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Chapter 16
Axial functions

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16.1 Introduction

Definition 16.1. Function $f : X \times Y \to X \times Y$ is axial if $f(x,y) = (x, g(x,y))$ for some $g : X \times Y \to Y$ (f is vertical) or $f(x,y) = (g(x,y), y)$ for some $g : X \times Y \to X$ (f is horizontal) (we use the notation $f(x,y)$ instead of $f((x,y))$).

Consideration of axial functions dates back to S. Banach and S. Ulam in Scottish book [9]. Our main question is which functions from $X \times Y$ to $X \times Y$ are finite composition of axial functions.

To show that most functions can not be obtained as a composition of just two axial functions consider the following example:

Example 16.2. Let $X = Y = \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f(0,0) = (1,1)$, $f(1,1) = (0,0)$ and $f(x,y) = (x,y)$ for all other $x, y \in \mathbb{R}$. Function f can not be obtained by a composition of only two axial functions.

If $f = h_2 \circ h_1$ and h_1 is, say, horizontal then $h_1(0,0) = (1,0)$ since then vertical function $h_2(1,0) = (1,1)$. But $h_1(1,0) = (x,0)$, where $x \neq 1$, and vertical function h_2 can not map it back to $(1,0)$.

16.2 Finite sets

By $|X|$ we denote the cardinality of the set X. In this section we consider functions $f : X \times Y \to X \times Y$, where X and Y are finite sets. Clearly every function is a composition of finitely many axial functions, it turns out that the number of axial functions can be bounded.

Theorem 16.3 ([3]). *Every function $f : X \times Y \to X \times Y$ is a composition of six axial functions $f = h_6 \circ \ldots \circ h_1$, moreover we can demand that h_1 is horizontal.*

The above Theorem is an answer to a question of Ulam ([14], VIII 2). In the same paper [3] a problem was stated if it is posibble to decrease number six. The positive answer appeared in [10].

Theorem 16.4. *Every function $f : X \times Y \to X \times Y$ is a composition of five axial functions $f = h_5 \circ \ldots \circ h_1$, moreover we can demand that h_1 is horizontal.*

The question whether number five is minimal remains open. The "sharp" result is the following

Theorem 16.5 ([10]). *For every $f : X \times Y \to X \times Y$ there is $g : X \times Y \to X \times Y$ such that $\forall_{(x,y)\in X\times Y} |f^{-1}(x,y)| = |g^{-1}(x,y)|$ and g is a composition of three axial functions $f = h_3 \circ h_2 \circ h_1$ and we may additionally assume h_1 is horizontal.*

As to bound the number of axial functions from below we have in [10] and later in [8] the following

Example 16.6 ([8]). There is a function from $\{1,2,3\}^2$ to $\{1,2,3\}^2$ which is <u>not</u> a composition of three axial functions.

Warning: in [8] and [10] the names horizontal and vertical have different (opposite) meaning than in Definition 16.1. In [8] we find yet another example.

Example 16.7. There is a function $f : X \times Y \to X \times Y$, where $|X| = 3$, $|Y| = 93$ (!), which is <u>not</u> a composition of four axial functions $h_4 \circ \ldots \circ h_1$ provided that h_1 is horizontal.

However, f is a composition of four axial functions when h_1 is vertical. We know only that four axial functions are enough in very special cases.

Theorem 16.8 ([8]). *Every function $f : X \times Y \to X \times Y$, where $|X| = 3$, is a composition of four axial functions $f = h_4 \circ \ldots \circ h_1$ provided that h_1 is vertical.*

We conclude this section with a remark that in case when f is a permutation the situation is easier.

Theorem 16.9 ([3]). *Every permutation $f : X \times Y \to X \times Y$ is a composition of only three axial permutations $f = h_3 \circ h_2 \circ h_1$ and we can additionally demand that h_1 is horizontal.*

(This is a difference with infinite sets - see next section).

16.3 Infinite sets

As we mentioned, first problems about axial functions appeared in Scottish Book and were about permutations, we start, however, with result about all functions. In contrast to finite sets, functions on infinite sets can be easily written as a composition of axial functions.

Theorem 16.10 ([3]). *If one of sets X, Y is infinite then every function $f : X \times Y \to X \times Y$ is a composition of three axial functions.*

We present the proof as it is simple and shows why axial permutations can not be used.

Proof. Assume $|X| \geq |Y|$, let $g : X \times Y \to X$ be a 1-1 function. We set $h_1 : X \times Y \to X \times Y$ as $h_1(x,y) = (g(x,y),y)$. Note that h_1 is 1-1, denoting $f = (f_1, f_2)$ we define $h_2(x,y) = (x, f_2(h_1^{-1}(x,y)))$ and $h_3(x,y) = (f_1[(h_2 \circ h_1)^{-1}(x,y)], y)$. It's direct to verify $f = h_3 \circ h_2 \circ h_1$. □

Obviously whether h_1 is horizontal or vertical depends on cardinalities of X and Y.

When we put restriction on the axial functions to be permutations then the situation gets harder (note the contrast with finite sets X,Y). In Scotisch Book [9] (problem 48) Banach asked if every permutation of $\mathbb{N} \times \mathbb{N}$ is a composition of finitely many axial permutations of $\mathbb{N} \times \mathbb{N}$. The affirmative answer has been given by Nosarzewska [7], she proved that five axial permutations are enough to obtain any permutation of $\mathbb{N} \times \mathbb{N}$. The proof uses induction on $\mathbb{N}$. Later Ehrenfeucht and Grzegorek [3] using algebraic argument generalized the result to any infinite sets.

Theorem 16.11. *When X and Y are infinite then every permutation of $X \times Y$ is a composition of five axial permutations, moreover we may assume h_1 is horizontal.*

Dropping the 'moreover' part it was possible to decrease number five. The strongest and "sharp" result is in [4].

Theorem 16.12. *Let X and Y are infinite.*

(i) Every permutation of $X \times Y$ is a composition of four axial permutations (we can not demand additionally that h_1 is, say, horizontal, it depends on function f).
(ii) There exists a permutation of $X \times Y$ that can not be represented as a composition of three axial permutations.

To finish we remark that we can adapt (simplify) the proof of Theorem 16.20 to obtain an alternative proof of a weaker statement than the above - every permutation is a composition of eleven axial permutations.

16.4 The plane

On the plane $\mathbb{R}^2$ we can consider various classes of functions and ask if they are composition of axial functions of the same classes.

16.4.1 Continuity

We define class of compositions of axial homeomorphisms of $\mathbb{R}^2$

$$\Theta = \{f : \mathbb{R}^2 \to \mathbb{R}^2 : f = h_n \circ ... \circ h_1, \text{ h}_\text{i} \text{ is an axial homeomprphism}\},$$

$$\Xi = \{f : [0,1]^2 \to [0,1]^2 : f = h_n \circ ... \circ h_1, \text{ h}_\text{i} \text{ is an axial homeomorphism}\}.$$

As example 16.16 shows there are homeomorphisms not in Θ or in Ξ.
Problem of Ulam in Scottish Book [9, problem 20] asks if it possible to approximate any homeomorphism of the plane by axial homeomorphisms. The answers gave Eggleston in [2].

Theorem 16.13. *Any homeomorphism of the plane $\mathbb{R}^2$ is a pointwise limit of members from Θ.*

Theorem 16.14. *There is a homeomorphism of $\mathbb{R}^2$ that is not a uniform limit (i.e. in supremum metric) of homeomorphisms in Θ.*

On a bounded set, however, we have the following

Theorem 16.15 ([2]). *Let $f\colon [0,1]^2 \to [0,1]^2$ be a homeomorphism of the square $[0,1]^2$ being identity on the boundary. Then f is a uniform limit (in supremum metric) of elements from Ξ.*

The assumption that $f|_{bd[0,1]^2}$ is identity can be only slightly relaxed to have vertices of $[0,1]^2$ as fixed points (then we can easily bring f to be identity on the boundary).

Example 16.16. There is a homeomorphism of $[0,1]^2$ that does not belong to Ξ.

This homeomorphism can be easily defined using polar coordinates (r,φ) and replacing $[0,1]^2$ by $[-1,1]^2$. Let $f:[-1,1]^2 \to [-1,1]^2$ and $f(r,\varphi) = (r,\varphi+\frac{1}{r})$ for $r \leq 1$ and extend it continuously to the rest of $[-1,1]^2$. The image of the interval $r \in [0,1],\ \varphi = 0$ is a spiral winding around $(0,0)$ infinitely many times. While by superposition of finitely many axial functions we can obtain only finitely many "twists" ([2]).
Similarly, homeomorphism $g:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(r,\varphi) = (r,\varphi+r)$ maps halfline $\varphi = 0$ to a spiral with infinitely many coils thus in supremum metric g is at infinite distance from Θ.

Knowing the results of Eggleston (especially Theorem 16.15) Ulam asked ([14], IV 2) if we can generalize Theorem 16.15 to continuous function. This time the answer is negative. Let

$$\Xi' = \{f\colon [0,1]^2 \to [0,1]^2 | f = f_n \circ .. \circ f_1,\ f_i \text{ is axial and continuous}\}.$$

Theorem 16.17. *[12] There is a continuous function $f:[0,1]^2 \to [0,1]^2$ (being identity on $bd[0,1]^2$) that is at least $\frac{1}{10}$ away from any function in Ξ' in supremum metric.*

If we consider only the images of sets, then we obtain the following Theorem:

Theorem 16.18. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be continuous and K any continuum. Then for arbitrary $\varepsilon > 0$ there is $g \in \Theta'$ which maps K onto a set closer to $f(K)$ than ε in Hausdorff metric.*

However, as Example 16.16 shows, there is a continuous mapping f such that $f([0,1]\times\{0\})$ is not equal to $g([0,1]\times\{0\})$ for any $g \in \Xi'$. We don't know if we can obtain any continuous image of $[0,1]^2$ by an element of Ξ' or Ξ.

16.4.2 Borelity

Theorem 16.19 ([11]). *Every Borel function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of three axial Borel functions. We can also demand that the first axial function is horizontal.*

The axial functions above are not onto, the question if we can require them to be onto (provided that f is) is harder. The next theorem is an answer to Ulam's question ([14], IV 2) and a question in [11].

Theorem 16.20 ([13]). *Every Borel permutation $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of eleven axial Borel permutations of $\mathbb{R}^2$.*

We can demand that the first axial permutation is, say, horizontal. Number eleven is surely not minimal.

16.4.3 Measurability

Theorems 16.19 and 16.20 hold for (Lebesgue) measurable functions and for functions with Baire property.

Theorem 16.21 ([11]). *Every function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a composition of three axial functions both measurable and with Baire property.*

16.4.4 Slides

Definition 16.22 ([1]). Function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a slide if $f(x,y) = (x, y+g(x))$ or $f(x,y) = (x+g(y), y)$ for some $g : \mathbb{R} \to \mathbb{R}$.

Slide is a very special case of axial function, note that it is a permutation of the plane (as a translation on horizontal or vertical lines). Very interesting (and surprising) result appeared in [1].

Theorem 16.23. *Every permutation of $\mathbb{R}^2$ is a composition of 209(!) slides.*

It is possible to decrease the number 209 to even below 100 (private communication with authors of [1]).

We may extend the definition of slide to any group.

Theorem 16.24 ([6]). *Let X be an infinite group and let $B \subset X^2$ such that $|B| = |X^2 \setminus B|$ then using five slides we can map set B onto a fixed set $D = \{(x,x) : x \in X\}$. Thus using ten slides we can map any set $A \subset X^2$ to a set $C \subset X^2$ provided that $|A| = |C|$ and $|X^2 \setminus A| = |X^2 \setminus C|$.*

The question that comes first to mind is if we can present continuous (or Borel, or measurable) permutations as a composition of such slides. The answer is mostly negative as measurable slide is a measure preserving mapping ([1]) so composition of slides preserves measure of every set as well. As example 16.16 shows, even measure preserving homeomorphism cannot be a composition of slides (even axial functions). It is not known which measure preserving homeomorphisms (or Borel isomorphisms) are compositions of continuous (or Borel) slides.

16.5 Higher dimensions

Definition 16.25 ([3]). Function $f : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ is axial if there exists $i \in \{1,...,n\}$ such that

$$f(x_1,...,x_n) = (x_1,..,x_{i-1}, g(x_1,...,x_n), x_{i+1},..,x_n)$$

for some $g : X_1 \times ... \times X_n \to X_i$.

Except [3] (and some questions in [14]) there is no literature about axial functions in higher dimensions.

Virtually repeating the proof of Theorem 16.10 we obtain

Theorem 16.26 ([3]). *If at least one of the sets $X_1,...,X_n$ is infinite, then every function $f : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ can be represented as a composition of $n+1$ axial functions $f = f_{n+1} \circ ... \circ f_1$.*

The choice of f_1 is determined by which X_i is the biggest (in cardinality), in particular, if $|X_1| = ... = |X_n|$ then f_1 may change for example the first coordinate.

Theorem 16.27 ([3]). *For any sets $X_1,...,X_n$ (finite or infinite) and any permutation $f : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ there is $k \in \mathbb{N}$ with $f = f_k \circ ... \circ f_1$, where all f_i are axial permutations.*

16.5.1 Borelity

The situation when we allow the axial functions to be not permutations is quite simple.

Theorem 16.28. *Every Borel function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a composition of $n+1$ axial Borel functions.*

The proof is almost identical to that of Theorem 16.19.

Borel isomorphisms

We prove a three dimensional analog of Theorem 16.20.

Theorem 16.29. *Any Borel isomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ is a composition of 22 axial Borel isomorphisms.*

Although the proof follows the proof of Theorem 16.20, it is more complicated, we suggest that the reader looks at the proofs in [13] first.

In order to prove Theorem 16.29 we list some useful facts, they are either well known or obvious.

Fact 16.30.
1. ([5], rem.1 §1 chp.13) If f is a 1-1 Borel function then f^{-1} is also Borel.
2. (Borel isomorphism theorem) ([5], cor.1 §1 chp.13) Any two Borel subsets of $\mathbb{R}$ or $\mathbb{R}^2$ of the same cardinality are Borel isomorphic.
3. For any Borel sets $A, B \subset \mathbb{R}$ with $|A| = |B|$ and $|\mathbb{R} \setminus A| = |\mathbb{R} \setminus B|$ there is a Borel permutation $f : \mathbb{R} \to \mathbb{R}$ with $f(A) = B$.
4. Composition of Borel functions is Borel.
5. Function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$, is Borel if and only if both f_1 and f_2 are Borel.
6. If a function f is axial so is f^{-1}, if f is a composition of axial functions so is f^{-1}.

We also list lemmas used to prove Theorem 16.20. In what follows set $C \subset [0,1]$ is a standard ternary Cantor set.

Lemma 16.31 ([13]). *There are three axial Borel isomorphisms $F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_3 \circ F_2 \circ F_1(\mathbb{R}^2 \setminus C \times \{0\}) = C \times \{0\}$ (thus $F_3 \circ F_2 \circ F_1(C \times \{0\}) = \mathbb{R}^2 \setminus C \times \{0\}$).*

Theorem 16.32 ([13]). *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a Borel permutation satisfying $f(C \times \{0\}) = C \times \{0\}$ (so $f(\mathbb{R}^2 \setminus C \times \{0\}) = \mathbb{R}^2 \setminus C \times \{0\}$) then f is a composition of eight axial Borel permutations of $\mathbb{R}^2$.*

Lemma 16.33 ([13]). *For every Borel permutation $f:\mathbb{R}^2\to\mathbb{R}^2$ there are four axial Borel permutations $g_3,g_2,g_1,g_0:\mathbb{R}^2\to\mathbb{R}^2$ such that $g_3\circ g_2\circ g_1\circ f\circ g_0(C\times\{0\})=C\times\{0\}$.*

We prove now three-dimensional counterparts of the statements above.

Lemma 16.34. *There are seven axial Borel isomorphisms $F_1,\dots,F_7:\mathbb{R}^3\to\mathbb{R}^3$ such that $F_7\circ\dots\circ F_1(\mathbb{R}^3\setminus C\times\{(0,0)\})=C\times\{(0,0)\}$ (thus $F_7\circ\dots\circ F_1(C\times\{(0,0)\})=\mathbb{R}^3\setminus C\times\{(0,0)\}$).*

Proof. The proof is very similar to the proof of Lemma 16.31. We may partition C into continuum many "subcantors" C_t for $t\in\mathbb{R}$ - see [13]. Sets C_t are labeled in a Borel way i.e. there is a Borel function $c:C\to\mathbb{R}$ such that $c^{-1}(t)=C_t$, moreover all C_t are translations of each other, that is $\forall_t\exists_{m_t}\ C_t-m_t=C_0$ (where $m_t=\min C_t$).
We shift the sets $C_t\times\{(0,0)\}$ on different 'z levels', let

$$F_1(x,y,z)=\begin{cases}(x,0,z+c(x)) & \text{if } x\in C, y=0\\ (x,y,z) & \text{otherwise}\end{cases}$$

(equivalently, we may write $F_1(x,0,z)=(x,0,z+t)$ for $x\in C_t$). F_1 is a slide thus a bijection, it is Borel since the function $c:C\to\mathbb{R}$ is Borel.
Now we shift sets $C_t\times\{(0,t)\}$ 'one over other' by a slide $F_2(x,0,t)=(x-m_t,0,t)$ for $x\in C_t$ and identity elsewhere, this way $F_2(C_t\times\{(0,t)\})=C_0\times\{(0,t)\}$. On every plane $z=t$ we use Lemma 16.31 with the set C replaced with C_0 to obtain three axial Borel permutations F_3,F_4,F_5 satisfying $F_5\circ F_4\circ F_3(C_0\times\{(0,t)\})=(\mathbb{R}^2\setminus(C_0\times\{0\}))\times\{t\}$.
The sixth permutation is to 'shift back' sets C_0 to the place of C_t. Define $F_6(x,0,z)=F_2^{-1}(x,0,z)=(x+m_t,0,t)$ for $x\in C_0$ and identity for other (x,y,z).
The last permutation $F_7=F_1^{-1}$ i.e.

$$F_7(x,y,z)=\begin{cases}(x,0,z-c(x)) & \text{if } x\in C, y=0\\ (x,y,z) & \text{otherwise}\end{cases}$$

(equivalently, we may write $F_7(x,0,z)=(x,0,z-t)$ when $x\in C_t$). $\square$

Theorem 16.35. *Let $f:\mathbb{R}^3\to\mathbb{R}^3$ be a Borel permutation satisfying $f(C\times\{(0,0)\})=C\times\{(0,0)\}$. Then f is a composition of sixteen axial Borel permutations of $\mathbb{R}^3$.*

Proof. The proof is almost a repetition of that of Theorem 16.32. The first seven functions $F_1,\dots,F_7$ are from Lemma above.
We define $\tilde{F}_8:C\times\{(0,0)\}\to C\times\{(0,0)\}$ as follows

$$\tilde{F_8} = F_7 \circ ... \circ F_1 \circ f \circ (F_7 \circ ... \circ F_1|_{C\times\{(0,0)\}})^{-1}.$$

It is easy to verify that $\tilde{F_8}$ is a permutation of $C\times\{(0,0)\}$ indeed. We extend $\tilde{F_8}$ to F_8 defined on the entire space $\mathbb{R}^3$ putting identity on $\mathbb{R}^3\setminus C\times\{(0,0)\}$.
Functions $F_9,...,F_{15}$ are defined so that $F_{15}\circ...\circ F_9 = (F_7\circ...\circ F_1)^{-1}$. We can verify that $F_{15}\circ...\circ F_1 = f$ on $\mathbb{R}^3\setminus C\times\{(0,0)\}$ and $F_{15}\circ...\circ F_1$ is identity on $C\times\{(0,0)\}$.
To finish we set $F_{16} = f$ on $C\times\{(0,0)\}$ and F_{16} is identity on $\mathbb{R}^3\setminus C\times\{(0,0)\}$. □

Lemma 16.36. *For every Borel permutation $f : \mathbb{R}^3 \to \mathbb{R}^3$ there are six axial Borel permutations $g_5,...,g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $g_5\circ...\circ g_1\circ f\circ g_0(C\times\{(0,0)\}) = C\times\{(0,0)\}$.*

Proof. We again follow the proof of Lemma 16.33. By a perfect set we understand additionally a compact set. By Borelity there is a perfect set $P \subset \mathbb{R}\times\{(0,0)\}$ such that $f|_P$ is continuous (thus homeomorphism). The set $f(P) \subset \mathbb{R}^3$ is a perfect set (thus of size continuum). The projection of $f(P)$ on XY-plane ($z = 0$) is a compact set, if it is of size continuum we set g_1 as identity, if not then $g_1(x,y,z) = (x+z,y,z)$ (planes perpendicular to the plane $z = 0$ become 'slant'), this way we assure that $g_1(f(P))$ has projection at XY-plane of size continuum and is still a compact set. Denote this projection by $\Pi_{XY}g_1(f(P)) \subset \mathbb{R}^2\times\{0\}$. Take a function $\tilde{g_2} : \Pi_{XY}g_1(f(P)) \to \mathbb{R}$ defined by $\tilde{g_2}(x,y) = \min\{z : (x,y,z) \in g_1(f(P))$, since $g_1(f(P))$ is compact it is a lower semicontinuous function thus Borel (even first Baire class) [5, chpt.11 §2]. We define slide $g_2(x,y,z) = (x,y,z-\tilde{g_2}(x,y))$ for $(x,y) \in \Pi_{XY}g_1(f(P))$ and $g_2(x,y,z) = (x,y,z)$ for other (x,y).

Using function g_3 we may ensure that the projection of $\Pi_{XY}g_1(f(P))$ on X-axis, denoted $\Pi_X[\Pi_{XY}g_1(f(P))]$, is compact and of size continuum. (take $g_2(x,y,z) = (x+y,y,z)$ if necessary or $g_2 =$ identity). Again the function $\tilde{g_4} : \mathbb{R} \to \mathbb{R}$ defined on $\Pi_X[\Pi_{XY}g_1(f(P))]$ by $\tilde{g_4}(x) = \min\{y : (x,y) \in \Pi_{XY}g_1(f(P))\}$ is lower semicontinuous and Borel. We define a slide $g_4(x,y,z) = (x,y-\tilde{g_4}(x),z)$ when $x \in \Pi_X[\Pi_{XY}g_1(f(P))]$ and identity for other (x,y,z).
Since $\Pi_X[\Pi_{XY}g_1(f(P))]$ is compact of cardinality continuum it contains a perfect set S, the set $P' = (g_4\circ g_3\circ g_2\circ g_1\circ f)^{-1}(S\times\{(0,0)\}) \subset P$ is perfect again (because functions g_i restricted to proper compact sets are continuous and 1-1).

Let $\tilde{g_5} : \mathbb{R} \to \mathbb{R}$ be a Borel permutation such that $\tilde{g_0}(S) = C$ (where C is the Cantor set), such permutation exists by Borel isomorphism theorem ([5], Cor.1, paragraf 1, Chapter 13). Axial function $g_5 : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $g_5(x,y,z) = (\tilde{g_5}(x),y,z)$.

Let $\tilde{g_0} : \mathbb{R} \to \mathbb{R}$ be a Borel permutation such that $\tilde{g_0}(C) = P'$. Again we define $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ as $g_0(x,y,z) = (\tilde{g_0}(x), y, z)$. We can verify that $g_5 \circ \ldots \circ g_1 \circ f \circ g_0(C \times \{(0,0)\}) = C \times \{(0,0)\}$. $\square$

Proof of Theorem 16.29. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a Borel permutation. Combining Lemma 16.36 and Theorem 16.35 (and using their notations) we obtain $g_5 \circ \ldots \circ g_1 \circ f \circ g_0 = F_{16} \circ \ldots \circ F_1$ thus $f = (g_5 \circ \ldots \circ g_1)^{-1} \circ F_{16} \circ \ldots \circ F_1 \circ g_0^{-1}$. Since g_5^{-1} and F_{16} are of the same type - they change x-coordinate, we treat $g_5^{-1} \circ F_{16}$ as one permutation and conclude the proof. $\square$

It is visible that applying the same method we obtain theorems for $\mathbb{R}^n$.

Theorem 16.37. *Any Borel isomorphism of $\mathbb{R}^n$ is a composition of finitely many axial Borel isomorphisms.*

16.5.2 Continuity

The author conjectures that Eggleston's Theorem 16.13 and 16.15 can be generalised to $\mathbb{R}^3$, however, the proof for the plane can not be applied for $\mathbb{R}^3$. As for $\mathbb{R}^n$ we do not dare to state any hypothesis.

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