not occur
CI(2,1)	R_1 dependencies $\mathbf{B}_1^T \mathbf{Y}_{t-1}$	do not occur	do not occur
CI(1,1)	do not occur	do not occur	P_1 dependencies $\mathbf{B}_{1\perp}^T \Delta\mathbf{Y}_{t-1}$
		polynomial cointegration	

Source: Own study

Colours description: flow relationships, flow relationships perpetuating to stock relationships, stock relationships

Table 4. Cointegrating relationships in I(2) models

Cointegration Type	Long-run relationships		Medium-run relationships
	static (simple)	dynamic (polynomial)	
CI(2,2)	R_0 dependencies $\mathbf{B}_0^T \mathbf{Y}_{t-1}$	do not occur	do not occur
CI(2,1)	R_1 dependencies $\mathbf{B}_1^T \mathbf{Y}_{t-1}$	do not occur	do not occur
CI(1,1)	do not occur	do not occur	P_1 dependencies $\mathbf{B}_{1\perp}^T \Delta \mathbf{Y}_{t-1}$
		polynomial cointegration	

Source: Own study

Colours description: relationships between flows, relationships between flows and stocks increments, relationships between stocks, relationships between accelerants (non-cointegrating)

Comparison of tables 1 and 4 as well as 2 of 3 allows to notice that in the I(2) system there are also relations between flow increments (accelerants) and these are strictly short-run relations. From an economic point of view, they are the least interesting (the interpretation of parameters is just acceleration, hence the proposed name), but they could not be ignored. Information about the acceleration (or slowing down, e.g. disinflation) of certain processes also carries important content.

It should be noted that the medium-run cointegration between flows is not the only type of CI(1,1) dependence in the I(2) system, but is the only form of immediate, simple cointegration in such a model. The second type of cointegration CI(1,1) is polynomial integration, which due to its specific nature requires a separate discussion.

4. Polynomial cointegration in the flows equilibrium context

From the interpretative point of view, medium-run cointegration differs from polynomial cointegration, that is a simple, one-step relationship between flow categories treated as starting categories (and not transformants of appropriate stocks). It is a medium-run relationship, so it takes place within a certain cycle and usually does not become permanent in the long run (stochastic cycles are dominated by the stochastic trends I(2) around which they circulate). Thus, it can be said that this is a classical, primary dependence of CI(1,1) very similar to that known from system I(1), with the difference that it concerns flows only, not stocks. Polynomial cointegration is a secondary form of the CI(1,1) relationship. It results from the fact that one of the crucial features that distinguishes the system I(2) from I(1) is the two-stage approach to achieving equilibrium. What is the essence of polynomial integration? Generally speaking, this is the relationship between non-integrated flows and non-integrated stock increments. P_1 directions of medium-run cointegration ($\mathbf{B}_{1\perp}^T \Delta \mathbf{Y}_{t-1}$ on Figures 1-4) may be interpreted as medium-run equilibrium. $\mathbf{B}_{1\perp}$ is $M \times P_1$ projection matrix of $\mathbf{B}_{\perp}$ into common stochastic I(1) trends subspace, where the latter is the orthogonal complement of the classical matrix of baseline cointegrating vectors defined in equation 3).

It remains, however P_2 noncointegrating directions ($\mathbf{B}_{2\perp}^T \Delta \mathbf{Y}_{t-1}$ on Figures 1-4), so these stocks increments are not stationary, but I(1). It is not difficult to notice that $\mathbf{B}_{2\perp}$ is $M \times P_2$ projection matrix of $\mathbf{B}_{\perp}$ into common stochastic I(2) trends.

I(1) trends can be interpreted on the one hand as the first sum of stationary stochastic shocks, but on the other hand as specifically integrated I(2) trends. This shows the next difference between classic, long-run cointegration dependencies $\mathbf{BY}_{t-1}$ and medium-run $\mathbf{B}_{1\perp} \Delta \mathbf{Y}_{t-1}$. The latter relate to medium-run relationships between increments of non-integrated I(2) stocks or medium-run relationships between flows treated as an independent category in the system. The treatment of medium-run cointegration as a stage of two-stage cointegration is incorrect because adaptation processes end in the medium-run horizon and there are no further (ie stock cointegration type) adjustments to long-run equilibrium.

The polynomial cointegration mechanism is the following (see Figure 1). In the zero step (in the medium period) follows:

1) Achieving medium-run equilibrium CI(2,2) between some stocks, which state is perpetuated in the long-run. Applicable R_0 linearly independent dependency directions in the system.

2) Achieving an equilibrium between the flows (but not yet between stocks) CI(2,1) for further R_1 linearly independent dependency directions in the system.

3) Achievement of medium-run CI(1,1) equilibrium between some stock increments, however this state is not permanent. This cointegration can be treated more as the annihilation of certain common stochastic cycles, not as the elimination of stochastic trends, because there are not such I(1) shocks in the case of I(2).

The first step of polynomial cointegration is to eliminate these (at least medium-run) disequilibria that have not been liquidated in the medium-run. These are:

a) continuing non-stationary (but "only" I(1)) discrepancies between I(2) stock levels,

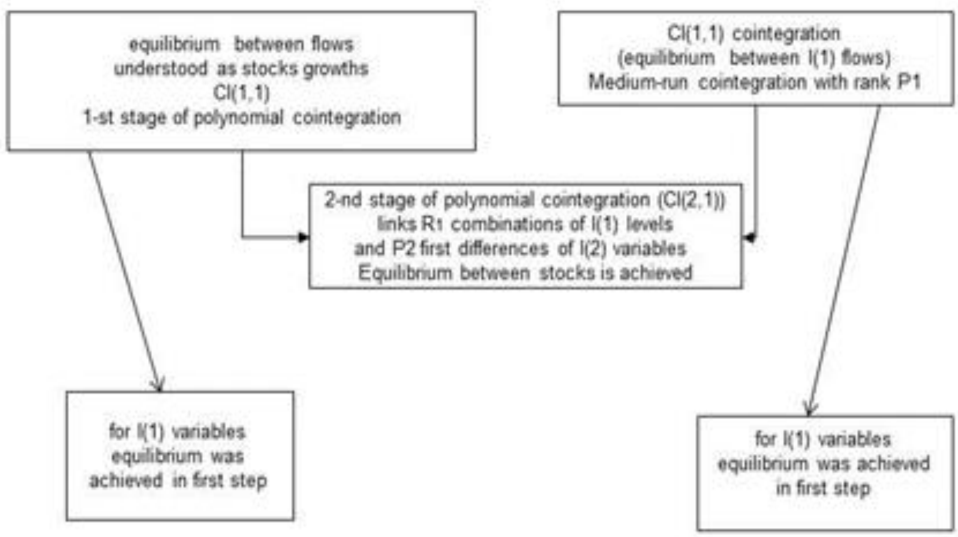
b) P_2 directions of dependencies, which could not be treated as cointegrating ($\mathbf{B}_{2\perp}^T \Delta \mathbf{Y}_{t-1}$ on Figures 1-4), that is, those stock increments (flows) that are not stationary, but still I(1).

The flows I(1) mentioned in a point b) are not mutually cointegrated. Medium-run cointegration is not, in contrast to CI(2,1), the first step to achieving stock equilibrium. $\mathbf{B}_{1\perp}^T \Delta \mathbf{Y}_{t-1}$ (in contrast to that in CI(2,1) $\mathbf{B}_1^T \mathbf{Y}_{t-1}$) shows that cointegration of variable levels will not be achieved. These flows, however, can cointegrate CI(1,1) with deviations from the relationship $\mathbf{B}_1^T \mathbf{Y}_{t-1}$, which are also (as noted) I(1). Therefore, these deviations play the role of specific catalysts for stock equilibrium in the system.

An interesting interpretation of polynomial integration is given by Juselius (1999). The variable, the variance of which can be explained by such a relationship, is subject to the error correction mechanism in relation to both the long-run equilibrium of the cointegration relationship and the dependence on the first increments.

Considerations for polynomial cointegration can be generalized to any order of integration of the variables in the model. In the case of the I(1) system, we are dealing with polynomial zero-order cointegration. The deviations from the cointegration of stock categories cointegrate with zero order increments (levels) of stocks. In the discussed case I(2) polynomial cointegration of the first stage takes place (deviations extinguish non-stationarity of the first increments of stocks). In the rather hypothetical case of I (3), it is necessary to consider the polynomial cointegration of the second order (deviations must extinguish non-stationarity of the stocks first increments or accelerants). The increasing order of integration of variables means, therefore, that the shocks affecting the economic variable are not so much intensifying in nature, but are becoming more and more complex. To describe them, an ARMA process of a higher and higher order is needed. The number of "steps" needed to achieve a full system equilibrium is exactly the highest order of integration of the variables used in the system, i.e. $N+1$, where N defines the order of cointegration polynomial.

Scheme 1. Mechanism of equilibrium achievements in I(2) model



Source: Own study

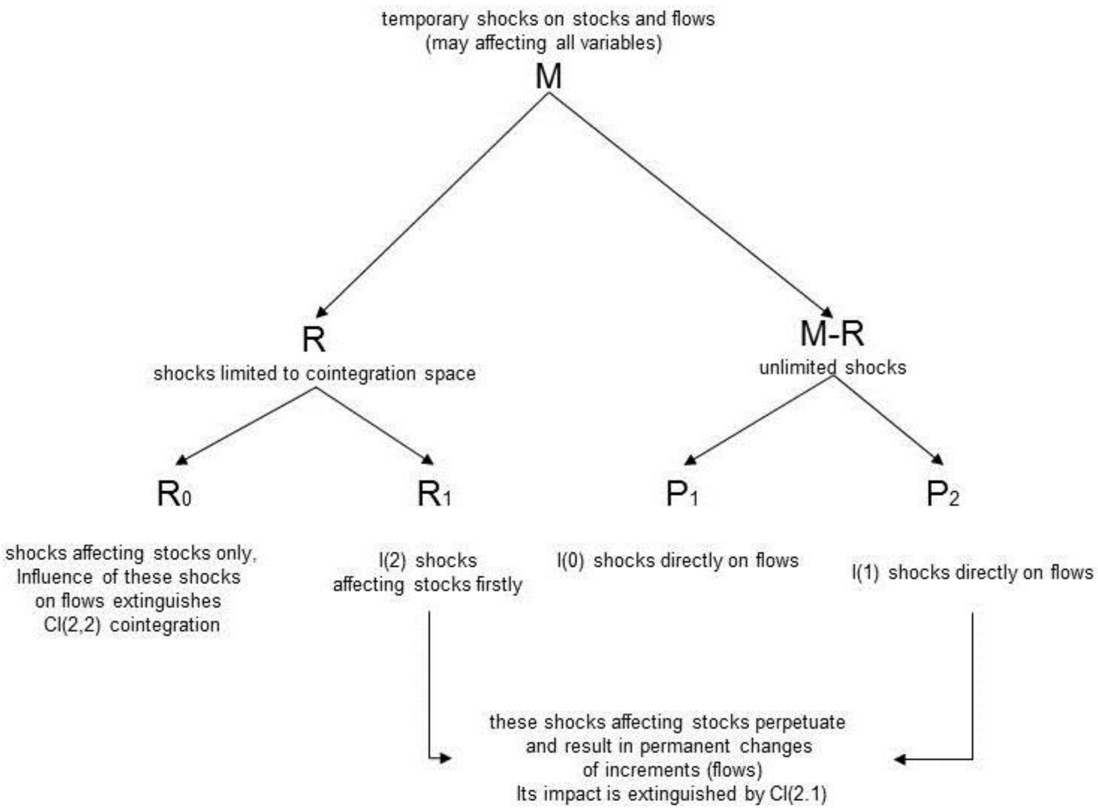
Polynomial cointegration is a cointegration of stock categories with flow categories (more precisely, the first flow category increments with "zero" flow category increments, hence: first-order polynomial cointegration), but strictly flow cointegration, i.e. CI(1,1). It occurs in the medium and becomes permanent in the long run. The polynomial cointegration relationship, in contrast to the medium-run cointegration, is therefore the leaven of stock cointegration (cointegration of flows consolidates to the cointegration of stocks). In the case of dependencies $\Gamma\Delta\mathbf{Y}_{t-1}$ we have the opposite situation: only the increments of the stock categories cointegrate, ie cointegrate their flows, here the integration of flows leads to a relatively faster integration of stock categories (elimination of the stochastic trends I(1) still present in them).

The polynomiality of this cointegration relation is also based on a few (in this case two, in the case of I(3) three levels) stages of achieving equilibrium in the economy. In the first one, the equilibrium of the flow is always achieved (and in the I(3) domain even the growth of the flow), in the second: the same equilibrium is achieved with respect to stocks.

5. Flows shocks and stock shocks. Shocks affecting flows and shocks affecting stocks

Considerations regarding flow and stock shocks as well as shocks affecting flows and shocks affecting stocks are, to a large extent, a mirror reflection of the considerations of achieving cointegration in the system. The direction of shocks impact (ie centrifugal forces) in the system is opposite to the direction of achieving equilibrium (ie centripetal forces). Shocks affecting stocks are more durable and their impact is visible faster (winning the lottery has an immediate and, under rationality assumption, a lasting effect on savings, but does not necessarily increase the flow of income, by changing the price we cause its permanent increase, but it does not necessarily mean consolidation of inflation shock). The mechanism for the precipitation of system I(2) from long-run equilibrium is shown in Scheme 2.

Scheme 2. Mechanism of system I(2) precipitation from long-run equilibrium



Source: Own study

In a multidimensional cointegration analysis, shocks affecting a given category are related to shock import matrices.

Consider the solution of the VECM model for case I(2) known as a representation of common stochastic trends:

$$\mathbf{Y}_t = C_1 \sum_{i=1}^{t} \Sigma_i + C_2 \sum_{j=1}^{t} \sum_{i=1}^{j} \Sigma_i + C(L)\Sigma_t, \tag{4}$$

where $C_2 = \mathbf{B}_{2\perp}(\mathbf{A}_{2\perp}^T(\Gamma\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\Gamma - \sum_{s=1}^{S-2}\Psi_s)\mathbf{B}_{2\perp}^T)^{-1}\mathbf{A}_{2\perp}^T$ is a matrix of parameters connecting with I(2) stochastic trends $\sum_{j=1}^{t}\sum_{i=1}^{j}\Sigma_i$.

$\mathbf{A}_{2\perp}^T$ matrix has a clear interpretation in the matrix category of basic stochastic trends I(2). $\mathbf{B}_{2\perp}$ matrix may be interpreted as the crucial component of weight matrix $\tilde{\mathbf{B}}_{2\perp} = \mathbf{B}_{2\perp}(\mathbf{A}_{2\perp}^T(\Gamma\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\Gamma - \sum_{s=1}^{S-2}\Psi_s)\mathbf{B}_{2\perp}^T)^{-1}$ connected with common stochastic trends I(2), defined as $\mathbf{A}_{2\perp}^T \sum_{j=1}^{t} \sum_{i=1}^{j} \Sigma_i$. The elements of this matrix inform about long-run shocks (the influence of stochastic trends I(2)) affecting system variables. $\tilde{\mathbf{B}}_{2\perp}$ matrix it is therefore a matrix of shock weights affecting stocks (and therefore these shock are permanent). It should be expected that for flow, and especially accelerant categories it is fulfilled $\tilde{\mathbf{B}}_{2\perp} \approx \mathbf{0}$. Consideration of the impact of such shocks therefore makes sense only for stock categories.

It should also be remembered that not all stocks in the system must be susceptible to long-run shocks, but only to cyclical shocks. There are also stock categories (see considerations in p.2) that are stationary changing around the deterministic trend.

With respect to flow categories, they may be sensitive to stochastic cyclical I(1) shocks and transition shocks I(0). The problem is that in the model of common stochastic trends I(2) it is not possible to determine the matrix responsible for the imports of such shocks.

For if the decomposition of the matrix C_2 is possible, with respect to C_1 econometricians could not extract the shock imports matrix. The matrix of coefficients with the I(1) trends identified with medium-run ones does not have a clear decomposition in contrast to a similar matrix from the representation of common stochastic trends for the model with I(1) variables only. In particular, it may not be clearly interpreted $\mathbf{B}_{1\perp}$ as the crucial component of weight matrix.

Thus, in the model with I(2) variables, the vulnerability of economic categories to the influence of stochastic cycles may not be directly examined. It is not such a solution to exclude from the system of I(2) variables and then to use a simpler representation of common stochastic trends I(1):

$$\mathbf{Y}_t = C\sum_{i=1}^{t} \Sigma_i + C(L)\Sigma_t, \qquad (5)$$

where due to decomposition $C = \mathbf{B}_\perp(\mathbf{A}_\perp^T(\sum_{s=1}^{S-1}\Gamma_s - \mathbf{I})\mathbf{B}_\perp)^{-1}\mathbf{A}_\perp^T = \tilde{\mathbf{B}}_\perp \mathbf{A}_\perp^T$ it is not difficult to determine the shock import matrix I(1) $\tilde{\mathbf{B}}_\perp$.

This is due to the following reasons. First of all, the removal of even one significant variable from the system will disrupt the entire system. C matrix is equivalent with C_1 only if C_2 is zero matrix, so model I(2) would be unnecessary. Secondly, in representation 5), the import of I(1) shocks is recorded, which consolidate into stochastic trends, i.e. to a long period, in the model 4) shocks I(1) have only a cyclical, oscillatory meaning around the trends dominating in the system I(2). The economic sense is quite different.

Shocks affecting the accelerants are only short-run, both in model I (1) and model I (2) are included in the $C(L)\Sigma_t$ component.

In I(1) domain shocks included in the matrix C they can affect both stocks (mainly) and flows. The difference is that we never treat these flows as stock increments, but as an intrinsic category. In turn in the component $C(L)\Sigma_t$ both shocks acting on flows and accelerants can be included.

From the point of view of the efficiency of economic activity, it is advantageous if, in the analyzed system, relevant rows of matrices $\tilde{\mathbf{B}}_{2\perp}$ related to key economic policy goals are non-zero, and especially if non-zero elements of this line correspond to the impact of those factors on which the decision maker has influence. This means that we can relatively easily and sustainably estimate influence key variables in the system.

In contrast to shocks affecting stock, shocks on flow and accelerant categories, much less regularity can be seen in stock and flow shocks. It is also worth noting that in this context it is better to use the terms: shocks from stock categories and shocks from flow categories. The shock by its nature is in fact a change, an impulse, and therefore a form of a flows. Stochastic shocks in the system can be treated exogenously. They are disturbances of the system equilibrium, contrary to the quite pejoratively sounding definition they can be both negative and desirable. They are complemented by shocks affecting the system through economic policy instruments or, more broadly, by any economic strategy (they can also affect the micro scale, where it is difficult to define economic policy).

The basic common stochastic trends I(2) can be defined as follows

$$\bar{\bar{a}}_{1n}\sum\sum\Sigma_{1t}+\ldots+\bar{\bar{a}}_{Mn}\sum\sum\Sigma_{Mt}\,, \qquad n=1,\ldots,P_2 \qquad (6)$$

where $\bar{\bar{a}}_{ij}$ is the element of $\mathbf{A}_{2\perp}$ matrix, which can be referred to as the exports shock matrix.

I(2) shocks do not have to be of stock character (in the sense of origin from the stock category). In fact, some seemingly weaker ones (derived from variables I(1), i.e. rather flow ones), can accumulate into I(2) shocks, i.e. long-run shocks.

In contrast to the analysis of the import of cyclic shocks, which as it was previously indicated is impossible, it is easy to analyze the sources of cyclic shocks I(1). The matrix serves this purpose is $\mathbf{A}_{1\perp}$. We can directly identify the elements of the $\mathbf{A}_{1\perp}$ matrix with coefficients defining common stochastic I(1) trends. The baseline common stochastic trends I(2) can be defined as follows:

$$\bar{a}_{1n}\sum\Sigma_{1t}+...+\bar{a}_{Mn}\sum\Sigma_{Mt}\ , \qquad n=1,...,P_1 \qquad (7)$$

where $\bar{a}_{ij}$ is the element of $\mathbf{A}_{1\perp}$ matrix.

Shocks I(1) have in the case of I(2) mostly sources in flow variables. However, this is not the rule. Sometimes shocks from stocks (derived from variable I(2) can be cointegrated with shocks I(2) sent by another variable of this system) and become only medium-run shocks. This means that such integrated shocks affect only the stocks, but not on the increment of variables, while the non-integrated shock I(2) permanently changes not only the level of the stock, but also the first increment (flow).

It is worth noting that shocks from stocks can be identified as growth-related shocks. An example is the fiscal shock of changing tax rates or the monetary shock associated with the change in interest rates.

Consequently, shocks from flow categories are often accelerated. This is particularly so when the flow can be reasonably considered in the growth category of the respective flow. For example, the intensification of inflation can be treated as a nominal acceleration shock coming from prices. However, there are such flow categories, from which the shock is very difficult to interpret in such a convention. For example, a real technological shock caused by the increase in labour productivity may be treated as a source of economic acceleration, but a large over-interpretation would be defining it as the second increase in accumulated efficiency.

On the other hand, it is difficult to relate shocks and flows with the commonly applied classification of shocks to nominal and real shocks. For example, both the change in both the nominal and the real money supply will be a flow shock (more specifically, growth-related).

6. Summary

Cointegration relationships of various orders are an immanent element of economic reality. The object of the discussion was therefore to translate complicated formulas of multi-dimensional cointegration analysis into more elementary concepts of economics.

The issue of inversion of analysis of common stochastic trends in relation to cointegration analysis is interesting. In the case of cointegration, it is the annihilation of common stochastic trends, so firstly such trends are removed from the flows, and only in the long-run stabilize each other (adjustment processes) stocks. If common trends are analyzed, the opposite sequence is true. It is much easier to permanently change the stock than the flow (for example, by changing the price we cause it to be permanently raised, but this does not necessarily mean consolidating the inflation shock). Changes affecting stock growth are therefore less long-run (in Juselius terminology: medium-run).

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