http://dx.doi.org/10.18778/7525-971-1.12

Chapter 12

On rings of Darboux-like functions. From questions about the existence to discrete dynamical systems

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2010 Mathematics Subject Classification: 26A15, 26A18, 54C40, 54C70, 54H25.
Key words and phrases: Darboux function, ring of functions, ideal of a ring of functions, Świątkowski property, strong Świątkowski property, Sharkovsky property, entropy point, generalized Fleissner condition, almost continuous function, first return continuity.

Introduction

A combination of considerations regarding algebraic structures of functions and topological properties of examined transformations is a common topic of many scientific papers. A lot of mathematical research and important theories are based on it. On the other hand, limiting considerations connected with topological aspects or measure theory to some algebraic structures gives completely new possibilities (e.g. in the context of dynamical systems it is visible in [2], [13], [14]). The facts mentioned above lead us in obvious way to the necessity of analyzing algebraic properties of classes of functions widely examined in the real functions theory. In this theory, Darboux-like functions play a particular role (e.g. basic properties of Darboux functions are presented at the beginning of the classical monograph connected with real functions theory [4]). Discovery that each real function of a real variable is a sum of two Darboux functions ([24]) became a starting point of looking for the answers to many questions connected with algebraic operations (addition, multiplication, lattice operations) performed on Darboux-like functions (e.g. [5], [6], [12], [20], [27], [29]).

In this chapter we will concentrate on the considerations connected with the rings of Darboux-like functions. It is a very wide issue so we have to limit it to basic topics. Note in fact, that with problems regarding rings of functions and its ideals (in algebraic sense) one can strictly relate the issues connected with ideals of sets and theory of density points and approximately continuous functions ([34]). However these considerations go beyond the scope of this chapter.

We will mainly focus on pointing out assumptions which guarantee the existence of rings of functions contained in fixed families of Darboux-like functions, examining its basic properties and, taking into account the directions signalled at the beginning, applying them in research connected with the discrete dynamical systems. To avoid analysis of very detailed issues we will sometimes only indicate the literature containing regarded facts.

Throughout this chapter we will use the classical symbols and notions. However, in order to avoid misunderstandings, we will present basic denotation, symbols and definitions which will be used in the next parts of the chapter.

Let f be a function. If A is a subset of the domain of f then the symbol $f \restriction A$ will stand for the restriction of f to A. The set of all continuity (discontinuity) points of f will be denoted by $\mathcal{C}(f)$ ($\mathcal{D}(f)$). Moreover we will use the notation $\mathcal{C}^*(f) = X \setminus \overline{\mathcal{D}(f)}$. If f is a real valued function then let us denote by $\mathcal{Z}(f)$ the zero set of f; i.e., $\mathcal{Z}(f) = f^{-1}(0)$. If $\mathcal{F}$ is a family of functions $f : X \to \mathbb{R}$ then put $\mathcal{D}[\mathcal{F}] = \bigcup_{f\in\mathcal{F}} \mathcal{D}(f)$ and $\mathcal{Z}[\mathcal{F}] = \{\mathcal{Z}(f) : f \in \mathcal{F}\}$.

For a function $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ we will use the following notations: $R^-(f,x_0) = \{\alpha \in \mathbb{R} : f^{-1}(\alpha) \cap (x_0 - \delta, x_0) \neq \emptyset$ for any $\delta > 0\}$ and $R^+(f,x_0) = \{\alpha \in \mathbb{R} : f^{-1}(\alpha) \cap (x_0, x_0 + \delta) \neq \emptyset$ for any $\delta > 0\}$.

If $f : X \to \mathbb{R}$ then $f^{<1>}(x) = f(x)$ and $f^{<n>}(x) = f^{<n-1>}(x) \cdot f(x)$ for $n > 1$.

If $f : X \to X$ then put $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$ for $n > 1$. A point x such that $f^M(x) = x$, but $f^n(x) \neq x$, for $n \in \{1,2,\ldots,M-1\}$ is called a *periodic point* of f of prime period M. The set of all periodic points of f of prime period M will be denoted by $\mathrm{Per}_M(f)$.

The symbol $\mathrm{const}_\alpha^{X,Y}$ will stand for the constant function from X to Y assuming value α.

If A is a subset of the domain of $f : X \to Y$ and $B \subset Y$, then we shall say that a set A f-covers a set B (denoted by $A \underset{f}{\to} B$) if $B \subset f(A)$.

The distance between a set $A \subset \mathbb{R}$ and a point $x \in \mathbb{R}$ (in the natural metric) will be denoted by $\operatorname{dist}(A,x)$.

In this paper we will consider several classes of functions, apart from the family of continuous functions, we will deal with Darboux functions or almost continuous functions. It should be noted that in our case, we limit most of these definitions (except continuous function and Darboux function) to the case of real functions of a real variable. However, these definitions can be naturally extended to the more general case. We start with definition of Darboux function in general case. Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be topological spaces. We shall say that $f : X \to Y$ is a Darboux function if an image of any connected set $A \subset X$ is a connected set. In the case of a real function f of a real variable the above definition is equivalent to the following intermediate value property: if x and y belong to the domain of f and α is any number between $f(x)$ and $f(y)$ then there exists a number z between x and y such that $f(z) = \alpha$.

Let $f : X \to \mathbb{R}$. We say that f is a Baire one function (or f is of the first class of Baire) if for any $a \in \mathbb{R}$ the sets $\{x \in X : f(x) < a\}$ and $\{x \in X : f(x) > a\}$ are F_σ type.

We say that a function f belongs to the class $\mathcal{B}_1^{**}$ if $\mathcal{D}(f) = \emptyset$ or $f \restriction \mathcal{D}(f)$ is continuous ([40]).

It is worth noting that the family $\mathcal{B}_1^{**}$ has been introduced in a connection with research regarding rings of functions and it is wider than the class of all continuous functions and is included in the class $\mathcal{B}_1^{*}$ ([31]).

Now, let X,Y be the unit intervals or $\mathbb{R}$ (with natural topology) and $f : X \to Y$ be a function.

A function f is approximately continuous if for any $x \in X$ there exists a Lebesgue measurable set $A_x \subset X$ such that

$$\lim_{h\to 0} \frac{\lambda(A_x \cap [x-h,x+h])}{2h} = 1 \text{ and } f(x) = \lim_{\substack{t\to x,\\ t\in A_x}} f(t).$$

Obviously if $X = [0,1]$ and $x = 0$ or $x = 1$ we consider $\lim_{h\to 0} \frac{\lambda(A_x\cap[x,x+h])}{h} = 1$ or $\lim_{h\to 0} \frac{\lambda(A_x\cap[x-h,x])}{h} = 1$, respectively. This kind of functions was considered for the first time by A. Denjoy in 1915 ([8]). Clearly, the family of all continuous functions from X to Y is a proper subset of the family of all approximately continuous functions from X to Y.

The next kind of functions we will consider are derivatives. It is known that the class of all approximately continuous functions is not contained in the class of derivatives but every bounded approximately continuous function is a derivative ([4]). In 1959 J. Stallings in paper [47] introduced the notion of almost continuity. We call a function f almost continuous if for any open set $U \subset X \times Y$ containing the graph of f, U contains the graph of some continuous function $g : X \to Y$. It is worth noting that every derivative (approximately continuous function) is an almost continuous function.

We shall say that f has the Świątkowski property (or is a Świątkowski function) if for any $x, y \in X$ such that $x < y$ and $f(x) \neq f(y)$ there exists $z \in \mathcal{C}(f)$ such that $z \in (x, y)$ and $f(z)$ belongs to the open interval of the ends $f(x)$ and $f(y)$. This kind of functions was introduced by T. Świątkowski and T. Mańk in 1982 ([28]). In the paper [26] one can find the following definition. A function f has the strong Świątkowski property if for any $x, y \in X$ such that $x < y$ and $f(x) \neq f(y)$ and any α between $f(x)$ and $f(y)$ there exists $z \in \mathcal{C}(f)$ such that $z \in (x, y)$ and $f(z) = \alpha$. This function is also called a strong Świątkowski function. If f is a strong Świątkowski function then it has the Świątkowski property and moreover it is a Darboux and quasi-continuous function ([27]).

We will use the following symbols for families of considered functions:
$Const(X)$ - the family of all constant functions defined on X,
$\mathcal{C}(X,Y)$ - the family of all continuous functions $f : X \to Y$,
$\mathcal{D}(X,Y)$ - the family of all Darboux functions $f : X \to Y$,
$\mathbb{S}(X,Y)$ - the family of all functions $f : X \to Y$ having the Świątkowski property,
$s\mathbb{S}(X,Y)$ - the family of all functions $f : X \to Y$ having the strong Świątkowski property,
$\mathcal{B}_1(X,Y)$ - the family of all functions $f : X \to Y$ of first Baire class,
$\mathcal{B}_1^{**}(X,Y)$ - the family of all functions $f : X \to Y$ from the class $\mathcal{B}_1^{**}$,
$\triangle'(X,Y)$ - the family of all derivatives from X to Y which are not approximately continuous functions,
$\mathcal{C}'_{ap}(X,Y)$ - the family of all approximately continuous functions $f : X \to Y$ which are not continuous functions,
$\mathcal{A}(X,Y)$ - the family of all almost continuous functions $f : X \to Y$.

In all the above notations if $X = Y$ we will write only one X, e.g. $\mathcal{D}(X)$ instead of $\mathcal{D}(X,X)$, $\mathbb{S}(X)$ instead of $\mathbb{S}(X,X)$ etc. If additionally $X = Y = \mathbb{R}$ then we will write shortly $\mathcal{D}$, $\mathbb{S}$ etc.

For brevity, if we wish to consider the intersection of two or three classes of functions, we shall write them next to each other (e.g. $\mathcal{D}\mathbb{S}(X,Y)$ or $\mathcal{D}\mathcal{B}_1(X)$).

The ring $\mathcal{R}$ of real functions defined on $[0,1]$ is called a complete ring if it contains the class of all continuous functions and the following condition is fulfilled:

$$\text{if } f,g \in \mathcal{R}, \text{ then } \max(f,g) \in \mathcal{R} \text{ and } \min(f,g) \in \mathcal{R}. \tag{12.1}$$

If $\mathcal{F}$ is a fixed class of functions and $f \in \mathcal{F}$ then the symbol $\mathfrak{R}_{\mathcal{F}}(f)$ will stand for the family of all rings of functions from $\mathcal{F}$ containing the function f. If we additionally assume that considered rings are extensions of some ring $\mathcal{W}$ then we will write $\mathfrak{R}_{\mathcal{F}}^{\mathcal{W}}(f)$. Moreover, if $\mathfrak{R}$ is a family of rings, then we will write $\widehat{\mathfrak{R}}$ to denote that all the rings belonging to $\mathfrak{R}$ satisfy condition (12.1).

For brevity of notation in the next parts of the chapter we will use the following rule. If $f : X \to Y$ then writing $\mathfrak{R}_{\mathcal{F}}^{\mathcal{W}}(f)$ we will assume that all the functions from the ring $\mathcal{W}$ and the family $\mathcal{F}$ are defined on X and their values belong to Y. For example, if $f : [0,1] \to \mathbb{R}$ is a Darboux function then we will write $\mathfrak{R}_{\mathcal{D}}^{\mathcal{C}}(f)$ instead of $\mathfrak{R}_{\mathcal{D}([0,1],\mathbb{R})}^{\mathcal{C}([0,1],\mathbb{R})}(f)$.

Let $\mathcal{R}$ be a ring. We will denote by $\mathfrak{I}(\mathcal{R})$ the set of all ideals of $\mathcal{R}$. If $f \in \mathcal{R}$ then the symbol $(f)_{\mathcal{R}}$ will stand for the ideal generated by f. An ideal $\mathcal{J} \in \mathfrak{I}(\mathcal{R})$ will be called an extension (restriction) of an ideal $\mathcal{J}_1 \in \mathfrak{I}(\mathcal{R})$ if $\mathcal{J}_1 \subset \mathcal{J}$ ($\mathcal{J} \subset \mathcal{J}_1$). An ideal $\mathcal{J}$ will be called a z-ideal if $f \in \mathcal{R}$ and $\mathcal{Z}(f) \in \mathcal{Z}[\mathcal{J}]$ implies $f \in \mathcal{J}$. Moreover, if $\mathcal{J}_2 \in \mathfrak{I}(\mathcal{R})$ is a z-ideal such that $\bigcap \mathcal{Z}[\mathcal{J}_2]$ is a nonempty closed set belonging to $\mathcal{Z}[\mathcal{J}_2]$, then we will called it z'-ideal. The set of all z'-ideals of $\mathcal{R}$ will be denoted by $\mathfrak{I}_{z'}(\mathcal{R})$. An ideal $\mathcal{J}$ is prime if $fg \in \mathcal{J}$ implies $f \in \mathcal{J}$ or $g \in \mathcal{J}$. A nonzero ideal $\mathcal{J}_0 \in \mathfrak{I}(\mathcal{R})$ is called essential if it intersects every nonzero ideal nontrivially. For $A \subset \mathcal{R}$ we write $Ann(A)$ to denote the set $\{\xi \in \mathcal{R} : \xi \cdot A = \{\mathrm{const}_0\}\}$, where const_0 stands for the constant function assuming value 0.

12.1 Rings of the real Darboux-like functions defined on topological spaces

The results presented in this part are based on the paper [43].

It is known that the family of all continuous functions defined on a topological space is a ring. Since each continuous function is a Darboux function, then for any topological space X one can create a ring of Darboux functions defined on X. In the context of our considerations this case is less interesting. That is why the question arises whether for each topological space there exists a ring of real Darboux functions defined on X containing at least one discontinuous function (we call such rings *essential Darboux rings*). We can extend the question: is there for any topological space a discontinuous Darboux function defined on it? The following theorem shows that even in the case of spaces with "very nice properties" such Darboux functions may not exist.

Theorem 12.1. *There exists a connected, uncountable, Hausdorff topological space X such that every Darboux function $f : X \to \mathbb{R}$ is constant.*

From the above theorem it is easy to conclude the following

Corollary 12.2. *There exists a connected, uncountable, Hausdorff topological space X for which there are no essential Darboux rings of real functions defined on X.*

In the context of the above results and the questions posed at the beginning of the section, the following problem seems to be fundamental: what kind of assumptions should we impose on the space X to obtain the existence of essential Darboux ring of real functions defined on X? The answer to this question is

Theorem 12.3. *If X is a connected and locally connected topological space such that there exists a nonconstant continuous function $f : X \to \mathbb{R}$, then there exists an essential Darboux ring of functions from X to $\mathbb{R}$.*

Of course obtaining the answer to one of the questions generates new problems, for example connected with the existence of essential Darboux rings consisting of such functions f that $\mathcal{D}(f) \subset \mathcal{Z}(f)$ (essential rings with this property will be called $*$-*rings*).

Theorem 12.4. *Let X be a non-singleton, connected and locally connected, perfectly normal topological space. Then for every point $x_0 \in X$ there exists a Darboux $*$-ring $\mathcal{R}$ of real functions defined on X such that $\mathcal{D}[\mathcal{R}] = \{x_0\}$.*

Of course the properties of such rings and properties of families of such rings may be examined. For example in [43] some properties of rings connected with cardinal functions were examined. However, the detailed considerations regarding these problems are beyond the scope of this chapter.

12.2 Rings of the real Darboux-like functions defined on the unit interval.

From now on till the end of the chapter we will refer the Darboux property exclusively to the natural topology. So if a topology T is given and we will write that each T-continuous function (i.e. continuous when we consider topology T in $[0,1]$) has the Darboux property then we will mean that each T-continuous function has the intermediate value property.

12.2.1 Rings of Darboux and Świątkowski functions.

The main results of this section are based on the statements contained in the papers [37] and [32].

At first one can notice that for Darboux functions, the Świątkowski property is equivalent to other properties frequently examined in real analysis.

Theorem 12.5. *A Darboux function $f : [0,1] \to \mathbb{R}$ has the Świątkowski property if and only if for any $x \in [0,1]$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(f)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = f(x)$.*

In [32] the statement being used in the proofs of theorems connected with rings of Świątkowski functions was proved. Before formulating this theorem we will briefly recall two notions. We call a function $f : [0,1] \to [0,1]$ *quasi-continuous* if for any $x \in [0,1]$ and any neighbourhood U of x and any neighbourhood V of $f(x)$ there exists a nonemty open set $W \subset U$ such that $f(W) \subset V$. We say that $f : [0,1] \to [0,1]$ has a *strong Blumberg set B* iff B is dense in $[0,1]$, $f \restriction B$ is continuous and for any nonempty open set $U \subset [0,1]$ the set $f(U \cap B)$ is dense in $f(U)$.

Theorem 12.6. *For Darboux function $f : [0,1] \to [0,1]$ the following conditions are equivalent:*

(i) f has the Świątkowski property,
(ii) f is quasi-continuous,
(iii) f has a strong Blumberg set.

It is worth noting that in [32] the Świątkowski property was defined also for functions defined on $\mathbb{R}^2$ and the theorem analogous to Theorem 12.6 was proved.

From Theorem 12.3 it follows immediately that there exists discontinuous Darboux function f such that $\mathfrak{R}_{\mathcal{D}}(f) \neq \emptyset$. In this case we can ask another question: what kind of assumptions should we impose on a Darboux function f to have $\mathfrak{R}^{\mathcal{C}}_{\mathcal{D}}(f) \neq \emptyset$? In particular one can ask whether the fact that f is a Darboux and Świątkowski function is a sufficient condition for the existence of a ring belonging to $\mathfrak{R}^{\mathcal{C}}_{\mathcal{D}}(f)$. The following theorem shows that the answer to this question is negative.

Theorem 12.7. *There exists a Darboux function $f : [0,1] \to \mathbb{R}$ having the Świątkowski property such that $\mathfrak{R}^{\mathcal{C}}_{\mathcal{D}}(f) = \emptyset$.*

Indeed, let $\mathfrak{C}$ denote the classical Cantor set and $\mathfrak{C}^*$ denote the set of all endpoints of the intervals "removed" from $[0,1]$ in construction of $\mathfrak{C}$ in even steps. For any component (a,b) of the set $[0,1]\setminus\mathfrak{C}$ "removed" from $[0,1]$ in the $(2n+1)$-th ($n=0,1,2,\dots$) step we will use the symbol h_a^b to denote a continuous function defined on (a,b) such that for any $c\in(a,b)$ we have $h_a^b((a,c))=h_a^b((c,b))=[0,a-\frac{a}{n+1}]$. For any component (a,b) of the set $[0,1]\setminus\mathfrak{C}$ "removed" from $[0,1]$ in the $2n$-th ($n=1,2,\dots$) step we will use the symbol h_a^b to denote a continuous function defined on (a,b) such that for any $c\in(a,b)$ we have $h_a^b((a,c))=h_a^b((c,b))=[b+\frac{b}{n},2]$. Define $f:[0,1]\to\mathbb{R}$ as follows:

$$f(x)=\begin{cases}2 & \text{for } x\in\mathfrak{C}^*,\\ 0 & \text{for } x\in(\mathfrak{C}\setminus\mathfrak{C}^*),\\ h_a^b(x) & \text{for } x\in(a,b),\ \text{where } (a,b) \text{ is a component of } [0,1]\setminus\mathfrak{C}.\end{cases}$$

The function f is a Darboux function and it has the Świątkowski property, but $\mathfrak{R}_{\mathcal{D}}^{\mathcal{C}}(f)=\emptyset$. The details of this example are presented in [37].

It is not difficult to check that the function constructed above is not of first Baire class. The question is whether the assumption that a considered function is of first class of Baire may improve the situation. The answer is positive:

Proposition 12.8. *If $f:[0,1]\to\mathbb{R}$ is Darboux and first class of Baire, then $\mathfrak{R}_{\mathcal{D}}^{\mathcal{C}}(f)\neq\emptyset$. Moreover if f also has the Świątkowski property, then $\mathfrak{R}_{\mathbb{S}}^{\mathcal{C}}(f)\neq\emptyset$.*

Indeed, let $\mathcal{K}$ be the set of all functions h of the form $h=h_0f^{<m>}+h_1f^{<m-1>}+\dots+h_{m-1}f+h_m$, where $h_0,h_1,\dots,h_m\in C([0,1],\mathbb{R})$ and $m\in\mathbb{N}$. It is easy to see that $f\in\mathcal{K}$, $C([0,1],\mathbb{R})\subset\mathcal{K}$ and $\mathcal{K}$ is a ring of functions. Applying the Young condition ([53]) we can show that $\mathcal{K}$ is a Darboux ring.

Similarly, Theorem 12.5 implies that $\mathcal{K}$ is a ring of functions having the Świątkowski property, whenever f has this property.

It should be mentioned here that in the case of Darboux Baire one functions it is possible to construct rings of functions from $\mathcal{DB}_1([0,1])$ in another way presented in [19] and [36].

However, let us notice that there exist Darboux (Świątkowski) functions not belonging to $\mathcal{B}_1([0,1],\mathbb{R})$ and for which construction of such a ring is possible.

Theorem 12.9. *Let $f:[0,1]\to\mathbb{R}$ be a Darboux function such that $\mathcal{D}(f)$ is a nowhere dense set and the following condition is satisfied* *for any* $x_0 \in [0,1]$ *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any component* J *of the set* $\mathcal{C}(f)$ *if* $\operatorname{dist}(J, x_0) < \delta$ *then* $\operatorname{dist}(f(J), f(x_0)) < \varepsilon$.

Then f *is a Świątkowski function and there exists a topology* $\mathcal{T}$ *in* $[0,1]$ *such that* $\mathcal{C}(([0,1],\mathcal{T}),\mathbb{R}) \in \widehat{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{D}}(f) \cap \widehat{\mathfrak{R}}^{\mathcal{C}}_{\mathbb{S}}(f)$.

All the above considerations regarding rings of Świątkowski functions were connected directly with the similar considerations regarding Darboux functions. This situation is not accidental, which is shown by the next theorem.

Theorem 12.10. *Let* $f : [0,1] \to \mathbb{R}$ *be a Świątkowski function such that* $\mathcal{D}(f) = \{x_0\}$. *Then* $\widehat{\mathfrak{R}}^{\mathcal{C}}_{\mathbb{S}}(f) \neq \emptyset$ *if and only if* f *is a Darboux function.*

In [17], [21], [52] the following issue was examined: at what assumptions regarding topology $\mathcal{T}$ finer than the natural topology of the real line do we have the equality $\mathcal{C} = \mathcal{C}(([0,1],\mathcal{T}),\mathbb{R})$ (this problem was also investigated in the case of more general spaces e.g. in [30])? The paper [22] presents a synthesis of the results on this issue. Natural complement to the considerations presented above is examining the possibility of creating a topology $\mathcal{T}$ finer than the natural topology of the real line such that the classes of real continuous and $\mathcal{T}$-continuous functions are different but the families of Baire one functions in both topologies coincide. Due to the considerations of this section, it seems to be natural to demand from the family of $\mathcal{T}$-continuous real functions to consist only of Darboux and Świątkowski functions. Then of course this family will be an essential and complete ring being an extension of the ring $\mathcal{C}([0,1],\mathbb{R})$.

Now let us formulate an adequate theorem.

Theorem 12.11. *There exists a topology* $\mathcal{T}^*$ *finer than a natural topology of* $[0,1]$ *fulfilling the following conditions:*

1. $\mathcal{C}([0,1],\mathbb{R}) \subsetneq \mathcal{C}(([0,1],\mathcal{T}^*),\mathbb{R})$,
2. *the families of all functions of first Baire class with respect to natural topology and to topology* $\mathcal{T}^*$ *coincide,*
3. *the ring* $\mathcal{C}(([0,1],\mathcal{T}^*),\mathbb{R})$ *consists of Darboux and Świątkowski functions.*

12.2.2 Rings of strong Świątkowski functions.

As it was pointed out in the introduction, functions with the strong Świątkowski property are Darboux functions, so considering rings consisting of strong Świątkowski functions in this chapter is entirely justified.

The considerations of this section are based on [23] and [44].

A sum of function with the strong Świątkowski property and linear function may not have the strong Świątkowski property. One can ask the following questions: for which functions f is there a complete ring of functions having the strong Świątkowski property and containing f and what is the form of functions belonging to such ring? Searching for answers to these questions has led to formulation of "generalized Fleissner condition".

We say that a function $f:[0,1]\to\mathbb{R}$ fulfills the *generalized Fleissner condition*, if f is a continuous function or $f\restriction\mathcal{D}(f)=\text{const}$
and for each $x\in\overline{D^{+}(f)}$ there exists a sequence $\{x_n\}\subset\mathcal{C}^{*}(f)$ such that $x_n\searrow x$ and $f(x_n)=f(x),n=1,2,\ldots$
and for each $x\in\overline{D^{-}(f)}$ there exists a sequence $\{y_n\}\subset\mathcal{C}^{*}(f)$ such that $y_n\nearrow x$ and $f(y_n)=f(x),n=1,2,\ldots.$

It is easy to show that the family of all functions fulfilling the generalized Fleissner condition is a proper subset of the class of all functions with the strong Świątkowski property.

Theorem 12.12. *If f fulfills the generalized Fleissner condition then*

$$\widehat{\mathfrak{R}}_{s\mathbb{SB}_1^{**}}(f)\neq\emptyset.$$

The proof of the above theorem is based on the observation that if f fulfills the generalized Fleissner condition and $\mathcal{D}(f)\neq\emptyset$ then there exists $\alpha\in\mathbb{R}$ such that $f\restriction\overline{\mathcal{D}(f)}=\text{const}_{\alpha}^{\overline{\mathcal{D}(f)},\mathbb{R}}$.

12.2.3 Ideals of rings of almost continuous functions.

According to the earlier considerations (e.g. Proposition 12.8) and due to the results presented in [19] and [36] it is easy to conclude the existence of rings of almost continuous functions containing discontinuous functions (so called: *essential almost continuous rings*). In reference to the previous section we can also formulate the following theorem.

Theorem 12.13. *If a function f fulfills the generalized Fleissner condition then* $\widehat{\mathfrak{R}}_{\mathcal{A}}(f)\neq\emptyset$.

Further considerations regarding existence of rings of almost continuous functions may be found in [44], while the next part of this section will be based on the paper [39].

In the study of algebraic properties of rings, ideals play a special role (e.g. [15]). In the remaining part of the section we examine this issue in relation to the rings of almost continuous functions. We shall consider the properties of ideals of some rings of almost continuous functions, being extensions of rings of continuous functions. Due to other observations in this section, these results can also be applied easily to the other classes of functions.

Let $f \in \mathcal{A}([0,1],\mathbb{R})$ be a function such that $\mathcal{D}(f) = \overline{\mathcal{D}(f)} \subset \mathcal{Z}(f)$. We will use the following notation $\widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f) = \{\mathcal{R} \in \mathfrak{R}^{\mathcal{C}}_{\mathcal{A}}(f) : \mathcal{D}(g) \subset \mathcal{D}(f) \text{ for } g \in \mathcal{R}\}$.

In the further considerations, if we write $\widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$ then we always assume that f is a fixed function belonging to $\mathcal{A}([0,1],\mathbb{R})$, such that $\emptyset \neq \mathcal{D}(f) = \overline{\mathcal{D}(f)} \subset \mathcal{Z}(f)$.

Our considerations start with the observation that the results included in the papers [3], [29], [33], [37], [38], [50] show, that for a function f satisfying the above assumptions, $\widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f) \neq \emptyset$ and, moreover, $\widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$ contains more than one ring.

The following theorem also shows some relationship between the ideals of the appropriate rings.

Theorem 12.14. *For each countable and closed set $P \subset [0,1]$, there exists a function $f : [0,1] \to \mathbb{R}$ such that $f \in \mathcal{A}([0,1],\mathbb{R})$ and $\mathcal{D}(f) = P$, for which there exist two families of rings $\{\mathcal{R}_\eta : \eta < \mathfrak{c}\}$, $\{\mathfrak{H}_\eta : \eta < \mathfrak{c}\} \subset \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$ such that $\mathcal{R}_{\eta_1} \neq \mathcal{R}_{\eta_2}$, $\mathfrak{H}_{\eta_1} \neq \mathfrak{H}_{\eta_2}$, $(f)_{\mathfrak{H}_{\eta_1}} \neq (f)_{\mathfrak{H}_{\eta_2}}$ ($\eta_1, \eta_2 < \mathfrak{c}$ and $\eta_1 \neq \eta_2$) and $(f)_{\mathcal{R}_{\eta_1}} = (f)_{\mathcal{R}_{\eta_2}}$ ($\eta_1, \eta_2 < \mathfrak{c}$).*

In many papers and monographs (e.g. [1], [15], [16]) the authors investigated the ideals of rings of continuous functions (often defined on more abstract space than $\mathbb{R}$). So, to begin with, let us note the relations between ideals of the rings of continuous functions and ideals of the rings belonging to $\widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$. First let us make some preliminary observations.

Remark 12.15. For an arbitrary ring $\mathcal{R} \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$, there exists an ideal $\mathcal{J}_0$ of the ring $\mathcal{C}([0,1],\mathbb{R})$ such that $\mathcal{J}_0 \notin \mathfrak{I}(\mathcal{R})$.

In fact. Let $[a,b] \subset (0,1)$ be a nondegenerate interval such that $[a,b] \cap \mathcal{D}(f) = \emptyset$ and $x_0 \in (a,b)$. Putting $\mathcal{J}_0 = \{h \in \mathcal{C}([0,1],\mathbb{R}) : h(x_0) = 0\}$ we obtain that $\mathcal{J}_0 \in \mathfrak{I}(\mathcal{C}([0,1],\mathbb{R}))$. Now, we consider a function $k : [0,1] \to \mathbb{R}$ defined by

$$k(x) = \begin{cases} 0 & \text{for } x = x_0, \\ 1 & \text{for } x \in [0,1] \setminus (a,b), \\ \text{linear} & \text{in the segments } [a,x_0] \text{ and } [x_0,b]. \end{cases}$$

Note that $k \in \mathcal{J}_0$ but $f \cdot k \notin \mathcal{J}_0$.

Since in this section rings of functions containing the family of all continuous functions are examined, so the following result seems to be interesting.

Theorem 12.16. *Let $\mathcal{R} \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$. For an arbitrary z'-ideal $\mathcal{J} \in \mathfrak{I}(\mathcal{C}([0,1],\mathbb{R}))$ for which $\bigcap \mathcal{Z}[\mathcal{J}]$ is not a singleton, there exist:*

(A) *an extension $\mathcal{J}^*$ of $\mathcal{J}$ which is a z'-ideal of $\mathcal{C}([0,1],\mathbb{R})$, such that $\mathcal{J}^* \notin \mathfrak{I}(\mathcal{R})$,*
(B) *a restriction $\mathcal{J}_*$ of $\mathcal{J}$ which is a z'-ideal of $\mathcal{C}([0,1],\mathbb{R})$, such that $\mathcal{J}_* \in \mathfrak{I}(\mathcal{R})$.*

The above theorem suggests considering in some sense opposite situation i.e. the following problem. We have a fixed ideal $\mathcal{J}$ of $\mathcal{R} \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$. Does there exist a restriction $\mathcal{J}_*$ of $\mathcal{J}$ such that $\mathcal{J}_*$ is an ideal of $\mathcal{C}([0,1],\mathbb{R})$ and $\mathcal{R}$? The following theorem gives the answer to this question.

Theorem 12.17. *Let f be a function for which $\mathcal{D}(f)$ is a countable set. For each ideal $\mathcal{J}$ of $\mathcal{R} \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$, there exists a restriction $\mathcal{J}_* \in \mathfrak{I}(\mathcal{C}([0,1],\mathbb{R})) \cap \mathfrak{I}(\mathcal{R})$. Moreover, if $\mathcal{J}$ is an essential ideal of $\mathcal{R}$, then we may assume that $\mathcal{J}_*$ is also an essential ideal of $\mathcal{R}$.*

Let us introduce some more notations. For a function $\xi : \mathbb{R} \to \mathbb{R}$ let ξ_α^β ($\alpha < \beta$) denote a function defined as follows (e.g. [4], p. 36):

$$\xi_\alpha^\beta(x) = \begin{cases} \beta & \text{if } \xi(x) \geq \beta, \\ \xi(x) & \text{if } \xi(x) \in [\alpha,\beta], \\ \alpha & \text{if } \xi(x) \leq \alpha. \end{cases}$$

Let $\mathcal{F}$ be a fixed family of functions. The symbol $\mathcal{F}_b$ will stand for the set $\{\xi_\alpha^\beta : \xi \in \mathcal{F} \wedge \alpha < 0 < \beta\}$. Moreover, if (X,ρ) is a metric space, $M \subset X$ and $x \in X$, then $p(M,x) = 2 \cdot \limsup_{R \to 0^+} \frac{\gamma(x,R,M)}{R}$, where for fix $R > 0$, $\gamma(x,R,M)$ is a supremum of the set of all positive r such that there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus M$ (here $B(y,\delta)$ denotes an open ball i.e. $B(y,\delta) = \{w \in X : \rho(y,w) < \delta\}$ for $y \in X$ and $\delta > 0$). We shall say that M is uniformly porous if there exists $m > 0$ such that $p(M,x) \geq m$ for any $x \in X$.

Theorem 12.18. *Let $\mathcal{J}$ be a nontrivial ideal of a ring $\mathcal{R} \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$. Then the set $A = Ann(\mathcal{J})$ has the following property: A_b is uniformly porous (in $\mathcal{R}_b$ which is endowed with the metric of uniform convergence).*

Theorem 12.19. *Let $\mathcal{R}_0 \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$. If $\mathcal{J} \in \mathfrak{I}_{z'}(\mathcal{R}_0)$, then $\mathcal{J}$ is an intersection of prime ideals.*

The proof is similar to that of Theorem 2.8 from [15] for rings of continuous functions.

Note that, if $\mathcal{R} \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$, then we can consider the set $\mathfrak{I}_{z'}(\mathcal{R})$ with the metric ρ_0 such that $\rho_0(\mathcal{J}_1,\mathcal{J}_2) = \rho_H(\bigcap \mathcal{Z}[\mathcal{J}_1],\bigcap[\mathcal{J}_2])$ for $\mathcal{J}_1,\mathcal{J}_2 \in \mathfrak{I}_{z'}(\mathcal{R})$, where $\rho_H(A,B) = \max(\sup_{a\in A}(\mathrm{dist}(a,B),\sup_{b\in B}(\mathrm{dist}(b,A)))$ for any closed sets $A,B \subset [0,1]$.

It is not hard to give an example of a z'-ideal which is not prime. So, the question arises whether this phenomenon is rare or frequent. The successive theorem is the answer to this question.

Theorem 12.20. *Let $\mathcal{P}$ be the set of all prime ideals of a ring $\mathcal{R}_0 \in \widetilde{\mathfrak{R}}^{\mathcal{C}}_{\mathcal{A}}(f)$. Then $\mathcal{P} \cap \mathfrak{I}_{z'}(\mathcal{R}_0)$ is a uniformly porous set in the space $(\mathfrak{I}_{z'}(\mathcal{R}_0),\rho_0)$.*

12.3 Rings of Darboux-like functions and problems connected with discrete dynamical systems.

12.3.1 The Sharkovsky property.

In [41] the following statement of M. Misiurewicz was quoted: *Combinatorial Dynamics has its roots in Sharkovsky's Theorem*. The basic version of this theorem concerns exclusively continuous functions. In [48] and [49] the theorem was generalized to the case of functions with connected and G_δ graphs (obviously such functions have the Darboux property).

This part of the chapter will be based on the papers [35] and [41].

Initial considerations are intended to highlight the main ideas connected with the issues presented in this section.

It is very useful to introduce the following notions. Let $(I_1,I_2,\dots,I_M)$ be a finite sequence of continuums ($I_i \subset \mathbb{R}$ for $i = 1,2,\dots,M$) and let $f_1,f_2,\dots,f_M : \mathbb{R} \to \mathbb{R}$. We say that $(I_1,I_2,\dots,I_M)$ is $(f_1,f_2,\dots,f_M)$-cycle if $I_1 \underset{f_1}{\to} I_2 \underset{f_2}{\to} I_3 \underset{f_3}{\to} \dots \underset{f_{M-1}}{\to} I_M \underset{f_M}{\to} I_1$. If $f_1 = f_2 = \dots = f_M = f$, we say that $(f_1,f_2,\dots,f_M)$-cycle $(I_1,I_2,\dots,I_M)$ is (f)-cycle. If $x_0 \in I_1$ is a point such that $(f_i \circ \dots \circ f_1)(x_0) \in I_{i+1}$ for $i \in \{1,2,\dots,M\}$, we shall say that x_0 is *connected with* an $(f_1,f_2,\dots,f_M)$-cycle $(I_1,I_2,\dots,I_M)$.

We shall say that (f)-cycle $(J_1,J_2,\dots,J_M)$ predominates $(f_1,f_2,\dots,f_M)$-cycle $(I_1,I_2,\dots,I_M)$ if for each $i \in \{1,2,\dots,M\}$, there exists a homeomorphic embedding $\xi_i : J_i \to I_i$ such that $(f_i \circ \cdots \circ f_1)(\xi_1(x)) = \xi_{i+1}(f^i(x))$ for each point x connected with (f)-cycle $(J_1, J_2, \ldots, J_M)$.

A family of functions $\mathcal{F}$ is substituted by a class of functions $\mathcal{F}_1$ if for any $M \in \mathbb{N}$ and any arbitrary $(f_1, f_2, \ldots, f_M)$-cycle $(I_1, I_2, \ldots, I_M)$, where $f_1, f_2, \ldots, f_M \in \mathcal{F}$, there exists an (f)-cycle $(J_1, J_2, \ldots, J_M)$ which predominates $(f_1, f_2, \ldots, f_M)$-cycle $(I_1, I_2, \ldots, I_M)$ such that $f \in \mathcal{F}_1$.

We shall say that a family of functions $\mathcal{F}$ has the property $\mathfrak{J}_1$ if for any (f)-cycle $(I_1, I_2, \ldots, I_M)$ $(f \in \mathcal{F}_1)$ there exists a point x_0 connected with this cycle and such that $f^M(x_0) = x_0$.

If $\mathcal{F}$ is a family of real functions of a real variable then we shall denote $\mathcal{F}^c = \{f_n \circ f_{n-1} \circ \cdots \circ f_1 : f_1, f_2, \ldots, f_n \in \mathcal{F}, n \geq 1\}$.

First, we are going to establish two classes of functions $\mathcal{P}_\mathcal{C}$ and $\mathcal{P}_\mathcal{D}$, which will form a model for our considerations. Let us note that some functions belonging to $\mathcal{P}_\mathcal{C}$ ($\mathcal{P}_\mathcal{D}$) were considered in many papers and monographs (e.g. [4], [5]).

Let P be an arbitrary Cantor-like set in $[0,1]$ (additionally we assume that $0, 1 \in P$) and let $P' \subset P$. Then we can distinguish some properties of functions $f_{P',P} : \mathbb{R} \to \mathbb{R}$ which are connected with the sets P' and P.

(P_1) $f_{P',P}(x) = 0$ if $x \in P \setminus P'$ and if $P' \neq \emptyset$, then $f_{P',P}(x) = 1$, if $x \in P'$.

(P_2) $f_{P',P} \restriction [a,b]$ is a continuous function and $f_{P',P}([a,b]) = [0,1]$ for any connected component (a,b) of $[0,1] \setminus P$.

(P_2') $f_{P',P} \restriction [a,b]$ is a continuous function, $f_{P',P} \restriction [a,b]$ is a Darboux function and $f_{P',P}([a,b]) = [0,1]$ for any connected component (a,b) of $[0,1] \setminus P$.

(P_3) $f_{P',P}(x) = f_{P',P}(0)$, for $x < 0$ and $f_{P',P}(x) = f_{P',P}(1)$, for $x > 1$.

Let us denote by $\mathcal{P}_\mathcal{C}$ ($\mathcal{P}_\mathcal{D}$) a family of all functions $f_{P',P}$ fulfilling conditions (P_1), (P_2), (P_3) ((P_1), (P_2'), (P_3)) for all possible pairs of sets (P, P'). It is easy to see that $\mathcal{P}_\mathcal{C} \subset \mathcal{D}$ ($\mathcal{P}_\mathcal{D} \subset \mathcal{D}$) and, moreover, both classes contain nonmeasurable (in the Lebesgue sense) functions, if the measure of P is positive and P' is a nonmeasurable set. Moreover, one can remark that the family $\mathcal{P}_\mathcal{C}$ is substituted by the family $\mathcal{C}$ and the family $\mathcal{P}_\mathcal{D}$ is substituted by the family $\mathcal{DB}_1$.

It should be mentioned here that we can consider various modifications of our models. For example, we can replace the condition (P_2) (and (P_2')) with

(P_2'') $f_{P',P} \restriction [a,b] \in \mathcal{DB}_1$ and $f([a,b]) = [0,1]$, for any component of $[0,1] \setminus P$.

Then such a family is also substituted by the family $\mathcal{DB}_1$.

Moreover, the assumption 3 suggests that one can consider the functions mapping $[0,1]$ into itself.

Now, we can define the notion of Sharkovsky function. First we should consider the following *Sharkovsky ordering* of the set of all positive integers:

$$3 \prec 5 \prec 7 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec \cdots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 2^0 = 1.$$

We shall say that f is a *Sharkovsky function* provided that if $\mathrm{Per}_M(f) \neq \emptyset$ and $M \prec N$, then $\mathrm{Per}_N(f) \neq \emptyset$.

Theorem 12.21. *Let us suppose that $\mathcal{F}$ is a family of functions substituted by a family $\mathcal{F}_1$ and the family $\mathcal{F}_1$ has the property $\mathfrak{J}_1$. Then each function $f \in \mathcal{F}^c$ is a Sharkovsky function.*

Since $\mathcal{C}$ and $\mathcal{DB}_1$ have the property $\mathfrak{J}_1$, then both families $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{D}}$ consist of Sharkovsky functions.

At the end of twentieth century, a team of American mathematicians considered issues related to the theory, which can generally be called: "first return" ([9], [10], [11]). It is worth noting that the first return continuous functions have the Darboux property. In the next part we will use this theory to build our own solutions leading to defining wide class of Darboux functions (see Theorem 12.22).

A set $H \subset \mathbb{R}$ is called an od-set if H is an open and dense subset of $\mathbb{R}$. Let H be an od-set and $f : \mathbb{R} \to \mathbb{R}$ be a function. We shall say that a set H f-replaces $\mathbb{R}$ (denoted by $H \underset{f-r}{\to} \mathbb{R}$) if for any nondegenerated interval $[\alpha,\beta] \subset \mathbb{R}$ there exists $(a,b) \subset [\alpha,\beta] \cap H$ such that $[a,b] \underset{f}{\to} f([\alpha,\beta])$.

The idea of the notions below derives from [7], [9], [10] and [11]. Let H be an od-set in $\mathbb{R}$ and $\{d_n\}_{n\in\mathbb{N}} \subset H$ be a fixed H-trajectory (i.e. $\{d_n\}_{n\in\mathbb{N}}$ is a sequence of distinct points such that $\{d_n : n \in \mathbb{N}\}$ is dense set in H). For $x \in \mathbb{R}$ the left first return path to x based on $\{d_n\}_{n\in\mathbb{N}}$, $P_x^l = \{t_k : k \in \mathbb{N}\}$ is defined as follows: t_1 is the first element of the sequence $\{d_n\}_{n\in\mathbb{N}}$ in the set $(-\infty, x)$, for $t \in \{2,3,\dots\}$ the element t_{k+1} is the first element of the sequence $\{d_n\}_{n\in\mathbb{N}}$ in the set (t_k, x). The right first return path to x based on $\{d_n\}_{n\in\mathbb{N}}$, $P_x^r = \{s_k : k \in \mathbb{N}\}$ is defined analogously. A function $f : \mathbb{R} \to \mathbb{R}$ is first return continuous from the left (right) at x with respect to the H-trajectory $\{d_n\}_{n\in\mathbb{N}}$ if

$$\lim_{\substack{t\to x\\ t\in P_x^l}} f(t) = f(x) \left(\lim_{\substack{t\to x\\ t\in P_x^r}} f(t) = f(x) \right).$$

A function $f:\mathbb{R}\to\mathbb{R}$ is an $(H,\{d_n\}_{n\in\mathbb{N}})$-first return continuous function ($f\in FRC(H,\{d_n\}_{n\in\mathbb{N}})$) if it is first return continuous with respect to the H-trajectory $\{d_n\}_{n\in\mathbb{N}}$ from the left and right at each point $x\in H$ and for any component (a,b) of the set H, f is first return continuous with respect to the H-trajectory $\{d_n\}_{n\in\mathbb{N}}$ from the right (left) at a (b).

We shall call f an $\mathcal{S}(H,\{d_n\}_{n\in\mathbb{N}})$-function ($f\in\mathcal{S}(H,\{d_n\}_{n\in\mathbb{N}})$) if $H \underset{f-r}{\to} \mathbb{R}$ and $f\in FRC(H,\{d_n\}_{n\in\mathbb{N}})$.

We say that $f:\mathbb{R}\to\mathbb{R}$ is an $\mathcal{S}$-function ($f\in\mathcal{S}$) provided that there exists an od-set H and an H-trajectory $\{d_n\}_{n\in\mathbb{N}}$ such that $f\in\mathcal{S}(H,\{d_n\}_{n\in\mathbb{N}})$.

The following theorem justifies considering the class $\mathcal{S}$ in the context of the Darboux-like functions.

Theorem 12.22. *If $f:\mathbb{R}\to\mathbb{R}$ is an $\mathcal{S}$-function, then f is a Darboux function.*

With reference to our considerations and the above statement it seems to be interesting to ask the following question: what kind of assumption should we impose on f in order to have guaranteed the existence of a ring belonging to $\widehat{\mathfrak{R}}^{Const}_{\mathcal{S}}(f)$?

Theorem 12.23. *Let $f\in\mathcal{S}$. Then there exists a ring $\mathcal{R}\in\widehat{\mathfrak{R}}^{Const}_{\mathcal{S}}(f)$.*

12.3.2 Rings of Darboux-like functions and entropy points.

In the introduction to this chapter it was noted that in the case of dynamical systems, some algebraic structures of functions are often considered (e.g. [46], [51]). This section will deal with rings of Darboux-like functions in the context of local interpretation of entropy. The results presented here are based on the papers [42] and [45].

We will start with introducing the concept of almost fixed point. Let $f:[0,1]\to[0,1]$ be a Darboux function. We will say that a point x_0 is an *almost fixed point* of f if

$$x_0\in \operatorname{int}(R^-(f,x_0))\cup\operatorname{int}(R^+(f,x_0)).$$

If $x_0=0$ or $x_0=1$, then we only consider $R^+(f,x_0)$ or $R^-(f,x_0)$, respectively. From now on, $\operatorname{aFix}(f)$ stands for the set af all almost fixed points of f and $\operatorname{Fix}(f)$ denotes the set af all fixed points of f.

It should be mentioned here that the notion of almost fixed point was created on the basis of conception of Darboux point presented by J. Lipiński in [25].

In the theory of discrete dynamical systems, the question regarding symmetry of properties of conjugate functions is essential. The following statement refers to this question with respect to the possession of almost fixed points by such functions.

Theorem 12.24. *If $f, g : [0,1] \to [0,1]$ are topologically conjugate via a homeomorphism ϕ (i.e. $\phi \circ f = g \circ \phi$), and $x_0 \in \operatorname{aFix}(f)$, then $\phi(x_0) \in \operatorname{aFix}(g)$.*

It is easy to see that the function $f : [0,1] \to [0,1]$ defined by the formula: $f(0) = \frac{1}{2}$ and $f(x) = |\sin \frac{1}{x}|$ for $x \in (0,1]$ belongs to the class $\mathcal{DB}_1$ and $0 \in \operatorname{aFix}(f) \setminus \operatorname{Fix}(f)$. However, the next theorem shows, that in the case of function $f \in \mathcal{DB}_1([0,1])$, in every neighborhood of any almost fixed point of f one can find a fixed point of f.

Theorem 12.25. *Let $f \in \mathcal{DB}_1([0,1])$ and let $x_0 \in \operatorname{aFix}(f)$. Then $(x_0 - \varepsilon, x_0 + \varepsilon) \cap \operatorname{Fix}(f) \neq \emptyset$ for each $\varepsilon > 0$.*

Our considerations are limited to the real functions defined on the interval $[0,1]$. However, it should be noted that all the following definitions, Theorem 12.27 and Remark 12.28 may be formulated for more general spaces ([44]).

Let $f : [0,1] \to [0,1]$. An f-bundle B_f is a pair $(\mathcal{F}, J)$ consisting of a family $\mathcal{F}$ of pairwise disjoint (nonsingletons) continuums in $[0,1]$ and a connected set $J \subset [0,1]$ (fibre of bundle) such that $A \underset{f}{\to} J$ for any $A \in \mathcal{F}$. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $B_f = (\mathcal{F}, J)$ be an f-bundle. A set $M \subset \bigcup \mathcal{F}$ is (B_f, n, ε)-separated if for each $x, y \in M$, $x \neq y$ there is $0 \leq i < n$ such that $f^i(x), f^i(y) \in J$ and $\rho(f^i(x), f^i(y)) > \varepsilon$. Let

$$\operatorname{maxsep}[B_f, n, \varepsilon] = \max\{\operatorname{card}(M) : M \subset [0,1] \text{ is } (B_f, n, \varepsilon)\text{-separated set}\}.$$

The entropy of the f-bundle B_f is the number

$$h(B_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[\frac{1}{n} \log \left(\operatorname{maxsep}[B_f, n, \varepsilon] \right) \right].$$

We shall say that a sequence of f-bundles $B_f^k = (\mathcal{F}_k, J_k)$ converges to a point x_0 (written $B_f^k \underset{k\to\infty}{\longrightarrow} x_0$), if for any $\varepsilon > 0$ there exists $k_0 \in N$ such that $\bigcup \mathcal{F}_k \subset B(x_0, \varepsilon)$ and $B(f(x_0), \varepsilon) \cap J_k \neq \emptyset$ for any $k \geq k_0$.

Putting

$$E_f(x) = \{\limsup_{n \to \infty} h(B_f^n) : B_f^n \underset{n\to\infty}{\longrightarrow} x\}$$

we obtain a multifunction $E_f \colon X \multimap \mathbb{R} \cup \{+\infty\}$.

We shall say that a point $x_0 \in [0,1]$ is an entropy point of f if $h(f) \in E_f(x_0)$ (where $h(f)$ denotes an entropy[1] of a function f). If in addition we require that $x_0 \in \mathrm{Fix}(f)$, then such a point will be called a strong entropy point of f. The family of all functions $f : [0,1] \to [0,1]$ having an entropy point (a strong entropy point) will be denoted by $\mathfrak{E}([0,1])$ ($\mathfrak{E}_s([0,1])$).

Theorem 12.26. *Let f be a Darboux function. If $x_0 \in \mathrm{aFix}(f) \cap \mathrm{Fix}(f)$ then x_0 is a strong entropy point of f.*

The following theorem shows that the notion of an almost fixed point is "dynamically invariant".

Theorem 12.27. *Let functions $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ be topologically conjugate. Then $f \in \mathfrak{E}([0,1])$ if and only if $g \in \mathfrak{E}([0,1])$.*

The above theorem is still true if we replace $\mathfrak{E}([0,1])$ with $\mathfrak{E}_s([0,1])$.

Let $\mathcal{F}$ be some class of functions from the unit interval into itself. We shall say that a function $f : [0,1] \to [0,1]$ is $\mathcal{T}_\Gamma$-approximated by functions belonging to $\mathcal{F}$ if for each open set U_f containing the graph of f, there exists $g \in \mathcal{F}$ such that the graph of g is a subset of U_f. We shall say that a function $f : [0,1] \to [0,1]$ is $\mathcal{T}_u$-approximated by functions belonging to $\mathcal{F}$ if there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ uniformly convergent to f. If we consider the family $\mathcal{A}([0,1])$ or $\mathcal{C}([0,1])$ as the family $\mathcal{F}$ in above definitions, then we have

Remark 12.28. (a) If $f \in \mathcal{A}([0,1])$, then the function f can be $\mathcal{T}_\Gamma$-approximated by continuous functions from $\mathfrak{E}_s([0,1])$.
(b) If $f \in \mathcal{A}([0,1])$, then the function f can be $\mathcal{T}_\Gamma$-approximated by discontinuous but almost continuous functions from $\mathfrak{E}_s([0,1])$.
(c) If $f \in \mathcal{C}([0,1])$, then the function f can be $\mathcal{T}_u$-approximated by continuous functions from $\mathfrak{E}_s([0,1])$.

Let $\mathrm{Per}^\infty(f)$ denote the set of all points $x \in \mathrm{Fix}(f)$ such that for any open neighborhood V of x and each $n \in \mathbb{N}$ there exists $y_x \in \mathrm{Per}_n(f)$ for which $O_f(y_x) = \{f^n(y_x) : n = 0,1,2,\dots\} \subset V$. f If $\mathrm{Per}^\infty(f) \neq \emptyset$, then we will say that f has the *local periodic property*. The family of all functions having local periodic propery will be denoted by Per^∞.

Theorem 12.29. *If $f \in \mathcal{DB}_1([0,1])$ then there exists a ring $\mathcal{K} \in \mathcal{R}_{\mathcal{DB}_1([0,1],\mathbb{R})}(f)$ such that*

[1] A definition of entropy of a function can be found in section 11.3.

(a) *the function f can be $\mathcal{T}_\Gamma$-approximated by functions belonging to* $\mathcal{K}\mathfrak{E}_s\triangle'\,\mathrm{Per}^\infty([0,1])$.
(b) *the function f can be $\mathcal{T}_\Gamma$-approximated by functions belonging to* $\mathcal{K}\mathfrak{E}_s\mathcal{C}'_{ap}\,\mathrm{Per}^\infty([0,1])$.

Now, following [19], we will introduce another class of functions whose definition is based on the notions of an od-set, H-trajectory and $(H,\{d_n\}_{n\in\mathbb{N}})$-first return continuity presented in Section 12.3.1.

Let $H\subset[0,1]$ be an od-set in $[0,1]$, $\{d_n\}_{n\in\mathbb{N}}$ be an H-trajectory and $f:[0,1]\to\mathbb{R}$. We say that function f is $H_\mathcal{C}$-connected with respect to H-trajectory $\{d_n\}_{n\in\mathbb{N}}$ if $f\in FRC(H,\{d_n\}_{n\in\mathbb{N}})$, $\{d_n\}_{n\in\mathbb{N}}\subset\mathcal{C}(f)$ and for any $x\in[0,1]\setminus H$ and any $\varepsilon>0$ there exists $\delta\in(0,\varepsilon)$ such that for any component I of the set H the following condition is fulfilled:

$$(I\cap(x-\delta,x+\delta)\neq\emptyset)\Rightarrow$$

$$(f(\{d_n\colon n=1,2,...\}\cap I\cap(x-\delta,x+\delta))\cap(f(x)-\varepsilon,f(x)+\varepsilon)\neq\emptyset).$$

The symbol $Conn_\mathcal{C}$ will denote the family of all functions $f\colon[0,1]\to\mathbb{R}$ such that there exist an od-set $H(f)$ and an $H(f)$-trajectory $\{d_n\}_{n\in\mathbb{N}}$ such that f is $H(f)_\mathcal{C}$-connected with respect to $\{d_n\}_{n\in\mathbb{N}}$.

Theorem 12.30. *If $f\in Conn_\mathcal{C}([0,1])$ then there exists a ring $\mathcal{K}\in\mathcal{R}_{Conn_\mathcal{C}([0,1],\mathbb{R})}(f)$ such that*

(a) *the function f can be $\mathcal{T}_\Gamma$-approximated by functions belonging to $\mathcal{K}\mathfrak{E}_s([0,1])$.*
(b) *the function f can be $\mathcal{T}_u$-approximated by functions belonging to $\mathcal{K}\mathfrak{E}_s([0,1])$.*

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