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Chapter 13
$\mathcal{I}$-approximate differentiation of real functions

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The notion of $\mathcal{I}$-approximate differentiation [9] is based upon the notion of an $\mathcal{I}$-density point which was introduced in [10].
For any $a \in \mathbb{R}$ and $A \subset \mathbb{R}$, we denote

$$aA = \{ax : x \in A\} \quad \text{and} \quad A - a = \{x - a : x \in A\}.$$

Definition 13.1 ([10]). Let $A \subset \mathbb{R}$ be a set having the Baire property and $x \in \mathbb{R}$. We say that x is an $\mathcal{I}$-density point of a set A $(\mathcal{I}\text{-}d(A,x) = 1)$ if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers, there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\left\{t \in [-1,1] : \chi_{n_{m_p}\cdot(A-x)\cap[-1,1]}(t) \nrightarrow 1\right\}$$

is a set of the first category. A point x is called an $\mathcal{I}$-dispersion point of a set A $(\mathcal{I}\text{-}d(A,x) = 0)$ if x is an $\mathcal{I}$-density point of the set $\mathbb{R} \setminus A$.

If in the above definition, the interval $[-1,1]$ is replaced either by $[0,1]$ or by $[-1,0]$, we obtain the definitions of right-hand $\mathcal{I}$-density and $\mathcal{I}$-dispersion points $(\mathcal{I}\text{-}d^+(A,x) = 1 \text{ and } \mathcal{I}\text{-}d^+(A,x) = 0)$ or left-hand $\mathcal{I}$-density and $\mathcal{I}$-dispersion points $(\mathcal{I}\text{-}d^-(A,x) = 1 \text{ and } \mathcal{I}\text{-}d^-(A,x) = 0)$, respectively.

We shall need the following characterization of the $\mathcal{I}$-dispersion point of an open set.

Lemma 13.2 ([5]). *Let $G \subset \mathbb{R}$ be an open set. Then 0 is an $\mathcal{I}$-dispersion point of G if and only if for each $n \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and a real number $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $i \in \{1, ..., n\}$, there exist two numbers $j \in \{1, ..., k\}$ and $j' \in \{1, ...k\}$ such that*

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk} \right) h, \left(\frac{i-1}{n} + \frac{j}{nk} \right) h \right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j'}{nk} \right) h, -\left(\frac{i-1}{n} + \frac{j'-1}{nk} \right) h \right) = \emptyset.$$

For each $A \subset \mathbb{R}$ having the Baire property, let

$$\Phi_{\mathcal{I}}(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}\text{-density point of } A\}.$$

Recall that, for any sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ having the Baire property, we have the following:

1. $\Phi_{\mathcal{I}}(A) \triangle A$ is a set of the first category,
2. if $A \triangle B$ is a set of the first category, then $\Phi_{\mathcal{I}}(A) = \Phi_{\mathcal{I}}(B)$,
3. $\Phi_{\mathcal{I}}(\emptyset) = \emptyset$ and $\Phi_{\mathcal{I}}(\mathbb{R}) = \mathbb{R}$,
4. $\Phi_{\mathcal{I}}(A \cap B) = \Phi_{\mathcal{I}}(A) \cap \Phi_{\mathcal{I}}(B)$,

where $A \triangle B$ denotes the symmetric difference of the set A and B (see [10]). Further, the family

$$\mathcal{T}_{\mathcal{I}} = \{A \subset \mathbb{R} : A \text{ has the Baire property and } A \subset \Phi_{\mathcal{I}}(A)\}$$

is a topology on the real line, called $\mathcal{I}$-density topology (see [10]). The topology $\mathcal{T}_{\mathcal{I}}$ is stronger then the natural topology. It is Hausdorff topology but not regular. The family of all functions continuous with respect to $\mathcal{I}$-density topology we call $\mathcal{I}$-approximately continuous.

We recall that a set $B \subset \mathbb{R}$ is said to be the Baire cover of a set $A \subset \mathbb{R}$ if the set B has the Baire property, $A \subset B$ and, for each set $C \subset B \setminus A$ having the Baire property, the set C is of the first category.

Definition 13.3. A point $x \in \mathbb{R}$ is called an exterior $\mathcal{I}$-density point of a set $A \subset \mathbb{R}$ ($\mathcal{I}$-$d_e(A,x) = 1$) if there exists the Baire cover B of the set A such that $\mathcal{I}$-$d(B,x) = 1$.

A point $x \in \mathbb{R}$ is called an exterior $\mathcal{I}$-dispersion point of a set $A \subset \mathbb{R}$ ($\mathcal{I}\text{-}d_e(A,x) = 0$) if there exists the Baire cover B of the set A such that $\mathcal{I}\text{-}d(B,x) = 0$.

A point $x \in \mathbb{R}$ is called an exterior right-hand $\mathcal{I}$-density point of a set $A \subset \mathbb{R}$ ($\mathcal{I}\text{-}d_e^+(A,x) = 1$) if there exists the Baire cover B of the set A such that $\mathcal{I}\text{-}d^+(B,x) = 1$.

A point $x \in \mathbb{R}$ is called an exterior right-hand $\mathcal{I}$-dispersion point of a set $A \subset \mathbb{R}$ ($\mathcal{I}\text{-}d_e^+(A,x) = 0$) if there exists the Baire cover B of the set A such that $\mathcal{I}\text{-}d^+(B,x) = 0$.

In a similar way we define and we denote exterior left-hand $\mathcal{I}$-density points and exterior left-hand $\mathcal{I}$-dispersion points of a set $A \subset \mathbb{R}$.

Definition 13.4. Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$.

The upper right-hand $\mathcal{I}$-approximate Dini derivative of a function f at a point x ($\mathcal{I}\text{-}D^+ f(x)$) is defined as the greatest lower bound of the set

$$\left\{\alpha \in \mathbb{R} : \mathcal{I}\text{-}d_e^+\left(\left\{t > x : \frac{f(t)-f(x)}{t-x} > \alpha\right\}, x\right) = 0\right\}.$$

The lower right-hand $\mathcal{I}$-approximate Dini derivative of a function f at a point x ($\mathcal{I}\text{-}D_+ f(x)$) is defined as the least upper bound of the set

$$\left\{\alpha \in \mathbb{R} : \mathcal{I}\text{-}d_e^+\left(\left\{t > x : \frac{f(t)-f(x)}{t-x} < \alpha\right\}, x\right) = 0\right\}.$$

The left-hand $\mathcal{I}$-approximate Dini derivatives are defined similarly and denoted by $\mathcal{I}\text{-}D^- f(x)$ and $\mathcal{I}\text{-}D_- f(x)$.

The ordinary Dini derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$, we denote by $D^+ f(x)$, $D_+ f(x)$, $D^- f(x)$ and $D_- f(x)$, respectively.

To study the properties of the $\mathcal{I}$-approximate Dini derivatives we shall consider only the upper right-hand $\mathcal{I}$-approximate Dini derivative, because we can obtain analogous properties for other $\mathcal{I}$-approximate Dini derivatives by the following

1. if for each $x \in \mathbb{R}$, $g(x) = -f(x)$ then for each $x \in \mathbb{R}$,

$$\mathcal{I}\text{-}D_+ f(x) = -\left(\mathcal{I}\text{-}D^+ g(x)\right),$$

2. if for each $x \in \mathbb{R}$, $g(x) = f(-x)$ then for each $x \in \mathbb{R}$,

$$\mathcal{I}\text{-}D_- f(x) = -\left(\mathcal{I}\text{-}D^+ g(-x)\right),$$

3. if for each $x \in \mathbb{R}$, $g(x) = -f(-x)$ then for each $x \in \mathbb{R}$,

$$\mathcal{I}\text{-}D^{-} f(x) = \mathcal{I}\text{-}D^{+} g(-x).$$

It is easy to see that

Theorem 13.5. *Let $f : \mathbb{R} \to \mathbb{R}$. Then for each point $x \in \mathbb{R}$,*

$$\mathcal{I}\text{-}D^{+} f(x) \leq D^{+} f(x).$$

Moreover we have the following theorem

Theorem 13.6 ([4]). *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the set*

$$\left\{x \in \mathbb{R} : \mathcal{I}\text{-}D^{+} f(x) \neq D^{+} f(x)\right\}$$

is of the first category.

The above theorem is not true for an arbitrary function having the Baire property. If we consider the characteristic function of the set rational numbers then for each irrational numbers x, we have $D^{+} f(x) = +\infty$ and $\mathcal{I}\text{-}D^{+} f(x) = 0$.

Theorem 13.7 ([9]). *If $f : \mathbb{R} \to \mathbb{R}$ is a monotone function, then*

$$\mathcal{I}\text{-}D^{+} f(x) = D^{+} f(x),$$

for each point $x \in \mathbb{R}$.

Theorem 13.8 ([4]). *If $f : \mathbb{R} \to \mathbb{R}$ and $m \in \mathbb{R}$. If a set*

$$A \subset \left\{x \in \mathbb{R} : \mathcal{I}\text{-}D^{+} f(x) < m\right\}$$

is of the second category and the function $f_{|A}$ is continuous, then there exists an interval $[a,b] \subset \mathbb{R}$ such that the set $[a,b] \cap A$ is of the second category and the function $h(x) = f(x) - mx$ is nonincreasing on $[a,b] \cap A$.

By taking into consideration the characteristic function of the Bernstein set it is easy to see that the $\mathcal{I}$-approximate Dini derivatives of the function which does not have the Baire property may not have this property, either. But the following theorem is true.

Theorem 13.9 ([3]). *If a function $f : \mathbb{R} \to \mathbb{R}$ has the Baire property, then its $\mathcal{I}$-approximate Dini derivatives have the Baire property, too. Moreover, if f is continuous, then they are of the Baire class 3.*

Additionally, by the following theorem

Theorem 13.10 ([3]). *Let* $f:\mathbb{R}\to\mathbb{R}$ *and*

$$A\subset\left\{x\in\mathbb{R}:\mathcal{I}\text{-}D^{+}f(x)<+\infty\right\}.$$

If A *is a set of the second category then there exists a set* $W\subset A$ *such that the set* $A\setminus W$ *is of the first category and the function* $f_{|W}$ *is continuous.*

we obtain

Theorem 13.11 ([3]). *Let* $f:\mathbb{R}\to\mathbb{R}$. *If*

$$\mathbb{R}\setminus\left\{x\in\mathbb{R}:\mathcal{I}\text{-}D^{+}f(x)<+\infty\right\}$$

is a set of the first category, then the function f *has the Baire property.*

The relations between the ordinary Dini derivatives of an arbitrary real function of real variable were described in the Denjoy-Young-Saks Theorem:

Theorem 13.12 ([11]). *Let* $f:\mathbb{R}\to\mathbb{R}$ *and*

$$E_1=\{x\in\mathbb{R}:\ f\text{ is differentiable at }x\},$$

$$E_2=\left\{x\in\mathbb{R}:D^{+}f(x)=D^{-}f(x)=+\infty,\ D_{+}f(x)=D_{-}f(x)=-\infty\right\},$$

$$E_3=\left\{x\in\mathbb{R}:D^{+}f(x)=D_{-}f(x)\text{ are finite},\ D_{+}f(x)=-\infty,D^{-}f(x)=+\infty\right\},$$

$$E_4=\left\{x\in\mathbb{R}:D_{+}f(x)=D^{-}f(x)\text{ are finite},\ D^{+}f(x)=+\infty,D_{-}f(x)=-\infty\right\}.$$

Then the set $\mathbb{R}\setminus(E_1\cup E_2\cup E_3\cup E_4)$ *has Lebesgue measure zero.*

Theorem 13.13 ([12]). *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a Lebesgue measurable function and*

$$E_1=\{x\in\mathbb{R}:\ f\text{ is differentiable at }x\},$$

$$E_2=\left\{x\in\mathbb{R}:D^{+}f(x)=D^{-}f(x)=+\infty,\ D_{+}f(x)=D_{-}f(x)=-\infty\right\}.$$

Then the set $\mathbb{R}\setminus(E_1\cup E_2)$ *has Lebesgue measure zero.*

It is worth mentioning that the theorems remain true if we replace there the ordinary Dini derivatives by the approximate Dini derivatives (see [1]).

The following example shows that the relations given in Danjoy-Young-Saks Theorem are not satisfied for $\mathcal{I}$-approximate Dini derivatives even if we assume the measurability in the sense of Baire and Lebesgue.

Example 13.14 ([4]). Let A be a set of the first category such that $\mathbb{R}\setminus A$ has Lebesgue measure zero and f be the characteristic function of the set A. Then for each $x\in A$,

$$\mathcal{I}\text{-}D_{+}f(x)=\mathcal{I}\text{-}D^{+}f(x)=-\infty$$

and

$$\mathcal{I}\text{-}D_{-}f(x)=\mathcal{I}\text{-}D^{-}f(x)=+\infty.$$

By Theorem 13.6, it immediately follows that if the upper and lower Dini derivatives of a continuous function $f:\mathbb{R}\to\mathbb{R}$ are equal to $+\infty$ and $-\infty$, resp., then the upper and lower $\mathcal{I}$-approximate Dini derivatives are equal to $+\infty$ and $-\infty$, respectively, on a residual set. Moreover, we have

Theorem 13.15 ([4]). *If a continuous function $f:\mathbb{R}\to\mathbb{R}$ has no finite derivative at any point, then there exists a residual set $E\subset\mathbb{R}$ such that, for each $x\in E$*

$$\mathcal{I}\text{-}D_{+}f(x)=\mathcal{I}\text{-}D_{-}f(x)=-\infty \textit{ and } \mathcal{I}\text{-}D^{-}f(x)=\mathcal{I}\text{-}D^{-}f(x)=\infty.$$

Theorem 13.16 ([4]). *If $f:\mathbb{R}\to\mathbb{R}$ has the Baire property, then the sets*

$$\left\{x\in\mathbb{R}:\mathcal{I}\text{-}D^{+}f(x)\neq\mathcal{I}\text{-}D^{-}f(x)\right\},\ \{x\in\mathbb{R}:\mathcal{I}\text{-}D_{+}f(x)\neq\mathcal{I}\text{-}D_{-}f(x)\}$$

are of the first category.

The above theorem is not true for an arbitrary real function of a real variable, for example if we consider the characteristic function of the Bernstein set, then for each $x\in\mathbb{R}\setminus B$,

$$\mathcal{I}\text{-}D^{-}f(x)=\mathcal{I}\text{-}D_{+}f(x)=0,\ \mathcal{I}\text{-}D^{+}f(x)=+\infty \text{ and } \mathcal{I}\text{-}D_{-}f(x)=-\infty.$$

Theorem 13.16 is a category version of the Denjoy-Young-Saks Theorem, for functions having the Baire property, establishing a relation between the $\mathcal{I}$-approximate Dini derivatives. In the next theorem, it is shown that this result cannot be improved even if we assume continuity of the function f.

Theorem 13.17 ([4]). *For any a and b such that $-\infty\leq a<b\leq+\infty$, there exists a continuous function $f:[0,1]\to\mathbb{R}$ and a residual set E on an interval $[0,1]$ such that for each point $x\in E$,*

$$\mathcal{I}\text{-}D_{+}f(x)=\mathcal{I}\text{-}D_{-}f(x)=a \textit{ and } \mathcal{I}\text{-}D^{+}f(x)=\mathcal{I}\text{-}D^{-}f(x)=b.$$

Definition 13.18. Let $f:\mathbb{R}\to\mathbb{R}$ and $x\in\mathbb{R}$.

We say that a function f has a right-hand $\mathcal{I}$-approximate derivative at a point x $\left(\mathcal{I}\text{-}f'_+(x)\right)$, if $\mathcal{I}\text{-}D^+ f(x) = \mathcal{I}\text{-}D_+ f(x)$. Then $\mathcal{I}\text{-}f'_+(x)$ is the common value of $\mathcal{I}\text{-}D^+ f(x)$ and $\mathcal{I}\text{-}D_+ f(x)$.

We say that a function f has a left-hand $\mathcal{I}$-approximate derivative at a point x $\left(\mathcal{I}\text{-}f'_-(x)\right)$, if $\mathcal{I}\text{-}D^- f(x) = \mathcal{I}\text{-}D_- f(x)$. Then $\mathcal{I}\text{-}f'_-(x)$ is the common value of $\mathcal{I}\text{-}D^- f(x)$ and $\mathcal{I}\text{-}D_- f(x)$.

We say that a function f has an $\mathcal{I}$-approximate derivative at a point x $(\mathcal{I}\text{-}f'(x))$, if $\mathcal{I}\text{-}f'_+(x) = \mathcal{I}\text{-}f'_-(x)$. Then $\mathcal{I}\text{-}f'(x)$ is the common value of $\mathcal{I}\text{-}f'_+(x)$ and $\mathcal{I}\text{-}f'_-(x)$.

We say that a function f is $\mathcal{I}$-approximately differentiable at a point x if $|\mathcal{I}\text{-}f'(x)| < +\infty$.

We say that a function f is $\mathcal{I}$-approximately differentiable if f is $\mathcal{I}$-approximately differentiable everywhere.

The ordinary derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$, we denote by $f'(x)$.

Lemma 13.19. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and*

$$A \subset \left\{x \in \mathbb{R} : \mathcal{I}\text{-}D^+(x) < +\infty\right\}.$$

If the set A *is dense subset of* $\mathbb{R}$ *and the function* $f_{|A}$ *is differentiable, then* $\mathcal{I}\text{-}D^+ f(x) \le f'_{|A}(x)$, *for each* $x \in A$.

Proof. Let $x \in A$ and $f'_{|A} = s$. We suppose that there exists $\beta > 0$ such that

$$\mathcal{I}\text{-}D^+ f(x) > s + \beta.$$

Then x is not an exterior right-hand $\mathcal{I}$-dispersion point of the set

$$W = \left\{t > x : \frac{f(t) - f(x)}{t - x} > s + \beta\right\}.$$

Therefore, for each $\delta > 0$, $W \cap (x, x + \delta)$ is a set of the second category.

Let $0 < \alpha < \beta$. By our assumption the function $f_{|A}$ is differentiable at x and therefore there exists a real number $\delta > 0$ such that

$$A \cap (x, x + \delta) \subset \left\{t > x : \frac{f(t) - f(x)}{t - x} < s + \alpha\right\}.$$

Let V be the Baire cover of the set W. Then there exists an open interval $(a, b) \subset (x, x + \delta)$ such that $(a, b) \setminus V$ is a set of the first category.

Let $y \in A \cap (a, b)$. Then $\mathcal{I}\text{-}d_e^+(W, y) = 1$ and for each $t \in (y, b) \cap W$,

$$f(t)-f(x)>(s+\beta)(t-x)$$

and

$$f(x)-f(y)>(-s-\alpha)(y-x).$$

Therefore

$$\frac{f(t)-f(y)}{t-y}>s+\frac{t-x}{t-y}\left(\beta-\alpha\frac{y-x}{t-x}\right),$$

for each $t\in(y,b)\cap W$. Hence

$$\lim_{t\to y^+,\ t\in W}\frac{f(t)-f(y)}{t-y}=+\infty$$

and $\mathcal{I}\text{-}D^+f(y)=+\infty$, a contradiction. Thus $\mathcal{I}\text{-}D^+f(x)\le f'_{|A}(x)$. □

Lemma 13.20. *Let $f:\mathbb{R}\to\mathbb{R}$ and*

$$A\subset\left\{x\in\mathbb{R}:\mathcal{I}\text{-}D^+(x)<+\infty\right\}.$$

If there exists the Baire cover B of the set A such that $\mathbb{R}\setminus B$ is a set of the first category and the function $f_{|A}$ is differentiable, then

$$\mathcal{I}\text{-}D^+f(x)\ge f'_{|A}(x),$$

for each $x\in A$.

Proof. Let $x\in A$ and $f'_{|A}=s$. We suppose that there exists $\eta>0$ such that

$$\mathcal{I}\text{-}D^+f(x)<s-\eta.$$

Then x is an exterior right-hand $\mathcal{I}$-dispersion point of the set

$$S=\left\{t>x:\frac{f(t)-f(x)}{t-x}>s-\eta\right\}.$$

Therefore there exists the Baire cover P of a set S such that x is a right-hand $\mathcal{I}$-density point of the set $W=\mathbb{R}\setminus P$ and

$$W\subset\left\{t>x:\frac{f(t)-f(x)}{t-x}\le s-\eta\right\}.$$

Let $n\in\mathbb{N}$. The set W is a subset of the second category of the interval $\left(x,x+\frac{1}{n}\right)$, hence there exists an open interval $(a_n,b_n)\subset\left(x,x+\frac{1}{n}\right)$ such that $(a_n,b_n)\setminus W$ is a set of the first category. Since $A\cap(a_n,b_n)$ is a subset of the second category of the interval (a_n,b_n), we have

$$W \cap A \cap \left(x, x + \frac{1}{n}\right) \neq \emptyset.$$

We choose a point $t_n \in W \cap A \cap \left(x, x + \frac{1}{n}\right)$. In this way we define a sequence of real numbers $\{t_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} t_n = x$, for each $n \in \mathbb{N}$, $t_n \in A \cap (x, +\infty)$ and

$$\frac{f(t_n) - f(x)}{t_n - x} \leq s - \eta.$$

By the assumption the function $f_{|A}$ is differentiable at x, therefore

$$f'_{|A}(x) \leq f'_{|A}(x) - \eta,$$

a contradiction. Hence $\mathcal{I}\text{-}D^+ f(x) \geq f'_{|A}(x)$. □

By Lemmas 13.19 and 13.20 we have the following

Theorem 13.21. *Let $f : \mathbb{R} \to \mathbb{R}$ and $A \subset \mathbb{R}$. If there exists the Baire cover B of the set A such that $\mathbb{R} \setminus B$ is a set of the first category, the function $f_{|A}$ is differentiable and, for each $x \in A$,*

$$-\infty < \min\{\mathcal{I}\text{-}D_+(x), \mathcal{I}\text{-}D_-(x)\} \leq \max\left\{\mathcal{I}\text{-}D^+(x), \mathcal{I}\text{-}D^-(x)\right\} < +\infty,$$

then the function f is $\mathcal{I}$-approximately differentiable at each point $x \in A$.

We observe that in the definition of $\mathcal{I}$-approximate derivative of a function f at a point x in [9], [2], [5], [6] and [8] it was assumed that f has the Baire property in some neighborhood of x. We have defined $\mathcal{I}$-approximate derivative without this assumption. But by Theorem 13.11 we know that every $\mathcal{I}$-approximately differentiable function has the Baire property. Therefore if a function f is $\mathcal{I}$-approximately differentiable then it is $\mathcal{I}$-approximately continuous function. Moreover we have the following theorem.

Theorem 13.22 ([8]). *For every $\mathcal{I}$-approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$ there exists an $\mathcal{I}$-approximately differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - h(x)| < \varepsilon$, for each $x \in \mathbb{R}$.*

Corollary 13.23 ([8]). *The uniform closure of the family of all $\mathcal{I}$-approximately differentiable functions coincides with the family of all $\mathcal{I}$-approximately continuous functions.*

Now we give the several properties of a function $f : \mathbb{R} \to \mathbb{R}$ which is $\mathcal{I}$-approximately differentiable.

Theorem 13.24 ([9]). *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable then it is Baire ^{*1}, which means that there exists a sequence of closed sets $\{A_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $f_{|A_n}$ is a continuous function and $\mathbb{R} = \bigcup_{n\in\mathbb{N}} A_n$.*

By the above we know that if a function f has a finite $\mathcal{I}$-approximate derivative at each point $x \in \mathbb{R}$ then the function f is of the first class of Baire. This result is not true if a function possesses infinite $\mathcal{I}$-approximate derivatives. Then we have the following theorems:

Theorem 13.25 ([2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function having the Baire property. If a function f has an $\mathcal{I}$-approximate derivative at each point $x \in \mathbb{R}$ then it is of the second class of Baire.*

Theorem 13.26 ([2]). *There exists a function $f : \mathbb{R} \to \mathbb{R}$ having the Baire property such that f has an $\mathcal{I}$-approximate derivative at each point $x \in \mathbb{R}$ and f is not of the first class of Baire.*

Theorem 13.27 ([9]). *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable then it has the Darboux property.*

Theorem 13.28 ([9]). *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable and $\mathcal{I}$-$f'(x) \geq 0$ at each $x \in \mathbb{R}$, then f is nondecreasing.*

Theorem 13.29 ([9]). *Let $f : \mathbb{R} \to \mathbb{R}$ be Baire 1 and Darboux. Suppose that*

1. *$\mathcal{I}$-$f'(x)$ exists except on a denumerable set,*
2. *$\mathcal{I}$-$f'(x) \geq 0$ almost everywhere (with respect to the Lebesgue measure).*

Then f is a nondecreasing and continuous function.

Now we give the several properties of a function $\mathcal{I}$-f'.

Theorem 13.30 ([9]). *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable then the function $\mathcal{I}$-f' has the Darboux property.*

Theorem 13.31 ([5]). *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable then the function $\mathcal{I}$-f' is of Baire class one.*

Theorem 13.32 ([7]). *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable then there exists a sequence of perfect sets $\{H_n\}_{n\in\mathbb{N}}$ and a sequence of differentiable functions $\{h_n\}_{n\in\mathbb{N}}$ defined on $\mathbb{R}$ such that*

1. *$h_n = f$ over H_n,*
2. *$h_n' = \mathcal{I}$-f' over H_n,*

3. $\bigcup_{n\in\mathbb{N}} H_n = \mathbb{R}$.

By Theorem 13.28 and Theorem 13.7 we have the following

Theorem 13.33. *Let $f : \mathbb{R} \to \mathbb{R}$. If a function f is an $\mathcal{I}$-approximately differentiable and $\mathcal{I}$-f' is bounded above or below then for each $x \in \mathbb{R}$, $\mathcal{I}$-$f'(x) = f'(x)$.*

We assume that a function f is $\mathcal{I}$-approximately differentiable. Since the $\mathcal{I}$-approximate derivative of f possesses the Darboux property, the above theorem forces $\mathcal{I}$-f' to attain every value indeed infinitely often on any interval where $\mathcal{I}$-f' is not f'. Thus $\mathcal{I}$-f' must oscillate between positive and negative values whose absolute value may be as large as desired.

On the other hand, since $\mathcal{I}$-approximate derivative of f is a function of Baire class one, we know that there exists an open dense set G on which $f' = \mathcal{I}$-f'. So the question arises whether the oscillation mentioned in the above occurs on the component intervals of the set G. In what follows, an affirmative answer is furnished to this question.

Theorem 13.34 ([6]). *Let $f : \mathbb{R} \to \mathbb{R}$. Suppose f has a finite $\mathcal{I}$-approximate derivative $\mathcal{I}$-$f'(x)$ at each point $x \in (a,b)$ and let $M \geq 0$. If $\mathcal{I}$-$f'(x)$ attains both M and $-M$ on (a,b), then there exists a subinterval $(c,d) \subset (a,b)$ on which $\mathcal{I}$-$f' = f'$ and f' attains both M and $-M$ on (c,d).*

Now we give applications of the above theorem.

Theorem 13.35 ([6]). *Let $f : \mathbb{R} \to \mathbb{R}$. Suppose f has a finite $\mathcal{I}$-approximate derivative $\mathcal{I}$-$f'(x)$ at each point $x \in (a,b)$ and let α be a real number. If*

$$\left\{x \in (a,b) : \mathcal{I}\text{-}f'(x) = \alpha\right\} \neq \emptyset$$

then there exists $x_0 \in \operatorname{int}(\{x \in (a,b) : f \text{ is differentiable function at } x\})$ such that $f'(x_0) = \alpha$.

Corollary 13.36 ([6]). *Let $f : \mathbb{R} \to \mathbb{R}$. Suppose f has a finite $\mathcal{I}$-approximate derivative $\mathcal{I}$-$f'(x)$ at each point $x \in (a,b)$. If a set $\{x \in (a,b) : f(x) = 0\}$ is dense in (a,b), then f is identically zero on (a,b).*

Corollary 13.37 ([6]). *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose f and g have a finite $\mathcal{I}$-approximate derivative $\mathcal{I}$-$f'(x)$ and $\mathcal{I}$-$g'(x)$ at each point $x \in (a,b)$. If a set $\{x \in (a,b) : f(x) = g(x)\}$ is dense in (a,b), then $f = g$ on (a,b).*

Corollary 13.38 ([6]). *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$. We assume that a function f has a finite $\mathcal{I}$-approximate derivative $\mathcal{I}\text{-}f'(x)$ and a function g has a finite derivative g', at each point $x\in(a,b)$. If $f'=g'$ on*

$$\operatorname{int}\left(\{x\in(a,b):\ f \text{ is differentiable function at } x\}\right),$$

then $f'=g'$ on (a,b).

Theorem 13.39 ([6]). *Let $\mathcal{W}$ be a property of functions saying that any function which is differentiable and possesses $\mathcal{W}$ on an interval (c,d) is monotone on (c,d).*

Let $f:\mathbb{R}\to\mathbb{R}$. If a function f has a finite $\mathcal{I}$-approximate derivative $\mathcal{I}\text{-}f'(x)$ at each $x\in(a,b)$ and if f has the property $\mathcal{W}$ on (c,d), then the function f is monotone on (a,b).

Now we shall prove the relationships between $\mathcal{I}$-approximate derivative and ordinary derivative.

Lemma 13.40 ([3]). *Let $h:\mathbb{R}\to\mathbb{R}$. If $D\subset\mathbb{R}$ is a residual set such that the function $h_{|D}$ is continuous, then, for each open interval $J\subset[0,+\infty)$, the set*

$$A=\{x\in D:(x+J)\cap\{t>x:h(t)>h(x)\} \text{ is a set of the second category }\}$$

is open relative to D.

Lemma 13.41. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function having the Baire property, (c,d) be an open interval and $b\in\mathbb{R}$. Let*

$$E=\left\{x\in(c,d): f \text{ is } \mathcal{I}\text{-approximately differentiable at } x \text{ and } \mathcal{I}\text{-}f'(x)<b\right\}.$$

If $(c,d)\setminus E$ is a set of the first category, then there exist an open interval $(a,b)\subset(c,d)$ and a set $D\subset(a,b)$ such that $(a,b)\setminus D$ is a set of the first category and for any $x\in D$ and $y\in D$, if $x\neq y$ then

$$\frac{f(x)-f(y)}{x-y}<b.$$

Proof. Put $g(x)=f(x)-bx$ for each $x\in\mathbb{R}$. For each $x\in\mathbb{R}$, let

$$P(x)=\{t>x:g(t)<g(x)\} \text{ and } L(x)=\{t<x:g(t)>g(x)\}.$$

For $n\in\mathbb{N}$, $p\in\mathbb{N}$ and $h>0$ we define a set A_{nph} in the following way: $x\in A_{nph}$ if and only if there exists $j\in\{1,...,n\}$ such that

$$\left(\frac{j-1}{n}\cdot h+x, \frac{j}{n}\cdot h+x\right)\setminus P(x)$$

is a set of the first category and, for each $j\in\{1...,n\}$,

$$\left(\frac{-j}{n}\cdot h+x, \frac{-j+1}{n}\cdot h+x\right)\cap L(x)$$

is a set of the second category. By Lemma 13.40 we know that for each $n\in\mathbb{N}$, $p\in\mathbb{N}$ and $h>0$, a set A_{nph} has the Baire property.

Let $x\in E$. Then x is a right-hand $\mathcal{I}$-density point of the set $P(x)$ and x is a left-hand $\mathcal{I}$-density point of the set $L(x)$. Therefore by Lemma 13.2, we have that

$$E\subset\bigcup_{n\in\mathbb{N}}\bigcup_{p\in\mathbb{N}}\bigcap_{0<h<\frac{1}{p}}A_{nph}.$$

Since $(c,d)\setminus E$ is a set of the first category, there exist $n\in\mathbb{N}$, $p\in\mathbb{N}$ and an open interval $(a,b)\subset(c,d)$ such that the set

$$D=(a,b)\cap\bigcap_{0<h<\frac{1}{p}}A_{nph}$$

is a residual subset of (a,b) and $b-a<\frac{1}{p}$. Let $x\in D$, $y\in D$ such that $x<y$. Put $h=y-x$. Then there exists $j\in\{1,...,n\}$ such that

$$\left(\frac{j-1}{n}\cdot h+x, \frac{j}{n}\cdot h+x\right)\setminus P(x)$$

is a set of the first category and

$$\left(\frac{j-1}{n}\cdot h+x, \frac{j}{n}\cdot h+x\right)\cap L(y)$$

is a set of the second category. Thus $P(x)\cap L(y)\cap(x,y)\neq\emptyset$ and there exists $t\in(x,y)$ such that $g(x)>g(t)$ and $g(t)>g(y)$. Hence, for any $x\in D$ and $y\in D$, if $x<y$ then $g(x)>g(y)$. Therefore, for any $x\in D$ and $y\in D$, if $x\neq y$ then $\frac{f(y)-f(x)}{y-x}<b$. □

Lemma 13.42. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. If*

$$\mathbb{R}\setminus\{x\in\mathbb{R}: f \text{ is } \mathcal{I}\text{-approximately differentiable at } x\}$$

is a set of the first category, then the set of all points of continuity of upper and lower derivatives of f is everywhere dense on $\mathbb{R}$.

Proof. Let (a,b) be an arbitrary open interval. Put

$$E = \{x \in (a,b) : f \text{ is } \mathcal{I}\text{-approximately differentiable at } x\}$$

and for each $n \in \mathbb{N}$, $E_n = \{x \in E : |\mathcal{I}\text{-}f'(x)| < n\}$.

By Theorem 13.9 we know that, for each $n \in \mathbb{N}$, the set E_n has the Baire property and by our assumption $E = \bigcup_{n\in\mathbb{N}} E_n$. Therefore there exist a positive integer n and an open interval $(a_1,b_1) \subset (a,b)$ such that $(a_1,b_1) \setminus E_n$ is a set of the first category. Thus, by Lemma 13.41 and by continuity of the function f, there exists a closed interval $[c_1,d_1] \subset (a_1,b_1)$ such that for any $x \in [c_1,d_1]$ and $y \in [c_1,d_1]$, if $x \neq y$ then $|\frac{f(x)-f(y)}{x-y}| \leq n$. Hence

$$-n \leq \inf\left\{\underline{f}'(x) : x \in [c_1,d_1]\right\} \leq \sup\left\{\overline{f}'(x) : x \in [c_1,d_1]\right\} \leq n,$$

where $\underline{f}'$ and $\overline{f}'$ denote the lower and the upper derivative of the function f, respectively.

Let

$$A = \left\{x \in (c_1,d_1) \cap E : -\frac{1}{2}n < \mathcal{I}\text{-}f'(x) < n\right\}$$

and

$$B = \left\{x \in (c_1,d_1) \cap E : -n < \mathcal{I}\text{-}f'(x) < \frac{1}{2}n\right\}.$$

Since for each $x \in E$, $\mathcal{I}\text{-}f'(x) = \mathcal{I}\text{-}D^+f(x)$, then by Theorem 13.9, the sets A and B have the Baire property and one of these is a set of the second category.

We assume it is the former. Then there exists an open interval $(a_2,b_2) \subset [c_1,d_1]$ such that

$$(a_2,b_2) \setminus \left\{x \in (c_1,d_1) \cap E : -\frac{1}{2}n < \mathcal{I}\text{-}f'(x) < n\right\}$$

is a set of the first category. In the similar way as the above, by Lemma 13.41, we can show that there exists a closed interval $[c_2,d_2] \subset (a_2,b_2)$ such that

$$-\frac{1}{2}n \leq \inf\left\{\underline{f}'(x) : x \in [c_2,d_2]\right\} \leq \sup\left\{\overline{f}'(x) : x \in [c_2,d_2]\right\} \leq n.$$

If the second set is a set of the second category, then we have a closed interval $[c_2,d_2] \subset (a_2,b_2)$ such that

$$-n \leq \inf\left\{\underline{f}'(x) : x \in [c_2,d_2]\right\} \leq \sup\left\{\overline{f}'(x) : x \in [c_2,d_2]\right\} \leq \frac{1}{2}n.$$

Thus

$$\sup\left\{\overline{f}'(x) : x \in [c_2, d_2]\right\} - \inf\left\{\underline{f}'(x) : x \in [c_2, d_2]\right\} \leq \frac{3}{4} \cdot 2n.$$

By induction, we may define a sequence of closed intervals $\{[c_k, d_k]\}_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $[c_{k+1}, d_{k+1}] \subset [c_k, d_k] \subset (a, b)$ and

$$\sup\left\{\overline{f}'(x) : x \in [c_2, d_2]\right\} - \inf\left\{\underline{f}'(x) : x \in [c_2, d_2]\right\} \leq 2n\left(\frac{3}{4}\right)^{k-1}.$$

Let $x \in \bigcap_{k\in\mathbb{N}}[c_k, d_k]$. Then $x \in (a, b)$ and functions $\underline{f}'$ and $\overline{f}'$ are continuous at x. □

Theorem 13.43. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If*

$$\mathbb{R} \setminus \{x \in \mathbb{R} : f \textit{ is } \mathcal{I}\textit{-approximately differentiable at } x\}$$

is a set of the first category, then

$$\mathbb{R} \setminus \{x \in \mathbb{R} : f \textit{ is differentiable at } x\}$$

is a set of the first category, too.

Proof. By Lemma 13.42, a set A of points of continuity of the function $\underline{f}'$ is dense and, of course, a G_δ set. Therefore A is a residual subset of $\mathbb{R}$. We know that the function f is differentiable at each point of continuity of $\underline{f}'$. Thus f is differentiable at each point belonging to A. □

Theorem 13.44. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function having the Baire property and*

$$E = \{x \in \mathbb{R} : f \textit{ is } \mathcal{I}\textit{-approximately differentiable at } x\}.$$

If $\mathbb{R} \setminus E$ is a set of the first category, then there exists a set M such that $\mathbb{R} \setminus M$ is a set of the first category, the function $f_{|M}$ is differentiable and at each point $x \in M$, $f'_{|M}(x) = \mathcal{I}\text{-}f'(x)$.

Proof. We consider a sequence of sequences of open intervals

$$\{\{(a_k^n, b_k^n\}_{k\in\mathbb{N}} : n \in \mathbb{N}\}$$

such that

1. for each $n \in \mathbb{N}$, $\mathbb{R} = \bigcup_{k\in\mathbb{N}}(a_k^n, b_k^n)$,
2. for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $b_k^n - a_k^n < \frac{1}{n}$.

Let $n \in \mathbb{N}$. We define $\mathcal{K}_n$ to be the family of all open intervals $J^n \subset \mathbb{R}$ such that there exist $k(J^n) \in \mathbb{N}$ and a set $E(J^n) \subset J^n$ for which

a. $J^n \setminus E(J^n)$ is a set of the first category,
b. for any $x \in E(J^n)$ and $y \in E(J^n)$, if $x \neq y$ then

$$a^n_{k(J^n)} \leq \frac{f(y)-f(x)}{y-x} \leq b^n_{k(J^n)}.$$

By Lemma 13.41, there exists a sequence $\{J^n_p\}_{p\in\mathbb{N}} \subset \mathcal{K}_n$ such that $\mathbb{R} \setminus \bigcup_{p\in\mathbb{N}} J^n_p$ is a set of the first category and for any $p \in \mathbb{N}$ and $p' \in \mathbb{N}$, if $p \neq p'$ then $J^n_p \cap J^n_{p'} = \emptyset$. We put $M_n = E \cap \bigcup_{p\in\mathbb{N}} E(J^n_p)$. Then $\mathbb{R} \setminus M_n$ is a set of the first category.

Let $x \in M_n$. There exists $p \in \mathbb{N}$ such that $x \in E(J^n_p)$. Therefore for each $y \in E(J^n_p)$, if $x \neq y$ then

$$a^n_{k(J^n_p)} \leq \frac{f(y)-f(x)}{y-x} \leq b^n_{k(J^n_p)}.$$

We suppose that $\mathcal{I}\text{-}f'(x) < a^n_{k(J^n_p)}$. Then there exists $\lambda > 0$ such that $(x-\lambda, x+\lambda) \subset J^n_p$ and

$$(x-\lambda, x+\lambda) \cap \left\{ y \in \mathbb{R} : x \neq y \text{ and } \frac{f(y)-f(x)}{y-x} < a^n_{k(J^n_p)} \right\}$$

is a set of the second category. It is impossible since $\mathbb{R} \setminus M_n$ is a set of the first category. Therefore $\mathcal{I}\text{-}f'(x) \geq a^n_{k(J^n_p)}$. In a similar way we can show that $\mathcal{I}\text{-}f'(x) \leq b^n_{k(J^n_p)}$.

Hence for each $y \in E(J^n_p)$, if $y \neq x$ then

$$\left| \frac{f(y)-f(x)}{y-x} - \mathcal{I}\text{-}f'(x) \right| < b^n_{k(J^n_p)} - a^n_{k(J^n_p)} < \frac{1}{n}.$$

Let $M = \bigcap_{n\in\mathbb{N}} M_n$. Then $\mathbb{R} \setminus M$ is a set of the first category. Let $x \in M$ and $n \in \mathbb{N}$. There exists $p \in \mathbb{N}$ such that $x \in E(J^n_p)$. Then for each $y \in J^n_p \cap M \subset J^n_p \cap M_n = E(J^n_p)$ such that $x \neq y$, we have

$$\left| \frac{f(y)-f(x)}{y-x} - \mathcal{I}\text{-}f'(x) \right| < b^n_{k(J^n_p)} - a^n_{k(J^n_p)} < \frac{1}{n}.$$

Therefore $f'_{|M}(x) = \mathcal{I}\text{-}f'(x)$. □

Now we shall consider functions $\mathcal{I}\text{-}f'_+$ and $\mathcal{I}\text{-}f'_-$. By Theorem 13.9 we know that if a function $f:\mathbb{R}\to\mathbb{R}$ has the Baire property then $\mathcal{I}\text{-}f'_+$ and $\mathcal{I}\text{-}f'_-$ have the Baire property, too.

Theorem 13.45. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. If the function f has a right-hand (left-hand) $\mathcal{I}$-approximate derivative (finite or infinite) at each $x\in\mathbb{R}$, then the function $\mathcal{I}\text{-}f'_+$ ($\mathcal{I}\text{-}f'_-$) is of the first class of Baire.*

Proof. Consider to fix the ideas, the derivative $\mathcal{I}\text{-}f'_+$. Now we suppose that f is not of the first class of Baire. Then there exist a perfect set P and a real numbers b and d such that $d<b$ and

$$D=\left\{x\in P:\mathcal{I}\text{-}f'_+(x)<d\right\}$$

is a set of the second category in P and

$$B=\left\{x\in P:\mathcal{I}\text{-}f'_+(x)>b\right\}$$

is dense in P.

We denote, for any $x\in\mathbb{R}$, $n\in\mathbb{N}$ and $r\in\mathbb{N}$,

$$D(x)=\{t\in[x,+\infty):f(t)-f(x)\le d(t-x)\},$$

$$B(x)=\{t\in[x,+\infty):f(t)-f(x)\ge b(t-x)\},$$

and

$$D_{nr}=\bigcap_{0<h<\frac{1}{r}}\bigcup_{i\in\{1,\dots,n\}}\left\{x\in P:\left(x+\frac{i-1}{n}h,x+\frac{i}{n}h\right)\subset D(x)\right\}.$$

Since $D(x)$ is a closed set, by Lemma 13.40, we have that, for any $n\in\mathbb{N}$ and $r\in\mathbb{N}$, the set D_{nr} is closed too.

Let $x\in D$. Then x is a right hand density point of the set $D(x)$. Therefore, by Lemma 13.2, there exist $n\in\mathbb{N}$ and $r\in\mathbb{N}$ such that $x\in D_{nr}$. Hence $D\subset\bigcup_{n\in\mathbb{N}}\bigcup_{r\in\mathbb{N}}D_{nr}$, and there exist $n\in\mathbb{N}$ and $r\in\mathbb{N}$ and an open interval (a,b) such that $P\cap(a,b)\subset D_{nr}$ and $P\cap(a,b)\neq\emptyset$.

Let $x\in B\cap P\cap(a,b)$. Then x is a right hand $\mathcal{I}$-density point of the set $B(x)$. Hence, by lemma 13.2, there exists $k\in\mathbb{N}$ and $p\in\mathbb{N}$ such that, for any $0<h<\frac{1}{p}$ and $i\in\{1,\dots,n\}$, there exists $j\in\{1,\dots,k\}$ such that

$$\left(x+\frac{(i-1)k+j-1}{nk}h,x+\frac{(i-1)k+j}{nk}h\right)\subset B(x).$$

Let $0 < h < \min\left\{\frac{1}{r}, \frac{1}{p}\right\}$. Then, by $x \in D_{nr}$, there exists $i \in \{1,...,n\}$ such that

$$\left(x + \frac{i-1}{n}h, x + \frac{i}{n}h\right) \subset D(x)$$

and there exists $j \in \{1,...,k\}$ such that

$$\left(x + \frac{(i-1)k + j - 1}{nk}h, x + \frac{(i-1)k + j}{nk}h\right) \subset B(x) \cap \left(x + \frac{i-1}{n}h, x + \frac{i}{n}h\right).$$

Hence $D(x) \cap B(x) \neq \{x\}$, a contradiction. Therefore $\mathcal{I}$-f'_+ is the first class of Baire. □

Corollary 13.46. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If the function f has an $\mathcal{I}$-approximate derivative (finite or infinite) at each $x \in \mathbb{R}$, then function $\mathcal{I}$-f' is of the first class of Baire.*

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