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Chapter 11
On Baire generalized topological spaces and some problems connected with discrete dynamical systems

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Introduction

Let us suppose that we have given a set X of some objects (e.g. events, information etc.) and some action (on the elements of the set X) whose result may be described by use of a function, multifunction or map. We repeat the action few or even several times. What can we conclude about complexity of these operations, if we extend this process to infinity? Will the process turn out to be chaotic or not?

It is not difficult to notice that in this way we will obtain some discrete dynamical system. From the mathematical point of view we need some topological structure in X and fixed notion of chaos to be able to describe this situation.

A special role in examining discrete dynamical systems (X, f) play the notion of f-covering, e.g. Itinerary Lemma ([4], [50], [16]); we say that set A f-covers set B and write $A \underset{f}{\rightarrow} B$ iff $B \subset f(A)$. So in X we can introduce a topological structure connected with this notion: $\gamma_f = \{A \subset X : A \underset{f}{\rightarrow} A\}$. The properties of the structure γ_f are close to topology, but γ_f may not be a topology. Indeed, if we consider a function $\zeta : [-1,1] \to [-1,1]$ defined as follows:

$$\zeta(x) = \begin{cases} \frac{1}{2}\sin\frac{1}{x} & \text{for } x \neq 0, \\ \frac{1}{2} & \text{for } x = 0, \end{cases}$$

then $(-\frac{1}{2},0],[0,\frac{1}{2}) \in \gamma_\zeta$, but $\{0\} = (-\frac{1}{2},0] \cap [0,\frac{1}{2}) \notin \gamma_\zeta$.

Fortunately, at the end of the 20th century Á. Császár introduced the new notion: generalized topology (and generalized topological space). That is, from the classical definition we leave only demands, that empty set is a generalized open set and union of generalized open sets is a generalized open set, too (we remove demands that the intersection of two generalized open sets is a generalized open set and whole space belongs to generalized topology). It is easy to notice that γ_ζ defined above is a generalized topology (a precise definition of generalized topological space will be given in section 11.1.1).

In addition to the theoretical results connected with generalized topological space we have J. Lee's observations in [28] connected with relationships between this theory and computer science. These observations are directly related to the information flow (e.g. [40], [41], [44]). Consequently they have become further motivation to study, among others, the relationship between generalized topological space and graph theory (e.g. [30]).

This section will be completed by the basic symbols and definitions used in the further sections of this chapter.

Let $X \neq \emptyset$, $f : X \to X$ be a function, $\Phi : X \multimap X$ be a multifunction and $\widetilde{\mathcal{P}}(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. The family of all continuous functions from X to X will be denoted by $\mathcal{C}(X)$. The symbol $\mathcal{C}(f)$ $(\mathcal{D}(f))$ stands for a set of all continuity (discontinuity) points of f. If $A \in \widetilde{\mathcal{P}}(X)$ then we will use the symbol $f \restriction A$ to denote a restriction of f to a set A. We will write $f(A)$ for an image of a set A by a function f. In the case of multifunction, we set $\Phi(A) = \bigcup_{a \in A} \Phi(a)$. Furthermore $f^0(x) = x$ and $f^i(x) = f(f^{i-1}(x))$ for $i \in \mathbb{N}$. Similarly, we put $\Phi^0(x) = \{x\}$ and $\Phi^i(x) = \Phi(\Phi^{i-1}(x))$ for $i \in \mathbb{N}$. Moreover, if $\Omega : X \multimap X$ then $(\Phi \circ \Omega)(x) = \Phi(\Omega(x))$.

From now on, the symbol $\mathrm{Fix}(f)$ $(\mathrm{Fix}(\Phi))$ will denote the set of all fixed points of function (multifunction) $f : X \to X$ $(\Phi : X \multimap X)$ i.e. the set of all points $x \in X$ such that $f(x) = x$ $(x \in \Phi(x))$.

This chapter is mainly based on the results contained in the papers [31], [30] and [25]. If we give statements from other sources, this will be marked by giving references to the relevant article.

11.1 Generalized topological spaces

As mentioned in the previous section the definition of generalized topology was introduced by Á. Császár in [10]. Generalized topological spaces are studied by many mathematicians (e.g. [10]-[15], [28], [5], [48], [25]). These studies are, for example, associated with different types of continuity in generalized topological spaces (e.g. [9], [35], [6], [45]), connectness (e.g. [11], [20]) or compactness of generalized topological spaces (e.g. [49], [24], [36]). In any case one can prove theorems analogous to well known statements from classical theory of topology as well as lead new considerations. This will be visible in the next parts of this chapter connected, among others, with nowhere dense sets, transitivity, etc.

It is interesting to note that every generalized topology in X can be associated with a monotonic map $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ (i.e a map such that $\eta(A) \subset \eta(B)$ if $A \subset B \subset X$). More precisely, in [10] one can find that every generalized topology γ in X can be generated by some monotonic map $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ in the following way $\gamma = \{A \subset X : A \subset \eta(A)\}$. On the other hand, if $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a monotonic map then $\gamma_\eta = \{A \subset X : A \subset \eta(A)\}$ is a generalized topology ([9]).

11.1.1 Basic notions and properties connected with generalized topological space

Let X be any nonempty set. We shall say that a family $\gamma \subset \mathcal{P}(X)$ is a *generalized topology* in X (in short GT) iff $\emptyset \in \gamma$ and $\bigcup_{t\in T} G_t \in \gamma$ whenever $\{G_t : t \in T\} \subset \gamma$. The pair (X,γ) is called a *generalized topological space* and it is denoted by GTS. Moreover, if $X \in \gamma$ we shall say that (X,γ) is a *strong generalized topological space* (briefly sGTS) and γ is a *strong generalized topology* (in short sGT). Obviously the space $([-1,1],\gamma_\zeta)$ described in Introduction is GTS and it is not sGTS. However, if we consider the function $\zeta \upharpoonright [-\frac{1}{2},\frac{1}{2}]$ then the space $\left([-\frac{1}{2},\frac{1}{2}],\gamma_{\zeta\upharpoonright[-\frac{1}{2},\frac{1}{2}]}\right)$ is sGTS. Moreover, if $\mathcal{K}$ is any family of sets containing $\emptyset$, then the family $\gamma_\mathcal{K}$ consisting of all sets that are unions of sets from the family $\mathcal{K}$ is GT.

From now on, we will consider a generalized topological space (X,γ). We will use the symbol $\widetilde{\gamma}$ to denote the family $\gamma \setminus \{\emptyset\}$ and $\gamma(x)$ to denote $\{U \in \gamma : x \in U\}$.

In many considerations related to the information flow theory, graph theory etc., particularly important are situations when considered spaces are finite. For this reason, in the papers on which this chapter is based often such kind of spaces are considered. Unlike, in this chapter we will consider mainly infinite spaces (with a few exception).

The basic definitions in generalized topological spaces are usually formulated in the same way as in topological spaces. For example, an interior of a set $A \subset X$ with respect to γ (in short $\mathrm{int}_\gamma(A)$) is a union of all γ-open sets B (i.e. $B \in \gamma$) such that $B \subset A$ and a closure of a set $A \subset X$ with respect to γ (in short $\mathrm{cl}_\gamma(A)$) is an intersection of all γ-closed sets B (i.e. $X \setminus B \in \gamma$) such that $A \subset B$. It is also worth noting that $\mathrm{cl}_\gamma(A)$ is a γ-closed set, $\mathrm{int}_\gamma(A) = X \setminus \mathrm{cl}_\gamma(X \setminus A)$, $\mathrm{cl}_\gamma(A) = X \setminus \mathrm{int}_\gamma(X \setminus A)$. Moreover we have, that $x \in \mathrm{cl}_\gamma(A)$ if and only if $U \cap A \neq \emptyset$ for any $U \in \gamma(x)$ ([13]). Separation axioms for generalized topological space are defined analogously to the case of a topological space (e.g. [12], [14], [48]). Here and subsequently, T_i-GTS denotes a generalized topological space which is T_i space (for $i = 1,2$). Moreover, we say that $A \subset X$ is γ-dense if $\mathrm{cl}_\gamma(A) = X$ or equivalently $A \cap U \neq \emptyset$ for any $U \in \widetilde{\gamma}$. From now on, the prefix connected with the symbol of a suitable generalized topology will be omitted when no confusion can arise.

However, despite identical definitions the properties of some mathematical objects in the case of standard topology may be quite different than the properties of respective objects in generalized topology. The examples of such situation are the notions of nowhere dense sets, which will be described in detail in the next section. Of course, there are more differences between topological spaces and generalized topological spaces. Now, we will present a few of them. Although the set X is always closed, an empty set is closed if and only if γ is sGT. Moreover, X is open if and only if γ is sGT. The union of a finite number of closed sets does not have to be closed. Indeed, let $\gamma_{\aleph_0} = \{A \subset \mathbb{Q} : \mathrm{card}(A) = \aleph_0\} \cup \{\emptyset\}$. Obviously, $(\mathbb{R}, \gamma_{\aleph_0})$ is GTS and sets $A_* = \mathbb{R} \setminus \{n \in \mathbb{Z} : n > -2\}$ and $B_* = \mathbb{R} \setminus \{n \in \mathbb{Z} : n < 2\}$ are closed. However, $A_* \cup B_* = \mathbb{R} \setminus \{-1,0,1\}$ is not closed because $\{-1,0,1\} \notin \gamma_{\aleph_0}$. Moreover, it is easy to see that $\mathrm{int}_{\gamma_{\aleph_0}}(\mathbb{R}) = \mathbb{Q}$ and $\mathrm{cl}_{\gamma_{\aleph_0}}(\emptyset) = \mathbb{R} \setminus \mathbb{Q}$.

From Property 1.3, Property 1.7 ([9]) and Lemma 1.1 ([10]) we obtain

Property 11.1. If (X, γ) is GTS then

(I.1) if $A \subset B \subset X$ then $\mathrm{int}_\gamma(A) \subset \mathrm{int}_\gamma(B)$,
(I.2) $\mathrm{int}_\gamma(A) \subset A$ for any $A \subset X$,
(I.3) $\mathrm{int}_\gamma(\mathrm{int}_\gamma(A)) = \mathrm{int}_\gamma(A)$ for any $A \subset X$,

(C.1) if $A \subset B \subset X$ then $\mathrm{cl}_\gamma(A) \subset \mathrm{cl}_\gamma(B)$,

(C.2) $A \subset \mathrm{cl}_\gamma(A)$ for any $A \subset X$,
(C.3) $\mathrm{cl}_\gamma(\mathrm{cl}_\gamma(A)) = \mathrm{cl}_\gamma(A)$ for any $A \subset X$.

Clearly, $\mathrm{cl}_\gamma(A) \cup \mathrm{cl}_\gamma(B) \subset \mathrm{cl}_\gamma(A \cup B)$ and $\mathrm{int}_\gamma(A \cap B) \subset \mathrm{int}_\gamma(A) \cap \mathrm{int}_\gamma(B)$. However, there exists GTS such that the above two inclusions are proper. Indeed, if we consider the space $(\mathbb{R}, \gamma_{\aleph_0})$ and sets A_*, B_* defined above, we obtain $\mathrm{cl}_{\gamma_{\aleph_0}}(A_*) \cup \mathrm{cl}_{\gamma_{\aleph_0}}(B_*) = A_* \cup B_* = \mathbb{R} \setminus \{-1, 0, 1\}$. Simultaneously, $\mathrm{cl}_{\gamma_{\aleph_0}}(A_* \cup B_*) = \mathbb{R}$. Moreover, $\mathrm{int}_{\gamma_{\aleph_0}}(\mathbb{R} \setminus A_*) \cap \mathrm{int}_{\gamma_{\aleph_0}}(\mathbb{R} \setminus B_*) = (\mathbb{R} \setminus A_*) \cap (\mathbb{R} \setminus B_*) = \{-1, 0, 1\}$ and $\mathrm{int}_{\gamma_{\aleph_0}}((\mathbb{R} \setminus A_*) \cap (\mathbb{R} \setminus B_*)) = \emptyset$.

11.1.2 Nowhere densities and Baire spaces

In case of topological space (X, τ), we can define a nowhere dense set in the following way: a set $A \subset X$ is nowhere dense if $\mathrm{int}_\tau(\mathrm{cl}_\tau(A)) = \emptyset$. This condition is equivalent to the following one: for any nonempty set $U \in \tau$ there exists a nonempty set $V \in \tau$ such that $V \subset U$ and $V \cap A = \emptyset$. However, in the case of generalized topological spaces these conditions lead to different concepts.

Let (X, γ) be GTS and $A \subset X$. We shall say that A is a *nowhere dense set* if $\mathrm{int}_\gamma(\mathrm{cl}_\gamma(A)) = \emptyset$ (e.g. [6], [31]). It is easy to see that a subset of a nowhere dense set is nowhere dense. However, a union of two nowhere dense sets does not have to be nowhere dense. Indeed, put $X = [0, 2]$ and $\mathcal{K} = \{\emptyset\} \cup \{[0, b) : b \in (0, 2]\} \cup \{(a, 2] : a \in [0, 2)\}$. Obviously $\mathrm{int}_{\gamma_\mathcal{K}}(\mathrm{cl}_{\gamma_\mathcal{K}}(\{0\})) = \emptyset$ and $\mathrm{int}_{\gamma_\mathcal{K}}(\mathrm{cl}_{\gamma_\mathcal{K}}(\{2\})) = \emptyset$, so $\{0\}$ and $\{2\}$ are nowhere dense sets. Moreover, it is easy to see that $\mathrm{int}_{\gamma_\mathcal{K}}(\mathrm{cl}_{\gamma_\mathcal{K}}(\{0, 2\})) = [0, 2]$. It implies that $\{0, 2\}$ is not a nowhere dense set. What is more, for $[0, 1) \in \gamma_\mathcal{K}$ there is no set $V \in \widetilde{\gamma}_\mathcal{K}$ such that $V \subset [0, 1)$ and $V \cap \{0\} \neq \emptyset$. Therefore $\{0\}$ is a nowhere dense set that does not satisfy the condition:

$$\text{for any } U \in \widetilde{\gamma}_\mathcal{K} \text{ there exists } V \in \widetilde{\gamma}_\mathcal{K} \text{ such that } V \subset U \text{ and } V \cap \{0\} = \emptyset.$$

For this reason, in the case of generalized topological space (X, γ) we introduce a new type of set. We shall say that $A \subset X$ is a *strongly nowhere dense set* if for any $U \in \widetilde{\gamma}$ there exists $V \in \widetilde{\gamma}$ such that $V \subset U$ and $V \cap A = \emptyset$.

The following statement shows a significant difference in the properties of nowhere dense sets and strongly nowhere dense sets in generalized topological spaces.

Theorem 11.2. (a) *There exists GTS (X, γ) and nowhere dense sets $A, B \subset X$ such that $A \cup B$ is not a nowhere dense set.*

(b) *For every two strongly nowhere dense sets A and B in an arbitrary GTS (X,γ) the union $A \cup B$ is a strongly nowhere dense set.*

Analogously to the topological case, we shall say that $A \subset X$ is a *meager* (*s-meager*) set if there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of nowhere dense (strongly nowhere dense) sets such that $A = \bigcup_{n\in\mathbb{N}} A_n$. A set A is called *second category* (*s-second category*) set if it is not a meager (s-meager) set. A set A is said to be *residual* (*s-residual*) if $X \setminus A$ is meager (s-meager).

The above considerations lead to the different notions connected with the notion of Baire space in standard topological spaces.
We shall say that GTS (X,γ) is

- a *weak Baire space* (in short wBS) if each set $U \in \widetilde{\gamma}$ is an s-second category set,
- a *Baire space* (in short BS) if each $U \in \widetilde{\gamma}$ is a second category set,
- a *strong Baire space* (in short sBS) if $V_1 \cap \cdots \cap V_n$ is a second category set for any $V_1, V_2, \ldots V_n \in \gamma$ such that $V_1 \cap \cdots \cap V_n \neq \emptyset$.

Obviously, if we consider a topological space (X,τ) instead of a generalized topological space (X,γ) then the above notions are equivalent. In the case of generalized topological spaces they are not. It is easy to see that if GTS is a strong Baire space then it is a Baire space and a weak Baire space. Further, each Baire space is a weak Baire space. The converse implications are not true. The detailed considerations of these relationships (presented in Figure 11.1) are contained in [25]

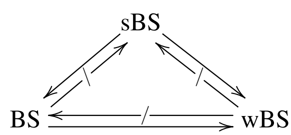

Fig. 11.1 The relationships between the different types of Baire GTS.

It should be noted that for some GTS these three notions may be equivalent. More specifically, it can be proved

Theorem 11.3. *If GTS (X,γ) satisfies the condition*

(KLP) $\mathrm{int}_\gamma(V_1 \cap V_2 \cap \cdots \cap V_m) \neq \emptyset$ *for any $m \in \mathbb{N}$ and $V_1, V_2, \ldots, V_m \in \gamma$ such that $V_1 \cap V_2 \cap \cdots \cap V_m \neq \emptyset$,*

then the notions of a strong Baire space, a Baire space and a weak Baire space are equivalent.

In [25] one can find the example of GTS satisfying the condition (KLP) which is not a topological space. Moreover, it is worth pointing out that GTS (X,γ), where $X=[0,3]$ and $\gamma=\{\emptyset,[0,2),(1,3],[0,3]\}$ is a strong Baire generalized topological space and it is not a topological space. What is more, if we consider sGTS $(X,\gamma_{\mathcal{K}})$, where $X=\{0,1\}\times\mathbb{R}$ and $\mathcal{K}=\{\emptyset\}\cup\{\{0\}\times V : V\in\mathcal{T}_{nat}\}\cup\{\{1\}\times V : V\in\mathcal{T}_{nat}\}\cup\{\{0\}\times((-\infty,-\alpha)\cup(\alpha,+\infty))\cup\{1\}\times[1,+\infty) : \alpha\geq 0\}\cup\{\{0\}\times((-\infty,-\alpha)\cup(\alpha,+\infty))\cup\{1\}\times(-\infty,1] : \alpha\geq 0\}$ we obtain that $V\cap U$ is a second category set for any $U,V\in\gamma_{\mathcal{K}}$ such that $V\cap U\neq\emptyset$. At the same time $\{1\}\times(0,2)\cap(\{0\}\times((-\infty,-1)\cup(1,+\infty))\cup\{1\}\times(-\infty,1])\cap(\{0\}\times((-\infty,-1)\cup(1,+\infty))\cup\{1\}\times[1,+\infty))$ is a meager set (detailed description of this example can be found in [25]). Therefore, it is reasonable to consider any finite intersection of open sets in the definition of a strong Baire space.

An interesting addition to our consideration would be introducing the following definition. We shall say that GTS (X,γ) is an *s-strong Baire space* if $V_1\cap\cdots\cap V_n$ is an s-second category set for any $V_1,V_2,\ldots V_n$ such that $V_1\cap\cdots\cap V_n\neq\emptyset$. Clearly if GTS is a strong Baire space then it is an s-strong Baire space and the converse implication is not true. Moreover, an s-strong Baire GTS is a weak Baire space. This kind of space has not been studied in detail in the literature previously. However, one can prove that if GTS satisfies the conditions (KLP) then the notion of an s-strong Baire space is equivalent to the notion of a strong Baire space, and in consequence, to the notion of a Baire space and a weak Baire space.

This section will be ended with the following property, which can be proved by methods described in the proof of Theorem 1.3 in [37].

Theorem 11.4. *Let (X,γ) be a Baire GTS. The intersection of any sequence of dense open sets is residual and each residual set is dense.*

More information about Baire spaces can be found in [31] and [25]. A deeper analysis of this topic is beyond the scope of this study, so we will omit it.

11.2 Generalized entropy

We indicated in Introduction, that we need a notion of chaos of a (discrete) dynamical system. Currently, there are many definitions of this notion and they essentially differ from each other (e.g. [29], [22], [51], [4], [18], [34]). However, it is commonly accepted that the entropy is some kind of measure of chaos. For this reason, there still appear new considerations related to those connected with entropy in relation to various problems (e.g. [23], [27]). For the need of our considerations we can assume that a function (multifunction, map) is chaotic if corresponding dynamical system has a positive entropy.

In the case of discrete dynamical systems there are two elementary (equivalent for the compact metric spaces) concepts of entropy: "covery" concept introduced by Adler, Konheim and McAndrew in [1] and Bowen-Dinaburg concept[1] based on notions of "separated set" or "span set" ([7], [19]). The basis of the first concept are coveries of a space and properties of a compact topological space. The fact that the intersection of finite number of open sets is an open set, plays an important role in this situation. That is why we can not adopt this definition to the case of GTS. The second concept is connected with compact metric spaces and, for obvious reasons, it is not proper in our case either. Although in [25] the idea of generalized metric space (briefly GMS) is presented, the problem of generalizing the notion of compactness of such spaces is still open.

Taking into account the above-mentioned aims we need to introduce new kind of entropy. The similar considerations, in the case of one dimension dynamical systems, one can find in [2].

We will present a definition of generalized entropy in the case of the map $\xi : \mathcal{P}(X) \to \mathcal{P}(X)$. However, if we have a multifunction ψ (or a function f) then we can consider a suitable map $\xi_\psi(A) = \psi(A)$ (generated by ψ) or $\xi_f(A) = f(A)$ (generated by f). In these cases the generalized entropy of a function (or a multifunction) will be the generalized entropy of a suitable map.

Let (X,γ) be GTS, $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ be a nonempty family and $\pi_\mathcal{K}$ be the set of all finite sequence of sets from $\mathcal{K}$ such that if $n \in \mathbb{N}$ and $(A_1,\dots,A_n) \in \pi_\mathcal{K}$ then $\mathrm{cl}_\gamma(A_i) \cap \mathrm{cl}_\gamma(A_j) = \emptyset$ whenever $i \neq j$ ($i,j \in \{1,\dots,n\}$). To each map $\xi : \mathcal{P}(X) \to \mathcal{P}(X)$ and each sequence $\mathcal{A} = (A_1,\dots,A_n)$ from $\pi_\mathcal{K}$, a matrix $\mathfrak{M}_{\mathcal{A},\xi} = [m_{i,j}]_{i,j\le n}$ such that $m_{i,j} = 1$ if $A_j \subset \xi(A_i)$ and $m_{i,j} = 0$ if $A_j \setminus \xi(A_i) \neq \emptyset$ will be assigned. Let $\mathfrak{M}^k_{\mathcal{A},\xi}$ and $\mathrm{tr}(\mathfrak{M}^k_{\mathcal{A},\xi})$ stand for k-times product of the matrix $\mathfrak{M}_{\mathcal{A},\xi}$ and the trace of the matrix $\mathfrak{M}^k_{\mathcal{A},\xi}$ for $k \in \mathbb{N}$, respectively.

A $(\mathcal{K},\xi,k)$-*entropy of the sequence* $\mathcal{A} \in \pi_\mathcal{K}$ ($k \in \mathbb{N}$) is the number

$$E^k_{\mathcal{K},\xi}(\mathcal{A}) = \begin{cases} 0 & \text{if } \mathrm{tr}(\mathfrak{M}^k_{\mathcal{A},\xi}) = 0, \\ \log(\mathrm{tr}(\mathfrak{M}^k_{\mathcal{A},\xi}))^{\frac{1}{k}} & \text{if } \mathrm{tr}(\mathfrak{M}^k_{\mathcal{A},\xi}) > 0. \end{cases}$$

[1] Bowen-Dinaburg concept of entropy will be presented in section 11.3.

Usually the base of logarithms is chosen either as 2 or as e. In fact, it does not matter which base we choose, as long as we use the same base greater then 1 all the time.

To illustrate this definition, consider the function $f : [0,1] \to [0,1]$ from Figure 11.2 and $\gamma_f = \{A \subset [0,1] : A \underset{f}{\to} A\}$. Obviously γ_f is GT, $A_1 \notin \gamma_f$

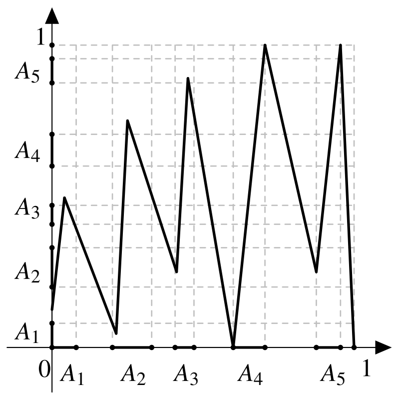

Fig. 11.2 The graph of same function $f : [0,1] \to [0,1]$.

and $A_i \in \gamma_f$ for $i \in \{2,\dots,5\}$. Moreover, for any $i \in \{1,\dots,5\}$ the set A_i is closed, because $f([0,1] \setminus A_i) \supset [0,1] \setminus A_i$. Putting $\mathcal{K}_0 = \{A_1,A_2,A_3,A_4,A_5\}$ we have that $\mathcal{A}_* = (A_1,A_2,A_3,A_4,A_5) \in \pi_{\mathcal{K}_0}$, $\mathcal{A}_{**} = (A_1,A_2,A_5) \in \pi_{\mathcal{K}_0}$ and $\mathcal{A}_{***} = (A_3,A_4) \in \pi_{\mathcal{K}_0}$. Then

$$\mathfrak{M}_{\mathcal{A}_*,f} = \begin{bmatrix} 0&1&1&0&0 \\ 0&1&1&1&0 \\ 0&0&1&1&0 \\ 1&1&1&1&1 \\ 0&0&1&1&1 \end{bmatrix}, \mathfrak{M}_{\mathcal{A}_{**},f} = \begin{bmatrix} 0&1&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}, \mathfrak{M}_{\mathcal{A}_{***},f} = \begin{bmatrix} 1&1 \\ 1&1 \end{bmatrix}.$$

Thus, for example considering the natural logarithm and $k = 1$ we have $E^1_{\mathcal{K}_0,f}(\mathcal{A}_*) = \log_e 4 \approx 1.386294361$ and $E^1_{\mathcal{K}_0,f}(\mathcal{A}_{**}) = E^1_{\mathcal{K}_0,f}(\mathcal{A}_{***}) = \log_e 2 \approx 0.69314718$. Moreover, if $k = 10$, then $E^{10}_{\mathcal{K}_0,f}(\mathcal{A}_*) \approx 1.153051977$, $E^{10}_{\mathcal{K}_0,f}(\mathcal{A}_{**}) \approx 0.069314718$ and $E^{10}_{\mathcal{K}_0,f}(\mathcal{A}_{***}) \approx 0.69314718$. Furthermore, it is easy to see that $E^k_{\mathcal{K}_0,f}(\mathcal{A}_{**}) = \frac{1}{k}\log_e 2$ and $E^k_{\mathcal{K}_0,f}(\mathcal{A}_{***}) = \log_e 2$ for any $k \in \mathbb{N}$.

The $(\mathcal{K},\xi)$*-entropy of the sequence* $\mathcal{A} \in \pi_{\mathcal{K}}$ is the number

$$E_{\mathcal{K},\xi}(\mathcal{A}) = \limsup_{k\to\infty} E^k_{\mathcal{K},\xi}(\mathcal{A}).$$

Clearly, $E_{\mathcal{K}_0,f}(\mathcal{A}_{**}) = 0$ and $E_{\mathcal{K}_0,f}(\mathcal{A}_{***}) > 0$.

Finally, the *$\mathcal{K}$-entropy of the map ξ* is the number

$$E_{\mathcal{K}}(\xi) = \sup_{\mathcal{A}\in\pi_{\mathcal{K}}} E_{\mathcal{K},\xi}(\mathcal{A}).$$

From above considerations one can conclude that $E_{\mathcal{K}_0}(f) > 0$ and so one can say that f is a chaotic function.

In the case of new concepts in the theory of discrete dynamical systems, it is important to check the "invariance with respect to certain homeomorphisms". In the case of GTS (X,γ) we consider γ-homeomorphisms. We shall call a bijection $f : X \to X$ a *γ-homeomorphism* if both f and the inverse function f^{-1} are γ-continuous. As in the case of the topological space a function $f : X \to X$ is *γ-continuous* iff $A \in \gamma$ implies $f^{-1}(A) \in \gamma$ (see [10]). The first statement presented below is related to a nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ invariant via γ-homeomorphism, i.e $\varphi(A) \in \mathcal{K}$ for any $A \in \mathcal{K}$ and any γ-homeomorphism $\varphi : X \to X$.

Theorem 11.5. *Let (X,γ) be GTS and a nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ be invariant via γ-homeomorphism. If $\mathcal{A} = (A_1, A_2, \ldots, A_n) \in \pi_{\mathcal{K}}$, then*

$$A_\varphi = (\varphi(A_1), \varphi(A_2), \ldots, \varphi(A_n)) \in \pi_{\mathcal{K}}$$

for any γ-homeomorphism $\varphi : X \to X$.

In the theory of discrete dynamical systems, invariance of certain properties of conjugate functions plays a special role. For this reason the next two theorems are devoted to such kind of problems with respect to our previous considerations. But first, we will give the definition of conjugate functions.

We will say that functions (multifunctions) $f,g : X \to X$ ($f,g : X \multimap X$) are *conjugate* iff there exists a γ-homeomorphism $\varphi : X \to X$ such that $\varphi \circ f = g \circ \varphi$. In this case, we will also say that functions (multifunctions) f,g are conjugate via γ-homeomorphism φ.

Theorem 11.6. *Let (X,γ) be GTS, a nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ be invariant via γ-homeomorphism and $f,g : X \to X$ be functions (or $f,g : X \multimap X$ be multifunctions) conjugate via γ-homeomorphism $\varphi : X \to X$. If $\mathcal{A} = (A_1, \ldots, A_n) \in \pi_{\mathcal{K}}$, then $\mathfrak{M}_{\mathcal{A},f} = \mathfrak{M}_{\mathcal{A}_\varphi,g}$ (where $A_\varphi = (\varphi(A_1), \ldots, \varphi(A_n))$).*

Now, let us assume that (X,γ) is GTS, $f,g : X \to X$ are conjugate via γ-homeomorphism $\varphi : X \to X$ and a nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ is invariant via γ-homeomorphism. According to Theorem 11.5 and Theorem 11.6, we obtain

$\mathrm{tr}(\mathfrak{M}^k_{\mathcal{A},f}) = \mathrm{tr}(\mathfrak{M}^k_{\mathcal{A}_\varphi,g})$ for any $k \in \mathbb{N}$. Therefore $E_{\mathcal{K}}(f) = \sup_{\mathcal{A}\in\pi_{\mathcal{K}}} E_{\mathcal{K},f}(\mathcal{A}) \leq \sup_{\mathcal{A}\in\pi_{\mathcal{K}}} E_{\mathcal{K},g}(\mathcal{A}) = E_{\mathcal{K}}(g)$. By a similar argumentation we obtain that $E_{\mathcal{K}}(g) \leq E_{\mathcal{K}}(f)$. In this way an important theorem justifying consideration of generalized entropy was proved.

Theorem 11.7. *Let (X,γ) be GTS. If $f,g : X \to X$ are conjugate and a nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ is invariant via γ-homeomorphism then*

$$E_{\mathcal{K}}(f) = E_{\mathcal{K}}(g).$$

The above theorem is still true if we replace functions $f,g : X \to X$ by multifunctions $f,g : X \multimap X$.

An important role in many considerations regarding practical use of mathematical theorems is played by finite spaces. Then the concept of entropy presented here makes possible some calculations. The following example is widely described in [30]. Here we will only indicate some elements related to these considerations.

Let $X = \{a_0,a_1,\ldots,a_{10}\}$ and $\nu : X \multimap X$ be defined in the following way: $\nu(a_i) = \{a_i,a_{i+1}\}$ for $i \in \{0,\ldots,4\}$, $\nu(a_5) = \{a_5,a_0\}$, $\nu(a_i) = \{a_{i+1}\}$ for $i \in \{6,\ldots,9\}$ and $\nu(a_{10}) = a_6$. Of course, $\mathrm{Fix}(\nu) = \{a_0,\ldots,a_5\}$. Putting $\gamma = \{B \subset X : B \subset \nu(B)\}$ we obtain that γ is GT in X and $\gamma = \{B \subset X : X \setminus B \subset \mathrm{Fix}(f) \vee \{a_6,a_7,a_8,a_9,a_{10}\} \subset X \setminus B\}$. Set $\mathcal{K} = \{\{a_0\},\{a_1\},\{a_2\},\{a_3\}\}$. It follows easily that $E^k_{\mathcal{K},\nu}(\mathcal{A}) \leq \frac{1}{k}\log 4$ for any $\mathcal{A} \in \pi_{\mathcal{K}}$ and $k \in \mathbb{N}$. This gives $E_{\mathcal{K}}(\nu) = 0$. If we consider a permutation $\Pi : X \to X$ such that $\Pi(\mathrm{Fix}(\nu)) = \mathrm{Fix}(\nu)$ and $\zeta = \Pi \circ \nu \circ \Pi^{-1}$ then we check at once that ν and ζ are conjugate. Since $\mathcal{K}$ is invariant via γ-homeomorphism, we can deduce, according to Theorem 11.7, that $E_{\mathcal{K}}(\zeta) = 0$.

The next theorem is connected with so called n-turbulent function (or n-horseshoe [2]) and it gives a convinient tool for lower estimation of a generalized entropy.

Theorem 11.8. *Let (X,γ) be GTS, $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ be a nonempty family, $\xi : \mathcal{P}(X) \to \mathcal{P}(X)$ and $n \in \mathbb{N}$. If there exists a sequence $\mathcal{A} = (A_1,A_2,\ldots,A_n) \in \pi_{\mathcal{K}}$ such that*

$$\bigcup_{i=1}^{n} A_i \subset \bigcap_{i=1}^{n} \xi(A_i),$$

then $E_{\mathcal{K}}(\xi) \geq \log n$.

11.3 Usual entropy and generalized entropy

There is a natural question about the relationship between generalized entropy (introduced in the previous part of this chapter) and standard entropy for usual functions. The "covery" definition of entropy is connected with continuous functions, because its essence is the demand that inverse image of an open set is an open set ([1]). As mentioned in section 11.2 in compact metric spaces the above definition is equivalent to Bowen-Dinaburg's definition of entropy ([7], [19]). So, we limit our considerations to compact metric spaces. Moreover, in recent times the results related to the entropy of discontinuous functions can be found in many papers. Therefore we will only give a definition of entropy formulated in the basic version in [7] and [19] (equivalent to the "covery" one) and transferred to a wider class of functions in [16].

Let (X,ρ) be a compact metric space, $f : X \to X$, $n \in \mathbb{N}$ and $\varepsilon > 0$. We shall say that a set $M \subset X$ is (n,ε)-*separated* if for any different points $x,y \in M$ there exists $i \in \{0,1,\dots,n-1\}$ such that $\rho(f^i(x), f^i(y)) > \varepsilon$. Moreover, a set $E \subset X$ will be called (n,ε)-*span* if for every $x \in X$ there is $y \in E$ such that $\rho(f^i(x), f^i(y)) \leq \varepsilon$ for any $i \in \{0,1,\dots,n\}$. Set $s_n(\varepsilon) = \max\{\mathrm{card}(M) : M \subset X \text{ is } (n,\varepsilon)-\text{separated set}\}$, $r_n(\varepsilon) = \min\{\mathrm{card}(E) : E \subset X \text{ is } (n,\varepsilon)-\text{span set}\}$, $\overline{s}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon)$ and $\overline{r}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon)$. The *topological entropy of* f is the number

$$h(f) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon).$$

It is possibile that $h(f) = +\infty$. Moreover, let us remark ([16], Lemma 3.2) that if $\varepsilon_1 < \varepsilon_2$, then $\overline{s}(\varepsilon_1) \geq \overline{s}(\varepsilon_2)$. Furthermore, in [16] one can find the following fact

Remark 11.9. For any compact metric space (X,ρ) and any $f : X \to X$ we have that

$$\lim_{\varepsilon\to 0^+} \overline{s}(\varepsilon) = \lim_{\varepsilon\to 0^+} \overline{r}(\varepsilon) = h(f).$$

Obviously, if (X,ρ) is a compact metric space and $f(x) = x$ for $x \in X$, then $h(f) = E_{\mathcal{K}}(f) = 0$ for any nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$. The following theorem shows that the generalized entropy is the lower estimation of standard one. We will present only a sketch of the proof of this theorem (the complete proof of this theorem one can find in [30]).

Theorem 11.10. *If (X,ρ) is a compact metric space and $f : X \to X$ then $E_{\mathcal{K}}(f) \leq h(f)$, for any nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$.*

Proof. If $E_{\mathcal{K}}(f)=0$ then the above inequality is obvious, so let us assume that $E_{\mathcal{K}}(f)=p>0$ and suppose contrary to our claim that $h(f)\in[0,p)$. There exists a sequence $\mathcal{A}_0=(A_1,A_2,\dots,A_k)\in\pi_{\mathcal{K}}$ such that

$$E_{\mathcal{K},f}(\mathcal{A}_0)>\frac{h(f)+p}{2}>0.$$

Moreover, there exists a strictly increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers such that

$$\log(\mathrm{tr}(\mathfrak{M}^{m_n}_{\mathcal{A}_0,f}))^{\frac{1}{m_n}}>\frac{h(f)+p}{2}.$$

Fix $n\in\mathbb{N}$. With the notation $\mathfrak{M}^{m_n}_{\mathcal{A}_0,f}=[x^{(m_n)}_{i,j}]_{i,j\le k}$ we have that if $x^{(m_n)}_{t,t}>0$, then there exist $x^{(m_n)}_{t,t}$ various sequences $(A_t,A_{s_1},\dots,A_{s_{m_n-1}},A_t)$ such that $s_1,\dots,s_{m_n-1}\in\{1,2,\dots,k\}$, $A_t\underset{f}{\to}A_{s_1}$, $A_{s_i}\underset{f}{\to}A_{s_{i+1}}$ for $i\in\{1,\dots,m_n-2\}$ and $A_{s_{m_n-1}}\underset{f}{\to}A_t$. Let us denote the set of these sequences by $\mathfrak{Y}^{(m_n)}_t$ (set $\mathfrak{Y}^{(m_n)}_t=\{Y^t_1,\dots,Y^t_{x^{(m_n)}_{t,t}}\}$). There exists an injective function $\xi_t:\mathfrak{Y}^{(m_n)}_t\to A_t$ such that if $Y^t_i=(A_t,A^i_{s_1},\dots,A^i_{s_{m_n-1}},A_t)\in\mathfrak{Y}^{(m_n)}_t$, then

$$\xi_t(Y^t_i)\in A_t,f(\xi_t(Y^t_i))\in A^i_{s_1},\dots,f^{m_n-1}(\xi_t(Y^t_i))\in A^i_{s_{m_n-1}},f^{m_n}(\xi_t(Y^t_i))\in A_t.$$

Putting $Z=\{t\in\{1,\dots,k\}:x^{(m_n)}_{t,t}>0\}$ and $\mathfrak{Q}_{m_n}=\bigcup_{t\in Z}\xi_t(\mathfrak{Y}^{(m_n)}_t)$ we obtain that

$$\mathrm{card}(\mathfrak{Q}_{m_n})=\mathrm{tr}(\mathfrak{M}^{m_n}_{\mathcal{A}_0,f}).$$

Moreover, there exist $\varepsilon^*>0$ such that $\mathfrak{Q}_{m_n}$ is (m_n,ε)-separated set for any $\varepsilon\in(0,\varepsilon^*)$. Therefore $s_{m_n}(\varepsilon)\ge\mathrm{tr}(\mathfrak{M}^{m_n}_{\mathcal{A}_0,f})$ for any $\varepsilon\in(0,\varepsilon^*)$. Thus

$$h(f)\ge\limsup_{n\to\infty}(\frac{1}{m_n}\log s_{m_n}(\varepsilon))\ge\limsup_{n\to\infty}(\frac{1}{m_n}\log\mathrm{tr}(\mathfrak{M}^{m_n}_{\mathcal{A}_0,f}))>h(f),$$

which is impossible. □

11.4 Generalized Vietoris topology and generalized entropy

As it was already indicated if we have fixed function then it is possible to consider some multifunction, map, etc. Opportunities in this area are much wider (e.g. [8]). For this reason, recently there appeared many papers in which authors compared the properties of dynamic of functions and suitable multifunctions and maps (e.g. [46], [21], [32]). In this section we refer to these ideas.

Let (X,γ) be GTS. We will denote by $\mathrm{CL}(X)$ the family of all nonempty, closed subsets of X. Consider the family $\mathcal{V}_\gamma \subset \mathcal{P}(\mathrm{CL}(X))$ consisting of sets $\alpha \in \mathcal{P}(\mathrm{CL}(X))$ such that $\alpha = \emptyset$ or for any $A \in \alpha$ there exist sets $U_1,\ldots,U_n \in \gamma$ such that $A \cap U_i \neq \emptyset$, for any $i \in \{1,\ldots,n\}$ and $\{B \in \mathrm{CL}(X) : B \subset \bigcup_{i=1}^n U_i$ and $B \cap U_i \neq \emptyset$ for any $i \in \{1,\ldots,n\}\} \subset \alpha$. We check at once that the family $\mathcal{V}_\gamma$ is a generalized topology in the space $\mathrm{CL}(X)$. This generalized topology can be called generalized Vietoris topology, because the above definition agrees with the classical definition of Vietoris topology in usual topological spaces.

Let $A \in \mathcal{P}(X)$. If there exists a set $B \in \mathrm{CL}(X)$ such that $B \subset A$, then put $\mathfrak{d}(A) = \{B \in \mathrm{CL}(X) : B \subset A\}$. Otherwise put $\mathfrak{d}(A) = \emptyset$. Moreover, set $\widehat{\mathrm{CL}}(X) = \{\mathfrak{d}(A) : A \in \mathrm{CL}(X)\}$.

Let $A \subsetneq X$ be a γ-closed set. Obviously, $\mathrm{CL}(X) \setminus \mathfrak{d}(A) = \{P \in \mathrm{CL}(X) : P \cap (X \setminus A) \neq \emptyset\} \neq \emptyset$ and $X \setminus A \in \gamma$. Moreover, we have that $W \cap X \neq \emptyset$ and $W \cap (X \setminus A) \neq \emptyset$ for any $W \in \mathrm{CL}(X) \setminus \mathfrak{d}(A)$. We check at once that

$$\{C \in \mathrm{CL}(X) : C \subset X \cup (X \setminus A) \wedge C \cap X \neq \emptyset \wedge C \cap (X \setminus A) \neq \emptyset\} = \mathrm{CL}(X) \setminus \mathfrak{d}(A).$$

Therefore, we obtain that $\mathrm{CL}(X) \setminus \mathfrak{d}(A) \in \mathcal{V}_\gamma$, so $\mathfrak{d}(A)$ is $\mathcal{V}_\gamma$-closed. Result of the above considerations can be saved in the form of theorem

Theorem 11.11. *Let (X,γ) be sGTS. If a set A is γ-closed, then $\mathfrak{d}(A)$ is $\mathcal{V}_\gamma$-closed.*

In the papers [46], [21], [47] a special kind of multifunction from $\mathrm{CL}(X)$ into itself connected with a function $f : X \to X$ was considered. The authors considered topological spaces or compact metric spaces X and the multifunction $\psi_f^{\mathfrak{d}} : \mathrm{CL}(X) \multimap \mathrm{CL}(X)$ defined by the formula $\psi_f^{\mathfrak{d}}(A) = \mathfrak{d}(f(A))$ for $A \in \mathrm{CL}(X)$ in connection with various problems of chaos. We can also investigate this multifunction for T_1-sGTS (X,γ) and $f : X \to X$ or $f : X \multimap X$. For example we have

Theorem 11.12. *Let (X,γ) be T_1-sGTS. For any multifunction $f : X \multimap X$ (function $f : X \to X$) we have*

$$E_{\mathrm{CL}(X)}(f) = E_{\widehat{\mathrm{CL}}(X)}(\psi_f^{\mathfrak{d}}). \tag{11.1}$$

The proof of this theorem was divided into three parts. In the first one, it was shown that a sequence $\mathcal{A} = (A_1, A_2, \ldots A_n)$ belongs to $\pi_{\mathrm{CL}(X)}$ if and only if $\mathcal{A}_\mathfrak{d} = (\mathfrak{d}(A_1), \mathfrak{d}(A_2), \ldots \mathfrak{d}(A_n))$ belongs to $\pi_{\widehat{\mathrm{CL}}(X)}$. Next, it was proved that

$\mathfrak{M}_{\mathcal{A},f} = \mathfrak{M}_{\mathcal{A}_{\mathfrak{d}},\psi_f^{\mathfrak{d}}}$. Finally, using the properties shown in the first and second parts, it was shown that $E_{\mathrm{CL}(X)}(f) \le E_{\widehat{\mathrm{CL}(X)}}(\psi_f^{\mathfrak{d}})$ and $E_{\widehat{\mathrm{CL}(X)}}(\psi_f^{\mathfrak{d}}) \le E_{\mathrm{CL}(X)}(f)$. For more details we refer the reader to [30].

Fixed points and periodic points of functions play an important role in the theory of combinatorial dynamics. It is particularly important to find a relationship between entropy of a function and the number of its fixed or periodic points. Now, we present the relationship between the fixed points of certain multifunctions and the generalized entropy of these multifunctions. We start with the definition.

Let (X,γ) be GTS and $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ be a nonempty family. We say that multifunction $\psi : X \multimap X$ has the property $\mathcal{I}_{\mathcal{K}}$ if for every sequence $(A_1, A_2, \ldots, A_n, A_1) \in \pi_{\mathcal{K}}$ such that $A_i \underset{\psi}{\to} A_{i+1}$ for $i = 1,2,\ldots,n-1$ and $A_n \underset{\psi}{\to} A_1$ there exists a sequence $(x_1, x_2, \ldots, x_n)$ such that $x_i \in A_i$ and $x_{i+1} \in \psi(x_i)$ for $i = 1,2,\ldots,n-1$ and $x_1 \in \psi(x_n)$. It is worth noting that if we consider a function $f : \mathbb{R} \to \mathbb{R}$ instead of multifunction ψ and a family $\mathcal{K}$ of all $\mathcal{T}_{nat}$-closed intervals then the above definition agrees with the one given in ([3], [51]).

In the context of these considerations the following theorem seems to be interesting.

Theorem 11.13. *Let (X,γ) be GTS and $\psi : X \multimap X$ be a multifunction having the property $\mathcal{I}_{\mathrm{CL}(X)}$. Then*

$$E_{\mathrm{CL}(X)}(\psi) \le \limsup_{n\to\infty} \max\left(0, \frac{1}{n}\log\left(\mathrm{card}\left(\mathrm{Fix}(\psi^n)\right)\right)\right).$$

As mentioned at the beginning of section 11.2, to each function $f : X \to X$, the map ξ_f generated by f may be assigned. Some properties of a function $\xi_f \restriction \mathrm{CL}(X)$ for a compact metric space X (so called "functions induced by f") one can find, for example, in papers [46], [21], [32], [47]. Moreover, if we consider $\mathrm{CL}(X)$ equipped with Vietoris topology then $\xi_f \restriction \mathrm{CL}(X)$ is a continuous function whenever f is a continuous function. From Theorem 11.10 and results contained in papers [42], [26], [32] it may be concluded

Theorem 11.14. *If (X,ρ) is a compact metric space and $f : X \to X$ is a continuous function, then for an arbitrary nonempty family $\mathcal{K} \subset \widetilde{\mathcal{P}}(X)$ we have*

$$E_{\mathcal{K}}(f) \le h(f) \le h(\xi_f \restriction \mathrm{CL}(X)). \tag{11.2}$$

Taking into account the examples presented in the papers [42], [32], [30], it is easily seen that the inequality in (11.2) may be strict. On the other hand we have the following theorem

Theorem 11.15. *Let $X = [0,1]^m$ and $\mathcal{A}r(X)$ be the set of all arcs in X. The set $\{f \in \mathcal{C}(X) : E_{\mathcal{A}r(X)}(f) = h(f) = h(\xi_f \restriction \mathrm{CL}(X))\}$ is dense in the space $\mathcal{C}(X)$ with the metric of uniform convergence.*

Therefore, for any continuous function $f : [0,1]^m \to [0,1]^m$ there exists a continuous function which is "arbitrarily close" to f and for which all entropies considered in Theorem 11.14 have the same value.

Notice that the above theorem is still true if we replace $\mathcal{A}r(X)$ with a family of all nonempty closed or connected or Borel subsets of X. Moreover, we can consider non-singleton, convex and compact subset of $\mathbb{R}^m$ in place of X or some manifold instead of $\mathbb{R}^m$. Of course, in last situation we must replace the set X with a set being a homeomorphic image of a suitable convex set. These considerations are a continuation of research initiated in the papers [44] and [39].

11.5 Transitive multifunctions

We start this section with some definitions. Let (X,γ) be GTS and $\Phi : X \multimap X$. Similarly to the case of usual functions, we shall say that a multifunction Φ is *transitive* if for any pair of nonempty open sets $U,V \subset X$ there exists $k \in \mathbb{N}$ such that $V \cap \Phi^k(U) \neq \emptyset$. Moreover, we shall say that a set (sequence) $\Theta_\Phi(x_0) = \{x_0, x_1, x_2, \dots\}$ is an *orbit of* x_0 *under* Φ if $x_i \in \Phi(x_{i-1})$ for any $i \in \mathbb{N}$. It is worth noting that, unlike in the case of a function, there may exist a lot of different orbits of x_0 under multifunction Φ. From now on, $\Theta^a_\Phi(x_0)$ stands for the family of all orbits $\Theta_\Phi(x_0)$ of x_0 under multifunction Φ. Clearly, we have the following property.

Property 11.16. Let (X,γ) be GTS. If $\Theta_\Phi(x_0) = \{x_0, x_1, x_2, \dots\}$ is an orbit of x_0 under $\Phi : X \multimap X$ then $x_i \in \Phi^i(x_0)$ for $i \in \mathbb{N}$.

In the paper [31] one can find the example showing that the converse statement is not true. However one can prove the following fact:

Theorem 11.17. *Let (X,γ) be GTS, $\Phi : X \multimap X$ and $x_0 \in X$. If $\alpha \in \Phi^m(x_0)$ for some positive integer m, then there exists an orbit $\Theta_\Phi(x_0) = \{x_0, x_1, x_2, \dots\}$ of x_0 under Φ such that $x_m = \alpha$.*

Many mathematicians have investigated relationship between transitivity and existence of a dense orbit of a function (e.g. [43], [17], [38], [33]). For instance, in [43] one can find the example of a transitive function f which does not have a dense orbit. On the other hand there exists a function f with dense orbit which is not a transitive function ([17]). Therefore these two notions are independent in general. However in some cases there are equivalent ([17], [38]).

It is clear that there is no connection between transitivity and existence of a dense orbit for multifunctions in general, too. To see this it suffices to consider multifunctions $\Phi(x) = \{f(x)\}$, where f is one of the functions mentioned above. However, we have the following theorem

Theorem 11.18. *Let (X,γ) be GTS and $\Phi : X \multimap X$. A multifunction Φ is transitive if and only if for any $U,V \in \widetilde{\gamma}$ there exist $x_0 \in U$ and the orbit $\Theta_\Phi(x_0)$ such that $\Theta_\Phi(x_0) \cap V \neq \emptyset$.*

The next theorem presents some condition equivalent to transitivity for a lower semicontinuous multifunction $\Phi : X \to X$ i.e multifunction $\Phi : X \to X$ such that for any $x \in X$ and any $U \in \gamma$ such that $\Phi(x) \cap U \neq \emptyset$ there exists $V \in \gamma(x)$ such that $\Phi(t) \cap U \neq \emptyset$ for any $t \in V$.

Theorem 11.19. *Let GTS (X,γ) be a Baire space with a countable base. A lower semicontinuous multifunction $\Phi : X \multimap X$ is transitive if and only if the set $\{x \in X : \mathrm{cl}_\gamma(\bigcup \Theta^a_\Phi(x)) = X\}$ is residual.*

One can ask if we can consider, in the above theorem, a set of all points $x \in X$ such that there exists a dense orbit of x under Φ instead of the set $\{x \in X : \mathrm{cl}_\gamma(\bigcup \Theta^a_\Phi(x)) = X\}$. The following example shows (see [31]) that answer to this question is negative.

If we consider $(\mathbb{R}, \mathcal{T}_{nat})$ and an arbitrary sequence $\{q_i\}_{i\in\mathbb{N}}$ of all rational numbers, then it is easy to see that the multifunction $\Phi : \mathbb{R} \multimap \mathbb{R}$ such that $\Phi(x) = \mathbb{Q}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\Phi(x) = \{q_1, q_2, \ldots, q_i\}$ if $x = q_i$ is transitive and lower semicontinuous. On the other hand there is no dense orbit of x under Φ for any $x \in \mathbb{R}$.

11.6 Strongly transitive multifunctions

Let (X,γ) be a Baire GTS and $\Phi : X \multimap X$ $(\Phi : X \to X)$. A multifunction (function) Φ is *strongly transitive* if for any $U,V \in \widetilde{\gamma}$ we have that $\{x \in U : \Theta_\Phi(x) \cap V \neq \emptyset$ for some $\Theta_\Phi(x) \in \Theta^a_\Phi(x)\}$ is a second category set.

Obviously, each strongly transitive multifunction (function) is transitive. On the other hand, there exists a transitive multifunction (function) which is not strongly transitive. Indeed, if we consider $(\mathbb{R}, \mathcal{T}_{nat})$ then the multifunction

$$\Omega(x) = \begin{cases} \{\pi\} & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}, \\ \mathbb{Q} & \text{for } x \in \mathbb{Q} \end{cases}$$

is transitive because $V \cap \Omega(U) \supset V \cap \mathbb{Q} \neq \emptyset$ for any $U, V \in \widetilde{\mathcal{T}}_{nat}$. Moreover, the set of all $x \in (0,1)$ such that $\Theta_\Omega(x) \cap (1,2) \neq \emptyset$ for some orbit $\Theta_\Omega(x) \in \Theta^a_\Omega(x)$ is countable, so we have that Ω is not strongly transitive.

The same is true if we consider the function $f : [0,2] \to [0,2]$ presented in Example in the paper [43] instead of the multifunction Ω.

Now we will focus on a particular type of multifunction, which has been also studied, for example by Crannell, Frantz and LeMasurier ([8]). Let us start with definition of the Cartesian product of generalized topological spaces introduced by Császár in [15]. Let T be a nonempty set and $X_t \neq \emptyset$ for any $t \in T$. Furthermore, let (X_t, γ_t) be GTS for any $t \in T$ and $X = \prod_{t \in T} X_t$ be the Cartesian product of the sets X_t. Moreover, let $\mathfrak{B}$ be a set of all sets of the form $\prod_{t \in T} M_t$, where $M_t \in \gamma_t$ for each $t \in T$ and $M_t \neq \bigcup \gamma_t$ only for a finite number of t from T. We call $\gamma_{\mathfrak{B}}$ *the product of generalized topologies* γ_t. Obviously, $\gamma_{\mathfrak{B}}$ is a generalized topology in X.

For any GTS (X, γ) and any function $f : X \to X$ we define a multifunction $\bar{f} : X \multimap X$ in the following way

$$\bar{f}(x) = \{y \in X : (x,y) \in \mathrm{cl}_{\gamma \times \gamma}(\Gamma(f))\},$$

where $\Gamma(f)$ is a graph of f and $\gamma \times \gamma$ is the product of generalized topologies. Our definition agrees with the one given in [8], in the case of topological spaces.

To illustrate this concept, consider the space $(\mathbb{R}, \mathcal{T}_{nat})$ and two functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined in the following way: $f_1(x) = f_2(x) = \sin \frac{1}{x}$ for $x \neq 0$, $f_1(0) = 0$ and $f_2(0) = 1$. Then $\bar{f}_1(x) = \bar{f}_2(x) = \left\{\sin \frac{1}{x}\right\}$ for $x \neq 0$ and $\bar{f}_1(0) = \bar{f}_2(0) = [-1,1]$.

The following two statements describe some connection between orbits and properties of a function f and orbits of a suitable function $\bar{f}$.

Proposition 11.20. *Let (X, γ) be GTS, $f : X \to X$ and $x_0 \in X$. The set $\Theta_f(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$ is an orbit of x_0 under multifunction $\bar{f}$.*

Proposition 11.21. *Let (X, γ) be T_2-GTS and $f : X \to X$.*

(a) *If $x_0 \in \mathcal{C}(f)$ then $\bar{f}(x_0) = \{f(x_0)\}$.*

(b) *If* $\Theta_f(x_0) \subset \mathcal{C}(f)$ *for some* $x_0 \in X$ *then* $\Theta_f(x_0)$ *is the unique orbit of* x_0 *under multifunction* $\bar{f}$.

Using the above statements, we can show the following theorem related to cm-function $f : X \to X$ i.e function $f : X \to X$ such that $\mathcal{D}(f)$ is a countable set and $f^{-m}(x) = \{z \in X : f^m(z) = x\}$ is a meager set for any $x \in \mathcal{D}(f)$ and $m \in \mathbb{N} \cup \{0\}$.

Theorem 11.22. *Let GTS* (X,γ) *be a strong Baire space with a countable base such that for any* $U \in \widetilde{\gamma}$ *and any finite set* $A \subset U$ *there exists a set* $V \in \widetilde{\gamma}$ *such that* $V \subset U \setminus A$. *Let* $f : X \to X$ *be a cm-function. The following conditions are equivalent:*

(A) f *is strongly transitive,*
(B) *there exists* $x_0 \in X$ *such that* $\Theta_f(x_0)$ *is a dense set and* $\Theta_f(x_0) \subset \mathcal{C}(f)$*,*
(C) $\bar{f}$ *is strongly transitive,*
(D) *there exists* $x_0 \in X$ *such that there exists an orbit* $\Theta_{\bar{f}}(x_0)$ *which is a dense set and* $\Theta_{\bar{f}}(x_0) \subset \mathcal{C}(f)$.

In the proof of this theorem an important role also plays the following property: if (X,γ) is GTS and $f : X \to X$ is a cm-function then the set $\{x \in X : \Theta_f(x) \cap \mathcal{D}(f) \neq \emptyset\}$ is a meager set.

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