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Chapter 9
Points of quasicontinuity and of similar generalizations of continuity

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9.1 Introduction

The points of quasicontinuity were characterized by J. S. Lipiński and T. Šalát for the first time in 1970. There exist two very good topical surveys on quasicontinuous functions [53], [47]. Unfortunately, the results concerning quasicontinuity (and similar) points are scattered throughout the literature.

Let $P(f)$ be the set of all x such that f has the property P in x. To characterize $P(f)$, it means:

Let X and Y be topological spaces (satisfying, if it necessary, some conditions), let A be a set in X. Find a family $\mathcal{K}$ of sets in X such that $A = P(f)$ for some $f : X \to Y$ if and only if A belongs to $\mathcal{K}$.

We can also characterize the pairs $(P_1(f), P_2(f))$ in the sense of finding conditions on sets A_1 and A_2 such that $A_1 = P_1(f)$ and $A_2 = P_2(f)$ for some $f : X \to Y$. Of course, it is also possible to characterize an n-tuple $(P_1(f), \dots, P_n(f))$. Clearly, in these cases the necessary conditions are usually easy, however, the sufficient condition is difficult.

In this paper, we will deal with the property P "near" to quasicontinuity. For the pair $(P_1(f), P_2(f))$, a very frequent case is $P_1(f) = C(f)$ ($C(f)$ is the set of all continuity points of f) and $P_2(f) \supset C(f)$, or $P_2(f) = X$. A special case (if $P_1(f) = C(f)$ and $P_2(f)$ is the whole X) is to characterize points of discontinuity of functions with the property P_2 at each point.

Of course, there are many generalizations of countinuity. E.g., in [45], J. S. Lipiński characterized the pair $(C(f), D_b(f))$, where $D_b(f)$ is the set of all points at which f is Darboux. There exist many papers investigating Darboux-like points, eg. [14], [38], [39], [40], [57], [59]. However, it is a subject for another paper.

We use standard topological denotations. If (X,d) is a metric space, then $S(x,\eta)$ is the open ball centered at x with radius η, and $\mathrm{diam}(A)$ is the diameter of A.

9.2 Continuity

It is well-known that the set of points of continuity of a real-valued function on a topological space X is the countable intersection of open sets. It is obvious that such a set must contain all isolated points of X. It is natural to ask this question: does every G_δ set in X (which contains all isolated points of X) coincide with the set af all continuity points of some real-valued function on X? An affirmative answer to this question was given in the case of real line by W. H. Young in 1907, [61]. In 1932, H. Hahn in [34] showed that in fact any metric space X has this property. Therefore, we have

Theorem 9.1 ([34]). *Let (X,d) be a metric space and let A be a subset of X. Then $A = C(f)$ for some real-valued function $f : X \to \mathbb{R}$ if and only if A is a G_δ-set containing all isolated points of X.*

Of course, a natural question arises for which larger class of spaces this assertion still holds.

On the one hand, for the sufficient condition, it is sufficient to assume that X is an almost resolvable space.

Theorem 9.2 ([2]). *Let X be an almost resolvable topological space and let A be a subset of X. Then $A = C(f)$ for some function $f : X \to \mathbb{R}$ if and only if A is a G_δ set in X.*

In fact, it is sufficient to assume that the range is a first countable Hausdorff topological space which contains a non isolated point. A space X is resolvable if it is the union of two disjoint dense sets. A family of resolvable spaces is very large. Every first countable topological space without isolated points, every locally compact Hausdorff space without isolated points is resolvable ([36]); every linear topological space is resolvable ([2]). A topological space is almost resolvable ([2]) if it is the countable union of sets with empty interiors. Every resolvable space is almost resolvable, however, there are almost resolvable spaces which are not resolvable. Every separable topological space without isolated points is almost resolvable. There is a space without isolated points which is not almost resolvable ([2]). For some spaces Y (e.g. if Y is countable) the condition on X to be almost resolvable is also necessary for the existence of required function.

On the other hand, the range Y also cannot be arbitrary. If e.g. $Y = \{a_0, a_1, \dots\}$ with the topology generated by the base consisting of the sets $\{a_i\}$, $i \neq 0$, then each subset of $\mathbb{R}$ is the set of all continuity points of some function $f : \mathbb{R} \to Y$. It is well-known that the set $C(f)$ is a G_δ-set for a metric space (Y,d). In [27] it is shown that it is sufficient to assume that Y is a developable space and in [37] that Y can be assumed weakly developable. A space Y is weakly developable ([15]) if there is a sequence $(\mathcal{G}_n)_n$ of open covers of Y such that if $y \in G_n \in \mathcal{G}_n$ for each n and V is an open set containing y then $\bigcap_{1\le i\le n} G_i \subset V$ for some n. So, we have

Theorem 9.3 ([37]). *Let X be an almost resovable space and let Y be a non-discrete weakly developable space. Let A be a set in X. Then $A = C(f)$ for some $f : X \to Y$ if and only if A is a G_δ-set.*

9.3 Semicontinuity

Let $S(f)$ be the set of all upper semicontinuity points of f, i.e.
$S(f) = \{x : f(x) \ge \limsup_{t\to x} f(t)\}$. Further, let $S^l(f) = \{x : f(x) \le \liminf_{t\to x} f(t)\}$, $T(f) = \{x : f(x) > \limsup_{t\to x} f(t)\}$ and $T^l(f) = \{x : f(x) < \liminf_{t\to x} f(t)\}$.
The quintuplet $(S(f), S^l(f), C(f), T(f), T^l(f))$ was characterized by T. Natkaniec in 1983.

Theorem 9.4 ([49]). *Let S, S^l, C, T, T^l be subsets of $\mathbb{R}$. Then $S = S(f)$, $S^l = S^l(f)$, $C = C(f)$, $T = T(f)$ and $T^l = T^l(f)$ for some function $f : \mathbb{R} \to \mathbb{R}$ if and only if $S \cap S^l = C$, C is dense in the set $\operatorname{int}(S) \cup \operatorname{int}(S^l)$, C is a G_δ-set, $T \subset S \setminus C$, $T^l \subset S^l \setminus C$ and the set $T \cup T^l$ is countable.*

For a real function $f : \mathbb{R} \to \mathbb{R}$ let us define the qualitative upper limit at the point x as

$q\text{-}\limsup_{t\to x} f(t) = \inf\{y \in \mathbb{R} : \{t \in \mathbb{R} : f(t) < y\}$ is residual at $x\}$.
Similarly let us define the qualitative lower limit of f at x. Denote
$C_q(f) = \{x \in \mathbb{R} : q\text{-}\limsup_{t\to x} f(t) = f(x) = q\text{-}\liminf_{t\to x} f(t)\}$,
$S_q(f) = \{x \in \mathbb{R} : q\text{-}\limsup_{t\to x} f(t) \le f(x)\}$,
$T_q(f) = \{x \in \mathbb{R} : q\text{-}\limsup_{t\to x} f(t) < f(x)\}$,
$S_q^l(f) = \{x \in \mathbb{R} : q\text{-}\liminf_{t\to x} f(t) \ge f(x)\}$ and
$T_q^l(f) = \{x \in \mathbb{R} : q\text{-}\liminf_{t\to x} f(t) > f(x)\}$.

The triplet $(C_q(f), S_q(f), T_q(f))$ was characterized by Z. Grande in 1985 in [32] and the quintuplet $(C_q(f), S_q(f), S_q^l(f), T_q(f), T_q^l(f))$ by T. Natkaniec. In the proof it is assumed that every subset of $\mathbb{R}$ of cardinality less than continuum is of first category. So, if we assume Continuum Hypothesis or Martin's Axiom, then we have

Theorem 9.5 ([50]). . *Assume CH or MA. For every sets S, S^l, C, T, T^l in $\mathbb{R}$ the following condintions are equivalent:*

(i) $S \cap S^l = C$, $T \cup T_1 \in \mathcal{B}$, $T \subset S \setminus C$, $T^l \subset S^l \setminus C$, the sets $S \setminus C$ and $S^l \setminus C$ do not contain sets of second category having Baire property and there exists a G_δ-set D such that $C = D \setminus (T \cap T^l)$,

(ii) there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $S = S_q(f)$, $S^l = S_q^l(f)$, $C = C_q(f)$, $T = T_q(f)$ and $T^l = T_q^l(f)$.

9.4 Quasicontinuity and cliquishness

Recall that a function $f : X \to Y$ (X and Y are topological spaces) is said to be quasicontinuous at a point x if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there is an open nonempty set $G \subset U$ such that $f(G) \subset V$ ([41]).

A function $f : X \to Y$ (X is a topological space and (Y, d) is a metric space) is said to be cliquish at a point $x \in X$ if for each neighbourhood U of x and each $\varepsilon > 0$ there is an open nonempty set $G \subset U$ such that $d(f(y), f(z)) < \varepsilon$ for each $y, z \in G$ ([53]).

Denote by $Q(f)$ the set of all quasicontinuity points of f and by $A(f)$ the set of all cliquishness points of f.

The sets $Q(f)$ and $A(f)$ were characterized for the first time by J. S. Lipiński and T. Šalát in 1970. They showed that $A(f)$ is always closed and gave the following characterizations:

Theorem 9.6 ([46]). *Let (X,d) be a metric space without isolated points and let (Y,p) be a metric space containing some one-to-one Cauchy sequence. Let A be a subset of X. Then $A = A(f)$ for some $f : X \to Y$ if and only if A is closed.*

Theorem 9.7 ([46]). *Let (X,d) be a complete metric space dense in itself and let (Y,p) be a metric space possesing at least one accumulation point. Let A be a subset of X. Then $A = Q(f)$ for some $f : X \to Y$ if and only if the set* $\operatorname{int}(\operatorname{cl}(A)) \setminus A$ *is of first category (in the sense of Baire).*

If Y is a metric space, then evidently $C(f) \subset Q(f) \subset A(f)$. A. Neubrunnová in 1974 showed (see [54]) that the set $A(f) \setminus C(f)$ is of first category. J. Ewert and J. S. Lipiński investigated the triplet $(C(f),\ Q(f),\ A(f))$. For the sets C, Q and A denote

(*) $C \subset Q \subset A$, C is a G_δ-set, A is closed and $A \setminus C$ is of first category.

Therefore (*) is a necessary condition for the triplet $(C(f),Q(f),A(f))$. In [23] they showed that if X is a Baire real normed space and Y is a normed space then $(C,Q,A) = (C(f),\ Q(f),\ A(f))$ for some function $f : X \to Y$ if and only if we have (*). In [22] they showed that if (*) implies the equality $(C,Q,A) = (C(f),\ Q(f),\ A(f))$ for some function $f : X \to Y$ then for each closed set A in X there is a decreasing sequence $(U_n)_n$ of open sets such that $A = \bigcap_n \operatorname{cl}(U_n)$ and for each closed nowhere dense subset $F \subset X$ there is a continuous function $g : X \setminus F \to \mathbb{R}$ such that the oscillation $\omega_g(x) > 0$ for each point $x \in F$. Further, in [24], they showed that the necessary condition on X is not only $A = \bigcap_n \operatorname{cl}(U_n)$ but even the sets U_n are the same as for $C(f)$, therefore

(**) there is a decreasing sequence $(W_n)_n$ of open subsets of X such that $\bigcap_n W_n = C \subset Q \subset A = \bigcap_n \operatorname{cl}(W_n)$.

By [24], (**) implies (*). By [9], (*) does not imply (**), however, if X is perfectly normal, then (*) and (**) are equivalent.

Further generalizations of conditions on a space X are investigated in [9].

Theorem 9.8 ([9]). *Let X be a Baire resolvable perfectly normal locally connected space (or let X be a Baire pseudometrizable space without isolated points). Let (Y,p) be a metric space containing a subspace isometric with $\mathbb{R}$. Let C, Q and A be subsets of X. Then the following conditions are equivalent:*

(i) there is a function $f : X \to Y$ such that $C = C(f)$, $Q = Q(f)$ and $A = A(f)$,
(ii) $C \subset Q \subset A$, C is a G_δ-set, A is closed and $A \setminus C$ is of first category,
(iii) there is a decreasing sequence $(W_n)_n$ of open subsets of X such that $\bigcap_n W_n = C \subset Q \subset A = \bigcap_n \operatorname{cl}(W_n)$.

If $Q = X$ then the assumption "X is resolvable" can be omitted. So, we have

Theorem 9.9 ([9]). *Let X be a Baire perfectly normal locally connected space or X be a Baire pseudometrizable space. Then the set M is the set of all discontinuity points of some quasicontinuous function $f : X \to \mathbb{R}$ if and only if M is an F_σ-set of first category.*

In [58], the question to characterize the sets of discontinuity points of quasicontinuous functions $f : X \to \mathbb{R}$ (X is a topological space) is posed. It was solved in [28] for $X = \mathbb{R}^2$. Theorem 9.9 is not true if X is normal is replaced with X is T_1 completely regular, as the Niemytzki plane shows ([9]). Other characterization (for spaces not Baire only) we can find in [48].

Theorem 9.10 ([48]). *Let for a Fréchet-Urysohn space X at least one of the following conditions holds:*

(i) X is a hereditarily separable perfectly normal;
(ii) X is hereditarily quasi-separable perfectly normal;
(iii) X is a regular space with a countable net;
(iv) X is a paracompact with a σ-locally finite net;
(v) X is metrizable.

Then a set M is the set of all discontinuity points of some quasicontinuous function $f : X \to \mathbb{R}$ if and only if M is an F_σ-set of first category.

It is interesting that the pairs $(C(f),A(f))$ and $(Q(f),A(f))$ can be characterized under very general conditions, while for the pair $(C(f),Q(f))$ I known the same conditions on X and Y as for the triplet $(C(f),Q(f),A(f))$ only.

Theorem 9.11 ([9]). *Let X be a resolvable topological space and let (Y,p) be a metric space with at least one accumulation point. Let Q and A be subsets of X. Then there is a function $f : X \to Y$ such that $Q = Q(f)$ and $A = A(f)$ if and only if there is a decreasing sequence $(W_n)_n$ of open sets such that $\bigcap_n W_n \subset Q \subset \bigcap_n \mathrm{cl}(W_n) = A$.*

Theorem 9.12 ([9]). *Let X be a resolvable topological space and let (Y,p) be a metric space with at least one accumulation point. Let C and A be subsets of X. Then there is a function $f : X \to Y$ such that $C = C(f)$ and $A = A(f)$ if and only if there is a decreasing sequence $(W_n)_n$ of open sets such that $C = \bigcap_n W_n$ and $A = \bigcap_n \mathrm{cl}(W_n)$.*

If $A = X$ then the assumption of resolvability of X can be omitted and we obtain a characterization of the set of discontinuity points of cliquish functions.

Theorem 9.13 ([9]). *Let X be a topological space and let Y be a metric space with at least one accumulation point. Then a set M is the set of all discontinuity points of some cliquish function $f: X \to Y$ if and only if M is an F_σ set of first category.*

9.5 Bilateral quasicontinuity and cliquishness

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be left (right) hand sided quasicontinuous at a point $x \in \mathbb{R}$ if for every $\delta > 0$ and for every open neighbourhood of $f(x)$ there exists an open nonempty set $G \subset (x-\delta, x)$ ($G \subset (x, x+\delta)$) such that $f(G) \subset V$ [33]. A function f is bilaterally quasicontinuous at x if it is simultaneously left and right hand quasicontinuous at x. Denote by $Q^-(f)$, $Q^+(f)$ and $BQ(f)$ the set of all left hand side quasicontinuity points, right hand side quasicontinuity points and bilateral quasicontinuity points of f. In this case we can find a characterization of the sixtuple $(C(f), BQ(f), Q^-(f), Q^+(f), Q(f), A(f))$.

Theorem 9.14 ([6]). *Let C, D, D_1, D_2, Q and A be subsets of $\mathbb{R}$. Then $C = C(f)$, $D = BQ(f)$, $D_1 = Q^-(f)$, $D_2 = Q^+(f)$, $Q = Q(f)$ and $A = A(f)$ for some $f: \mathbb{R} \to \mathbb{R}$ if and only if $C \subset D = D_1 \cap D_2 \subset D_1 \cup D_2 = Q \subset A$, C is a G_δ-set, A is closed, $A \setminus C$ is of first category and $Q \setminus D$ is countable.*

Let Y be a topological space. If (X, d) is a metric space, we can generalize the bilateral quasicontinuity as follows: a function $f: X \to Y$ is S-quasicontinuous at x if for every neighbourhood V of $f(x)$ and every $y \in X$, $y \neq x$, there exists an open nonempty set $G \subset S(y, d(x,y))$ such that $f(G) \subset V$. Denote the set of all S-quasicontinuity points of f by $QS(f)$.

If X is a topological space, another definition of bilateral quasicontinuity is possible, too. We will say that a function $f: X \to Y$ is B-quasicontinuous at x if for every neighbourhood V of $f(x)$ and for every open connected set A such that $x \in \mathrm{cl}(A)$ there exists an open nonempty set $G \subset A$ such that $f(G) \subset V$. Denote the set of all B-quasicontinuity points of f by $QB(f)$. Evidently, if $X = \mathbb{R}$, the notions of bilateral quasicontinuity, B-quasicontinuity and S-quasicontinuity coincide. The characterizations of $QB(f)$ and $QS(f)$ are similar.

Theorem 9.15 ([3]). *Let X be a locally connected perfectly normal almost resolvable topological space. Let B be a set in X. Then $B = QB(f)$ for some function $f: X \to \mathbb{R}$ if and only if the set $\mathrm{cl}(B) \setminus B$ is of first category.*

Theorem 9.16 ([3]). *Let (X,d) be a metric space without isolated points such that* $\mathrm{cl}(S(x,\delta)) = \{y \in X : d(x,y) \le \delta\}$ *for each $x \in X$ and each $\delta > 0$. Let S be a subset of X. Then $S = QS(f)$ for some $f : X \to \mathbb{R}$ if and only if the set* $\mathrm{cl}(S) \setminus S$ *is of first category.*

Similarly, we can define an one sided and bilateral cliquishness. A function $f : \mathbb{R} \to Y$ ((Y,d) is a metric space) is said to be left-side (right-side) cliquish at $x \in \mathbb{R}$ if for each $\delta > 0$ and $\varepsilon > 0$ there is nonempty open set $G \subset (x-\delta, x)$ ($G \subset (x, x+\delta)$) such that $d(f(y), f(z)) < \varepsilon$ for each $y,z \in G$. A function f is bilaterally cliquish at x if it is both right-side and left-side cliquish at x [26]. Denote by $A^+(f)$, $A^-(f)$ and $BA(f)$ the sets of all points at which f is right-side cliquish, left-side cliquish and bilaterally cliquish, respectively. For a set $M \subset \mathbb{R}$ denote by $L^+(M)$ ($L^-(M)$) the set of all right-sided (left-sided) cluster points of M.

Theorem 9.17 ([7]). *Let A,B,C,D be subsets of $\mathbb{R}$. Then $A = A(f)$, $B = A^+(f)$, $C = A^-(f)$ and $D = BA(f)$ for some function $f : \mathbb{R} \to \mathbb{R}$ if and only if $L^+(A) \subset B$, $L^-(A) \subset C$, $A = B \cup C$, $D = B \cap C$ and the set $A \setminus D$ is countable.*

If X is a topological space (and (Y,d) a metric one) we say that a function $f : X \to Y$ is B-cliquish at x if for each $\varepsilon > 0$ and for each open set V with $x \in \mathrm{cl}(V)$ there is a nonempty open set $G \subset V$ such that $\mathrm{diam}\, f(G) < \varepsilon$. Denote by $AB(f)$ the set of all B-cliquishness points of f. We have $AB(f) \subset A(f)$ and the set $A(f) \setminus AB(f)$ is nowhere dense.

Theorem 9.18 ([7]). *Let X be a resolvable space and let M be a subset of X. Then $M = AB(f)$ for some $f : X \to Y$ if and only if $M = \bigcap_n M_n$ where M_n are open and such that* $\mathrm{int}(\mathrm{cl}(M_{n+1})) \subset M_n$.

9.6 Upper and lower quasicontinuity

A function $f : X \to \mathbb{R}$ is said to be upper (lower) quasicontinuous at x if for each $\varepsilon > 0$ and for each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $f(y) < f(x) + \varepsilon$ ($f(y) > f(x) - \varepsilon$) for each $y \in G$ ([25]). Denote by $E(f)$ the set of all points of both upper and lower quasicontinuity of f. In [31] it is shown that if a function $f : X \to \mathbb{R}$ is upper and lower quasicontinuous at each point then it is cliquish. However, the inclusion $E(f) \subset A(f)$ does not hold. Nevertheless, the set $E(f) \setminus A(f)$ is nowhere dense ([10]). According to [21], the set $E(f)$ is the countable intersection of semi-open sets. A set M is semi-open ([44]) (or quasi-open, see [53]) if $M \subset \mathrm{cl}(\mathrm{int}(M))$. For $X = \mathbb{R}$ we have a characterization of the quadruplet $(C(f), Q(f), E(f), A(f))$.

Theorem 9.19 ([10]). *Let C, Q, E and A be subsets of $\mathbb{R}$. Then $C = C(f)$, $Q = Q(f)$, $E = E(f)$ and $A = A(f)$ for some $f : \mathbb{R} \to \mathbb{R}$ if and only if $C \subset Q \subset A \cap E$, C is a G_δ-set, A is closed, $A \setminus C$ is of first category and $E \setminus A$ is nowhere dense.*

This characterization is not true for functions $f : \mathbb{R}^2 \to \mathbb{R}$. In fact, for $X = \mathbb{R}^2$ this characterization is not true for the triplet $(C(f), Q(f), E(f))$. The remaining triplets can be characterized for Baire metric spaces without isolated points. The triplet $(C(f), Q(f), A(f))$ is characterized in Theorem 9.8. Remaining two cases:

Theorem 9.20 ([8]). *Let X be a Baire metric space without isolated points. Let C, E and A be subsets of X. Then $C = C(f)$, $E = E(f)$ and $A = A(f)$ for some function $f : X \to \mathbb{R}$ if and only if $C \subset A \cap E$, C is a G_δ-set, A is closed, $A \setminus C$ is of first category and $E \setminus A$ is nowhere dense.*

Theorem 9.21 ([8]). *Let X be a Baire metric space without isolated points. Let Q, E and A be subsets of X. Then $Q = Q(f)$, $E = E(f)$ and $A = A(f)$ for some function $f : X \to \mathbb{R}$ if and only if $Q \subset A \cap E$, A is closed, $A \setminus Q$ is of first category and $E \setminus A$ is nowhere dense.*

9.7 Strong quasicontinuity

The set $Q(f)$ of points of quasicontinuity of a real function $f : \mathbb{R} \to \mathbb{R}$, in general, need not be Lebesque measurable. If a function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable then the set $Q(f)$ is measurable [42]. Similarly, although the set $Q(f) \setminus C(f)$ is of first category, it need not be measurable or of measure zero [12]. Even, there is a Darboux function such that the measure of $Q(f) \setminus C(f)$ is positive [43].

Remind that λ (λ^*) denote the Lebesgue measure (outer Lebesgue measure) in $\mathbb{R}$. Denote by $d_u(A,x) = \limsup_{r \to 0^+} \frac{\lambda^*(A \cap (x-r,x+r))}{2r}$ the upper outer density of $A \subset \mathbb{R}$ at a point $x \in \mathbb{R}$; similarly $d_l(A,x) = \liminf_{r \to 0^+} \frac{\lambda^*(A \cap (x-r,x+r))}{2r}$ is the lower outer density at x. Denote $\mathcal{T}_d = \{A \subset \mathbb{R} : A$ is measurable and for every $x \in A$ we have $d_l(A,x) = 1\}$. $\mathcal{T}_d$ is a topology called the density topology.

Z. Grande in [29] has defined some ”stronger“ quasicontinuities.
A function $f : \mathbb{R} \to \mathbb{R}$ has the property $A(x)$ at a point x if there is an open set U such that $d_u(U,x) > 0$ and the restriction $f \restriction (U \cup \{x\})$ is continuous at

x. A function f has property $B(x)$ at x if for every $\eta > 0$ we have $d_u(\mathrm{int}(\{t : |f(t) - f(x)| < \eta\}), x) > 0$. A function f is strongly quasicontinuous at x if for every $\eta > 0$ and every $U \in \mathcal{T}_d$ containing x there is a nonempty open set G such that $U \cap G \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in U \cap G$. Denote by $Q_A(f)$ the set of all points with property $A(x)$, by $Q_B(f)$ the set of all points with property $B(x)$ and by $Q_s(f)$ the set of all strong quasicontinuity points of f. Obviously $C(f) \subset Q_A(f) \subset Q_B(f) \subset Q_s(f) \subset Q(f)$. The set $Q_s(f) \setminus C(f)$ need not be of measure zero ([29]), however, the set $Q_B(f) \setminus C(f)$ is of measure zero ([30]). Moreover, the sets $Q_A(f)$ and $Q_B(f)$ have Baire propery, however, they need not be borelian. Futher, he gave a characterization of the set $Q_A(f)$.

Theorem 9.22 ([30]). *Let $A \subset \mathbb{R}$. Then $A = Q_A(f)$ for some $f : \mathbb{R} \to \mathbb{R}$ if and only if $A = \bigcup_m \bigcap_n A_{m,n}$, where $A_{m,n}$ be such that there are open sets G_n such that for each $m, n \in \mathbb{N}$ we have $d_u(\mathrm{int}(A_{m,n}), x) \geq 1/m$ for each $x \in A$, $A_{m,n+1} \subset A_{m,n}$, $A_{m,n} \subset A_{m+1,n}$, $G_{n+1} \subset G_n$, $G_n \subset A_{m,n}$ and $d_u(G_n, x) \geq 1/m$ for all $x \in A_{m,n}$.*

Also, there exist characterizations of the pairs $(C(f), Q_A(f))$ and $(C(f), Q_B(f))$.

Theorem 9.23 ([5]). *Let A and C be subsets of $\mathbb{R}$. Then $C = C(f)$ and $A = Q_A(f)$ for some function $f : \mathbb{R} \to \mathbb{R}$ if and only if there exist open sets G_n such that $C = \bigcap_n G_n \subset A$, $G_{n+1} \subset G_n$ and $\inf\{d_u(G_n, x) : n \in \mathbb{N}\} > 0$ for each $x \in A$.*

Theorem 9.24 ([5]). *Let B and C be subsets of $\mathbb{R}$. Then $C = C(f)$ and $B = Q_B(f)$ for some function $f : \mathbb{R} \to \mathbb{R}$ if and only if there exist open sets G_n such that $C = \bigcap_n G_n \subset B$, $G_{n+1} \subset G_n$ and $d_u(G_n, x) > 0$ for each $x \in B$.*

9.8 Simple continuity

Let X and Y be topological spaces. A set A is simply open if it is the union of open set and a nowhere dense set. A function $f : X \to Y$ is said to be simply continuous if the inverse image $f^{-1}(V)$ is a simply open set in X for each open set V in Y ([1]). Evidently, each quasicontinuous function is simply continuous. A suitable pointwise definition of simple continuity is given in [13]. We say that $f : X \to Y$ is simply continuous at a point $x \in X$ if for each open neighbourhood V of $f(x)$ and for each neighbourhood U of x, the set $f^{-1}(V) \setminus \mathrm{int}\left(f^{-1}(V)\right)$ is not dense in U. Denote by $N(f)$ the set of all simple continuity points of f. Then f if simply continuous if and only if $N(f) = X$ and moreover $Q(f) \subset N(f)$ ([13]).

Theorem 9.25 ([13]). *Let X be a perfectly normal resolvable space. Let Y be a metric space with at least one accumulation point. Further, let moreover, X be a Baire space and Y be separable (or let Y be totally bounded). Let $N \subset X$. Then $N = N(f)$ for some function $f : X \to Y$ if and only if* $\mathrm{cl}(\mathrm{int}(N)) \subset N$ *and the set* $\mathrm{cl}(N) \setminus N$ *is of first category.*

9.9 Closed graph

A function $f : X \to Y$ has a closed graph if the set $Gr(f) = \{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$. For a function f denote $C(f,x) = \bigcap\{\mathrm{cl}(f(U)) : U$ is a neighbourhood of $x\}$. We say that a function $f : X \to Y$ has a closed graph at x if $C(f,x) = \{f(x)\}$. Denote by $H(f)$ the set of all closedness graph points of f. Then f has closed graph if and only if $H(f) = X$ ([35], [55]).

Then characterizations of the set $H(f)$ and the pair $(C(f), H(f))$ are following.

Theorem 9.26 ([4]). *Let X be an almost resolvable topological space. Let H be a subset of X. Then $H = H(f)$ for some function $f : X \to \mathbb{R}$ if and only if H is a G_δ-set.*

Theorem 9.27 ([4]). *Let X be a Baire almost resolvable perfectly normal topological space. Let C and H be subsets of X. Then $C = C(f)$ and $H = H(f)$ for some function $f : X \to \mathbb{R}$ if and only if $C \subset H$, C and H are G_δ-sets, C is open in H and* $\mathrm{int}(H \setminus C) = \emptyset$.

There are examples that any condition on X cannot be omitted. By [19], the set of discontinuity points of a function with the closed graph is of first category and closed: and if moreover X is a Baire space, then it is even nowhere dense. Nevertheless, the set $H(f) \setminus C(f)$ can be residual and not closed (even for a function $f : \mathbb{R} \to \mathbb{R}$).

9.10 Generalized topology

Generalized continuities in the above section usually are not continuous (in a some suitable topology). Nevertheless, many of them are "continuous" in some weaker "topology". Let us recall some notions. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. We call a class $\mathfrak{g} \subset \mathcal{P}(X)$ a generalized topology (briefly GT, see [16]), if $\emptyset \in \mathfrak{g}$ and the arbitrary union of elements of $\mathfrak{g}$ belongs to $\mathfrak{g}$. A GT $\mathfrak{g}$ is strong if $X \in \mathfrak{g}$. A set X with a GT $\mathfrak{g}$ is called a generalized topological space (briefly, GTS) and is denoted by $(X,\mathfrak{g})$. For $x \in X$ we denote $\mathfrak{g}(x) = \{A \in \mathfrak{g} : x \in A\}$.

By [17], if $(X,\mathfrak{g})$ and $(Y,\mathfrak{h})$ are GTS's, then a mapping $f : X \to Y$ is called $(\mathfrak{g},\mathfrak{h})$-continuous, if $f^{-1}(V) \in \mathfrak{g}$ for each $V \in \mathfrak{h}$. A function $f : X \to Y$ is $(\mathfrak{g},\mathfrak{h})$-continuous at $x \in X$ if for each $V \in \mathfrak{h}(f(x))$ there is $U \in \mathfrak{g}(x)$ such that $f(U) \subset V$. By [17], a function f is $(\mathfrak{g},\mathfrak{h})$-continuous if it is such at each point. Denote by $C_{(\mathfrak{g},\mathfrak{h})}(f)$ the family of all $(\mathfrak{g},\mathfrak{h})$-continuity points of f.

In generally, the set $C_{(\mathfrak{g},\mathfrak{h})}(f)$ can be arbitrary. However, if (Y,d) is a metric space then this set is the countable intersection of sets from $\mathfrak{g}$. From now, we will assume that (Y,d) is a metric space. We will use the notion $\mathfrak{g}$-continuity for $(\mathfrak{g},d)$-continuity and $C_{\mathfrak{g}}(f)$ for continuity points $C_{(\mathfrak{g},d)}(f)$. By [18], $\mathcal{H} \subset \mathcal{P}(X)$ is a hereditary class, if $B \subset A \in \mathcal{H}$ implies $B \in \mathcal{H}$.

Theorem 9.28 ([11]). *Let $\mathfrak{g}$ be a GT on X and let Y be a metric space. If there is a function $f : X \to Y$ such that $C_{\mathfrak{g}}(f) = \emptyset$ and the set $f(X)$ is countable then there is a hereditary class $\mathcal{A} \subset \mathcal{P}(X)$ such that $\mathcal{A} \cap \mathfrak{g} = \{\emptyset\}$ and $X = \bigcup_{n\in\mathbb{N}} X_n$, where $X_n \in \mathcal{A}$ for $n \in \mathbb{N}$.*

Theorem 9.29 ([11]). *Let $\mathfrak{g}$ be a GT on X and let (Y,d) be a metric space with at least one accumulation point. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a hereditary class such that $\mathcal{A} \cap \mathfrak{g} = \{\emptyset\}$ and $X = \bigcup_{i\in\mathbb{N}} X_i$, where $X_i \in \mathcal{A}$. Let $M \subset X$. Then $M = C_{\mathfrak{g}}(f)$ for some $f : X \to Y$ if and only if $M = \bigcap_{n\in\mathbb{N}} M_n$, where $M_n \in \mathfrak{g}$ and $M_{n+1} \subset M_n$ for $n \in \mathbb{N}$.*

If $\mathcal{A}$ is the family of sets with empty interiors, we obtain

Theorem 9.30 ([11]). *Let X be an almost resolvable topological space and let (Y,d) be a metric space with at least one accumulation point. Let $\mathfrak{g}$ be a GT on X such that the interior of A is nonempty for each nonempty $A \in \mathfrak{g}$. Let $M \subset X$. Then $M = C_{\mathfrak{g}}(f)$ for some $f : X \to Y$ if and only if $M = \bigcap_{n\in\mathbb{N}} M_n$, where $M_n \in \mathfrak{g}$ and $M_{n+1} \subset M_n$.*

Let $(X,\mathcal{T})$ be a topological space. A set A is said to be semi-open if $A \subset \mathrm{cl}(\mathrm{int}(A))$, pre-open if $A \subset \mathrm{int}(\mathrm{cl}(A))$, β-open if $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and α-open if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. Denote the family of semi-open sets by $SO(X)$, the family of pre-open sets by $PO(X)$, the family of β-open sets by $\beta(X)$ and the family of α-open sets by $\alpha(X)$. All $SO(X)$, $PO(X)$, $\beta(X)$ and $\alpha(X)$ are GT's (in fact, $\alpha(X)$ is a topology). A function $f : X \to Y$ (X and Y are topological spaces) is semi-continuous (pre-continuous, β-continuous, α-continuous) at x if for every open neighbourhood V of $f(x)$ there is a set A from $SO(X)$ ($PO(X)$,

$\beta(X)$, $\alpha(X)$) containing x such that $f(A) \subset V$, respectively. Denote by $SO(f)$, $PO(f)$, $\beta(f)$, $\alpha(f)$ the set of all semi-continuity, pre-continuity, β-continuity and α-continuity points of f, respectively. In fact $SO(f) = Q(f)$ ([54]).

Now, in Thereom 9.30, if $\mathfrak{g}$ is the family of all open sets in X, we obtain the characterization of continuity points, if $\mathfrak{g}$ is the family of all semi-open sets, we obtain the characterization of quasicontinuity points, if $\mathfrak{g}$ is the family of all α-sets, we obtain the characterization of α-continuity points.

If $\mathcal{M}$ is the family of nowhere dense sets, we obtain the characterization of pre-continuity and β-continuity points on spaces of first category ([11]).

A function $f : X \times Y \to Z$ (X, Y and Z are topological spaces) is said to be quasicontinuous at (x,y) with respect to first (second) coordinate if for all neighbourhoods U, V and W of x, y and $f(x,y)$, respectively, there are nonempty open sets G and H such that $x \in G \subset U, H \subset V$ ($G \subset U, y \in H \subset V$) and $f(G \times H) \subset W$. A function f is symmetrically quasicontinuous if it is quasicontinuous both with respect to the first and the second coordinate ([56]). Denote by $Q_{sx}(f)$, $Q_{sy}(f)$ and $Q_{ss}(f)$ the set of all points at which f is quasicontinuous with respect to first coordinate, quasicontinuous with respect to second coordinate, symmetrically quasicontinuous, respectively. Then $C(f) \subset Q_{ss}(f) = Q_{sx}(f) \cap Q_{sy}(f) \subset Q_{sx}(f) \cup Q_{sy}(f) \subset Q(f)$ ([20]).

For $A \subset X \times Y$ and $x \in X$ ($y \in Y$) let

$$A_x = \{v \in Y : (x,v) \in A\}, \quad A^y = \{u \in X : (u,y) \in A\}.$$

Denote

$$SO_1(X,Y) = \{A \subset X \times Y : \text{ if } (x,y) \in A \text{ then } y \in \text{cl}\,((\text{int}(A))_x)\}$$

and

$$SO_2(X,Y) = \{A \subset X \times Y : \text{ if } (x,y) \in A \text{ then } x \in \text{cl}\left((\text{int}(A))_y\right)\}\}.$$

Then $(x,y) \in Q_{sx}(f)$ ($(x,y) \in Q_{sy}(f)$, $(x,y) \in Q_{ss}(f)$) if and only if for each neighbourhood W of $f(x,y)$ there is a set $A \in SO_1(X,Y)$ ($A \in SO_2(X,Y)$, $A \in SO_1(X,Y) \cap SO_2(X,Y)$) containing (x,y) such that $f(A) \subset W$ ([60]). It is easy to see that $SO_1(X,Y)$, $SO_2(X,Y)$ and $SO_1(X,Y) \cap SO_2(X,Y)$ are GT's. So, according to Theorem 9.30 we obtain this characterization.

Theorem 9.31. *Let X and Y be topological spaces such that $X \times Y$ is an almost resolvable topological space. Let (Z,d) be a metric space with at least one accumulation point. Let $M \subset X \times Y$. Then $M = Q_{sx}(f)$ ($M = Q_{sy}(f)$, $M = Q_{ss}(f)$) for some $f : X \times Y \to Z$ if and only if M is the countable intersection of a de-*

creasing sequence of sets from $SO_1(X,Y)$ ($SO_2(X,Y)$, $SO_1(X,Y) \cap SO_2(X,Y)$), respectively.

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