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Chapter 18
Bilinear mappings – selected properties and problems

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Preliminaries

This work is a survey of results and problems connected with selected properties of bilinear mappings between function spaces. Section 1 is devoted to the lack of the counterpart of the Banach openness principle for bilinear surjections between Banach spaces. Some counterexamples and positive results are discussed. Then we deal with variants of openness for multiplication in spaces $C(K)$, for a compact K, and in other Banach algebras and function spaces. In Section 2, several phenomena of dichotomies for operators of multiplication and convolution in spaces of integrable or continuous functions are presented. Some natural properties of these operators hold either always or the sets of objects having the given property are small. The smallness is described by meagerness and porosity.

Let us introduce some basic notation. The ball with center x and radius r in a given metric space X is denoted by $B_X(x,r)$ (if X is fixed, index X will be omitted). The interior of a set $A \subset X$ will be written as $\operatorname{int}(A)$.

If (X,d_X) and (Y,d_Y) are two metric spaces, then we consider the Cartesian product $X\times Y$ as a metric space with the maximum metric

$$d((x_1,y_1),(x_2,y_2)) := \max\{d_X(x_1,x_2),d_Y(y_1,y_2)\}.$$

As usual, X^* denotes the dual space of a Banach space X.

If X,Y,Z are Banach spaces, then by $L(X,Y)$ we denote the space of all continuous linear mappings from X to Y and by $B(X\times Y,Z)$ we denote the space of all continuous, bilinear mappings from $X\times Y$ to Z. It is known that $B(X\times Y,Z)$ is a Banach space with the following norm (we denote the norms on X, Y, Z and $B(X\times Y,Z)$ by the same symbol $||\cdot||$):

$$||T|| := \sup\{||T(x,y)||\colon x\in X,\ y\in Y,\ ||x||\le 1,\ ||y||\le 1\}.$$

Note that $||T|| = \inf\{K>0\colon \forall_{x\in X}\ \forall_{y\in Y}\ ||T(x,y)||\le K||x||\,||y||\}$.

We will deal several times with Banach spaces of integrable functions. If (X,Σ,μ) is a measure space and $p\in[1,\infty]$, then by $L^p(X)$ (L^p in short) we denote the Banach space of all measurable real functions f (more formally, all equivalence classes of functions equal μ-a.e.) such that $||f||_p<\infty$, where $||f||_p$ is given by:

$$||f||_p := \begin{cases} \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}}, & \text{if } p\in[1,\infty);\\ \text{ess sup}\,|f|, & \text{if } p=\infty,\end{cases}$$

where ess sup$\,|f| := \inf\{u>0\colon |f|\le u,\ \mu\text{-a.e.}\}$.

18.1 Openness of bilinear mappings

18.1.1 The lack of the openness principle for bilinear mappings

A mapping f between topological spaces X and Y is called *open* if the image of every open set in X is open in Y. The mapping f is called *open at a point* $x\in X$ whenever $f(x)$ is in the interior of the image $f[U]$ for every open neighbourhood U of x. It follows that f is open if and only if f is open at every point of X. Easy examples show that a continuous mapping need not be open. However, there are important situations where continuous functions with additional properties are open or at least they have some local features of openness. The Banach openness principle in functional analysis and the openness principle for holomorphic complex-valued functions of several variables are well-known important instances of criteria of global openness of mappings. A local kind of openness plays a key role in the local inverse theorem and in the implicit function theorem used in classical real analysis.

Through this section, X, Y, Z will usually denote Banach spaces. If they are more general spaces, this will be specified.

The Banach openness principle states that a continuous linear surjection between Banach spaces is open. This can be extended to Fréchet spaces [37]. Some further extensions are known where the assumptions either on the operator or on the spaces have been relaxed. This series of results is cited in [33] where a new general criterion of openness for mappings is proved, with the assumed properties independent of continuity and linearity.

The bilinear counterpart of the Banach openness principle is false even for finite-dimensional Banach spaces. The full description of spaces and continuous surjections for which it is true, seems difficult. We will show some known counterexamples and positive results concerning selected special cases. Let us start with a simple positive observation which should be known.

Proposition 18.1. *Given a Banach space X, the bilinear continuous functional $T\colon X \times X^* \to \mathbb{R}$, defined by $T(x,y) := y(x)$ for $(x,y) \in X \times X^*$, is an open surjection.*

Proof. It is known that every nonzero functional $z \in X^*$ is an open surjection; cf. [20], Exercise 2.28. Hence T is a surjection. Fix balls $B(x,r) \subset X$, $B(y,R) \subset X^*$ and define $U := T[B(x,r) \times B(y,R)]$. We have

$$U = \bigcup_{z \in B(y,R)} z[B(x,r)],$$

so, if $0 \notin B(y,R)$ then U is open. Assume that $0 \in B(y,R)$. We have then

$$U = \bigcup_{z \in B(y,R) \setminus \{0\}} z[B(x,r)] \cup \{0\}.$$

Hence it suffices to show that $0 \in \operatorname{int} U$. Since $0 \in B(y,R)$, we can pick $\varepsilon > 0$ such that $B(0,\varepsilon) \subset B(y,R)$. Fix any $y' \in X^* \setminus \{0\}$ and pick $x' \in B(x,r)$ such that $\alpha := |y'(x')| > 0$. Then

$$U \supset \bigcup_{t \in (-\varepsilon/||y'||, \varepsilon/||y'||)} \{(ty')(x')\} = \left(-\frac{\varepsilon\alpha}{||y'||}, \frac{\varepsilon\alpha}{||y'||}\right).$$

□

Examples witnessing that a bilinear continuous surjection need not be open at the origin were given by Cohen [17] and Horowitz [27] – they answered a question of Rudin. All these examples violate the necessary condition for the openness of a bilinear surjection at $(0,0)$, described in the following lemma.

Lemma 18.2. *Let X,Y,Z be normed spaces and let $T\colon X\times Y\to Z$ be a bilinear surjection open at $(0,0)$. Then for every bounded set $E\subset Z$ there is a bounded set $A\subset X\times Y$ such that $T[A]=E$.*

Proof. Since T is open at $(0,0)$ and $T(0,0)=0$, there is $r>0$ such that

$$B_Z(0,r)\subset T[B_X(0,1)\times B_Y(0,1)]. \tag{18.1}$$

Let $E\subset Z$ be bounded and pick $R>0$ such that $E\subset B_Z(0,R)$. Using (18.1) and the bilinearity of T, we have

$$E\subset B_Z(0,R)=\frac{R}{r}B_Z(0,r)\subset\frac{R}{r}T[B_X(0,1)\times B_Y(0,1)]\subset$$
$$\subset T\left[B_X\left(0,\sqrt{\frac{R}{r}}\right)\times B_Y\left(0,\sqrt{\frac{R}{r}}\right)\right].$$

Then the set

$$A:=T^{-1}[E]\cap\left(B_X\left(0,\sqrt{\frac{R}{r}}\right)\times B_Y\left(0,\sqrt{\frac{R}{r}}\right)\right)$$

is as desired. □

Now, we sketch the counterexample from [27] where $\mathbb{C}$ was used instead of $\mathbb{R}$ (however, the reasoning is the same).

Example 18.3. Let $T\colon\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^4$ be defined by

$$T(x,y):=(x_1y_1,x_1y_2,x_1y_3+x_3y_1+x_2y_2,x_3y_2+x_2y_1)$$

where $x:=(x_1,x_2,x_3)$, $y:=(y_1,y_2,y_3)$. Obviously, T is bilinear and continuous. We omit the proof that T is surjective, given in [27]. Let

$$E:=\left\{\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},1\right)\in\mathbb{R}^4\colon n\in\mathbb{N}\right\}.$$

Then E is bounded. Suppose that T is open at $(0,0)$. By Lemma 18.2 pick a bounded set $A\subset\mathbb{R}^3\times\mathbb{R}^3$ such that $T[A]=E$. For each $n\in\mathbb{N}$, choose $(x^{(n)},y^{(n)})\in A$ such that $T(x^{(n)},y^{(n)})=(1/n,1/n,1/n,1)$. Let

$$w_n := (x_1^{(n)} + x_2^{(n)} + x_3^{(n)})(y_1^{(n)} + y_2^{(n)} + y_3^{(n)}).$$

By the calculations in [27], we have $w_n = (3/n) + 2 - n$. Since A is bounded, so is $\{w_n \colon n \in \mathbb{N}\}$ which yields the contradiction.

A simple example of a continuous bilinear surjection $T_0 \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, non-open at $(0,1,1)$, can be found in [37], Chapter 2, Exercise 11. Namely, let $T_0(t,x,y) := (tx,ty)$. Then

$$(0,0) = T_0(0,1,1) \notin \operatorname{int} T_0[(-1,1) \times (1/2,3/2)^2].$$

Moreover, it can be shown that T_0 is not open at any point $(0,x,y)$ with $x^2 + y^2 > 0$, and it is open at the remaining points.

Note that Horowitz in [27] modified his example to obtain the respective infinite-dimensional case. Another infinite-dimensional example was given earlier in [17] with a more involved construction. An important infinite-dimensional example, due to Fremlin, will be discussed in the next subsection. An interesting example dealing with the Banach algebra of operators on ℓ^∞ was given in [14].

Question 18.4. The Cauchy product $S \colon \ell_1 \times \ell_1 \to \ell_1$ *(which can be treated as a special case of convolution), given by*

$$S((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) = \left(\sum_{i=0}^{n} x_i y_{n-i} \right)_{n\geq 0},$$

is a continuous bilinear surjection. Is it an open mapping?

In [8], the following strengthened notion of openness for mappings was introduced. We say that a mapping f between metric spaces X and Y is *uniformly open* whenever

$$\forall_{\varepsilon>0}\, \exists_{\delta>0}\, \forall_{x\in X}\; B(f(x),\delta) \subset f[B(x,\varepsilon)].$$

It was observed that arctan is an open function from $\mathbb{R}$ into $\mathbb{R}$ which is not uniformly open. In [8], some important examples of multiplication were found where the uniform openness holds. Namely, we have

Theorem 18.5 ([8]). *Multiplication* $\Phi \colon X^2 \to X$ *is uniformly open in the following cases:*

(1) $X := \mathbb{R}$;
(2) X is the Banach algebra of real-valued bounded functions measurable with respect to a given σ*-algebra of subsets of a fixed set E;*

(3) $X :=$ bBor$_\alpha$ is the Banach algebra of real-valued bounded functions on a fixed metrizable space that are Borel measurable of class α (where α is an ordinal, $0 < \alpha < \omega_1$).

(In cases (2), (3)*, X is endowed with the supremum norm.)*

Note that, statement (1) holds with an analogous proof if $\mathbb{R}$ is replaced by $\mathbb{C}$.

Given a measure space (X, Σ, μ), consider the respective Banach spaces L^p, $p \in [1, \infty]$, of integrable functions. The main result of [8] is the following

Theorem 18.6 ([8]). *Multiplication from $L^p \times L^q$ to L^1 (for $p \in [1, \infty]$ and $1/p + 1/q = 1$) given by $(f, g) \mapsto fg$ is an open mapping, being a continuous bilinear surjection.*

Recently, this result has been improved in [9] by showing that multiplication in Theorem 18.6 is a uniformly open mapping.

18.1.2 Openness of multiplication in spaces of continuous functions

Let Z and T be topological spaces. We say that a function $\Phi\colon Z \to T$ is *weakly open* if, for every nonempty open set $U \subset Z$, the interior of $\Phi(U)$ is nonempty in T. Weak openness was considered by many authors (see, e.g., Burke [15]). We will mainly focus on the operation $\Phi\colon C(X) \times C(X) \to C(X)$ of multiplication $\Phi(f, g) := fg$ where $C(X)$ denotes the space of all continuous real functions defined on a topological space X, with the metric of uniform convergence

$$d(f, g) := \min\{1, \sup\{|f(x) - g(x)| \colon x \in X\}\}.$$

For $U, V \subset C(X)$ we write $U \cdot V$ instead of $\Phi[U \times V]$.

If X is a compact Hausdorff space then $C(X)$ forms a Banach algebra equipped with the supremum norm. It turns out that, for $X := [0, 1]$, multiplication in $C[0, 1]$ yields the following very simple example of the lack of openness for a continuous bilinear surjective mapping between infinite dimensional spaces (compare with the mentioned earlier examples by Cohen [17] and Horowitz [27]).

Example 18.7 ([12]). (due to D.H. Fremlin) Let $f(x) = x - 1/2$, $x \in [0, 1]$ and $r = 1/2$. Then $f^2 \notin \operatorname{int}(B(f, r) \cdot B(f, r))$. Indeed, it is easy to see that every function in $B(f,r)$ possesses zeros, whereas every neighbourhood of f^2 contains functions $f^2+\delta$, $\delta>0$, without zeros. Consequently, multiplication is not an open mapping in $C[0,1]$ since it is not open at (f,f).

Fremlin's example was an impetus for Wachowicz to examine the weak openness of multiplication in $C[0,1]$. In [44] he proved the following theorem.

Theorem 18.8 ([44], [12]). *Multiplication* $\Phi\colon C[0,1]\times C[0,1]\to C[0,1]$ *is a weakly open mapping.*

The proof of the above theorem presented in [44] was complicated and it was based on the density of the set of all polygonal functions in $C[0,1]$, and the fact that in any two open balls in $C[0,1]$, one can find two polygonals (respectively) with finite disjoint sets of zeros. To this aim, the technique of "gluing of open balls" on adjacent closed intervals was used. This proof of Theorem 18.8 has been nowhere published except for the PhD thesis [44], although the technique of gluing of open balls seems to be interesting and its variant was used by Balcerzak, Wachowicz and Wilczyński in [12] where a much shorter and more transparent proof of Theorem 18.8 was presented. This technique was also used by Wachowicz [43] in the proof of an analogous result obtained in the space $C^n[0,1]$ of all functions $f\colon [0,1]\to\mathbb{R}$ with continuous n-th derivative (here also Fremlin's example works).

A next essential step was made by Komisarski in [29] who considered the operation of multiplication in $C(X)$ where X is a compact Hausdorff space. Komisarski linked the concepts of openness and weak openness with the topological dimension of X, and he generalized Theorem 18.8 in the following way.

Theorem 18.9 ([29]). *Let X be a compact Hausdorff space. The following equivalences hold:*

(1) multiplication in $C(X)$ is open iff $\dim X<1$,
(2) multiplication in $C(X)$ is weakly open and not open iff $\dim X=1$,
(3) multiplication in $C(X)$ is not weakly open iff $\dim X>1$,

where $\dim X$ *denotes the topological (covering) dimension of X.*

Another result in that matter was obtained by Kowalczyk in [30] for the multiplication map in $C(X)$ where $X=(0,1)$. This multiplication is not continuous (cf. also [10], Prop. 3) which makes it different from the cases considered earlier. In spite of this, the following theorem holds.

Theorem 18.10 ([30]). *Multiplication in the space* $C(0,1)$ *is a weakly open mapping.*

It is also interesting that a technique used in the proof of Theorem 18.10 gives (with the aid of the Tietze extension theorem) a new proof of Theorem 18.8. The result established by Kowalczyk was then strengthened by Balcerzak and Maliszewski in [10]. The authors introduced a notion of *dense weak openness*, a stronger version of weak openness, cf. [30]. If Y, Z are topological spaces then a function $T\colon Y \to Z$ is called *densely weakly open* whenever $\operatorname{int}(T[U])$ is a dense set in $T[U]$ for any nonempty open set $U \subset Y$. The following theorem was proved.

Theorem 18.11 ([10]). *If $X \subset \mathbb{R}$ is an interval, then multiplication in $C(X)$ is densely weakly open.*

In particular, this yields a next proof of Theorem 18.8.

Now, let us turn to the article [13] by Behrends who characterized the pairs (f,g) at which multiplication in $C[0,1]$ is open. To formulate the main result of [13] we need some definitions. For every pair (f,g) of functions in $C[0,1]$, we consider a map $\gamma(t) = (f(t),g(t))$, $t \in [0,1]$, which generates the so called *path* in $\mathbb{R}^2$ associated with (f,g). Denote by Q^{++}, Q^{+-}, Q^{-+}, Q^{--} the four quadrants of the plane, i.e., the sets

$$\{(x,y)\colon x,y \geq 0\},\ \{(x,y)\colon x \geq 0,\ y \leq 0\},$$

$$\{(x,y)\colon x \leq 0,\ y \geq 0\},\ \{(x,y)\colon x,y \leq 0\},$$

respectively.

Definition 18.12 ([13]). Let $t_0 \in (0,1)$. We say that a path $\gamma = (f,g)$ has a *positive saddle point crossing at t_0* if $\gamma(t_0) = 0$ and there exists $\varepsilon > 0$ such that the following two conditions hold:

(1) $\gamma(t) \in Q^{++} \cup Q^{--}$ for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$;
(2) there are $t_1 \in [t_0 - \varepsilon, t_0]$ and $t_2 \in [t_0, t_0 + \varepsilon]$ such that $\gamma(t_1) \in Q^{++} \setminus \{(0,0)\}$ and $\gamma(t_2) \in Q^{--} \setminus \{(0,0)\}$ or vice versa.

A *negative saddle point* is defined similarly: then γ moves from Q^{-+} to Q^{+-} or from Q^{+-} to Q^{-+}.

Behrends obtained the following characterization.

Theorem 18.13 ([13]). *Let $f,g \in C[0,1]$ and let $\gamma = (f,g)$ be the associated path in $\mathbb{R}^2$. Then the following assertions are equivalent:*

1. *Multiplication in $C[0,1]$ is open at (f,g).*
2. *For every $r > 0$ there exists $\delta > 0$ such that the functions $fg + \delta$, $fg - \delta$ are in $B(f,r) \cdot B(g,r)$.*

3. *γ has no positive and no negative saddle point crossings.*

Behrends described details of his proof using a nice geometrical interpretation, connected with “walks in a landscape" with hills and valleys where an accompanying dog can move in a certain prescribed way. Behrends proved also the following generalization of Theorem 18.13 in his next article [14].

Theorem 18.14 ([14]). *Let $f_1,\dots,f_n \in C[0,1]$ be given. The following assertions are equivalent:*

1. *$f_1 \cdot f_2 \cdots f_n$ lies in the interior of $B(f_1,r)\cdot B(f_2,r)\cdots B(f_n,r)$ for every $r>0$.*
2. *The associated path $\gamma\colon t \mapsto (f_1(t),\dots,f_n(t))$ is admissible.*

The notion of *admissibility* defined in [14] is much more complicated than the respective description in [13], and indeed the above theorem is not a simple generalization of Theorem 18.13. In [14] Behrends considered also multiplication in $C_{\mathbb{C}}[0,1]$, the Banach algebra of continuous complex-valued functions on $[0,1]$, endowed with the supremum norm. He proved the following theorem which makes a contrast to the real-valued case.

Theorem 18.15 ([14]). *If $O_1,\dots,O_n$ are open subsets of $C_{\mathbb{C}}[0,1]$ then $O_1\cdots O_n$ is also open.*

Currently, we focus on extensions of Theorem 18.8 to the case where, instead of multiplication, we consider some other functions of two variables. This was investigated in general by Kowalczyk [31], and also by Wachowicz [44] who considered the operations of addition, minimum, maximum and composition. Kowalczyk in [31] proposed for $C(X)$, where X is a compact Hausdorff space, studies of openness for the operation $\Phi\colon C(X)\times C(X)\to C(X)$ determined by a continuous function $\varphi\colon \mathbb{R}^2\to\mathbb{R}$ by the formula

$$\Phi(f,g)(x) := \varphi(f(x),g(x)),\ x\in X.$$

Kowalczyk obtained a number of interesting results which generalize the above-mentioned theorems by Wachowicz and Komisarski. He introduced the following definition.

Definition 18.16 ([31]). Let $\varphi\colon \mathbb{R}^2\to\mathbb{R}$ be a continuous function and let K be a nonempty subset of $\mathbb{R}^2$. We say that a function $\alpha\colon K\to\mathbb{R}^2$ is *φ-increasing* (*φ-decreasing*, respectively) if it is continuous and for every $v\in K$ the function $\varphi_v\colon [0,1]\to\mathbb{R}$ defined by $\varphi_v(t)=\varphi((1-t)v+t\alpha(v))$ is strictly increasing (strictly decreasing, respectively). If α is φ-increasing or φ-decreasing, it will be called *φ-monotone*.

Let us observe (see [31], Example 1) that, for some φ and $K \subset \mathbb{R}^2$, a φ-increasing (or φ-decreasing) function α does not exist (it is the case for $\varphi(x,y) = x \cdot y$ and $K = [0,1] \times \{0\}$).

Theorem 18.17 ([31]). *Let X be a compact Hausdorff space, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, $\Phi\colon C(X) \times C(X) \to C(X)$ be the operation determined by φ, and $K \subset \mathbb{R}^2$ be nonempty such that there exist a φ-increasing function and a φ-decreasing function defined on K. Then for each pair of continuous functions $f, g \in C(X)$ such that $\{(f(x), g(x))\colon x \in X\} \subset K$ and for each $r > 0$, the image $\Phi[B(f,r) \times B(g,r)]$ contains $B(\Phi(f,g), \varepsilon)$ for some $\varepsilon > 0$.*

Corollary 18.18 ([31]). *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. If there exist a φ-increasing function defined on the whole plane, and a φ-decreasing function defined on the whole plane, then for every Hausdorff compact topological space X, the operation $\Phi\colon C(X) \times C(X) \to C(X)$ determined by φ is open.*

The next result is connected with a strengthening of assumptions on a function φ.

Theorem 18.19 ([31], Theorem 4). *Let X be a Hausdorff compact space. If $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function such that $\nabla\varphi(v) \neq (0,0)$ for $v \in \mathbb{R}^2$, then the operation $\Phi\colon C(X) \times C(X) \to C(X)$ determined by φ is open.*

On the other hand, the following result holds.

Theorem 18.20 ([31], Theorem 5). *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let X be a topological Hausdorff compact space. If φ has a local extremum, then the operation $\Phi\colon C(X) \times C(X) \to C(X)$ determined by φ is not open.*

In further considerations, Kowalczyk observed that, if there are φ-increasing and φ-decreasing functions defined not on the whole plane but on a “big" set, then properties of Φ determined by φ, depend on the dimension of X. Kowalczyk showed a few facts on that topic which we join together in the following theorem. For this we need one else definition.

Definition 18.21. We say that a function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ has a *constant point* $x_0 \in \mathbb{R}$ if $\varphi(x_0,t) = \varphi(t,x_0) = \varphi(x_0,x_0)$ for $t \in \mathbb{R}$.

Theorem 18.22 ([31], Corollary 2). *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function without local extremum and with a constant point $x_0 \in \mathbb{R}$, X be a Hausdorff compact space and $\Phi\colon C(X) \times C(X) \to C(X)$ be the operation determined by φ. Moreover, assume that for some boundary subsets $A, B \subset \mathbb{R}$, there exist a φ-increasing function $\alpha\colon \mathbb{R}^2 \setminus (A \times B) \to \mathbb{R}^2$ and a φ-decreasing function $\beta\colon \mathbb{R}^2 \setminus (A \times B) \to \mathbb{R}^2$. Then*

(1) Φ is open iff $\dim X \le 0$,
(2) Φ is weakly open iff $\dim X \le 1$,
(3) Φ is not weakly open iff $\dim X \ge 2$.

Theorem 18.9 by Komisarski is a particular case of the above theorem since it suffices to take $\varphi(x,y) = x \cdot y$ which fulfills the assumptions. However, if $\varphi(x,y) = x^2 - y^2$, Theorem 18.22 is also applicable (for details, see [31], Theorem 11 and Example 2).

In the last part of the paper [31] Kowalczyk gave some necessary conditions for the openness and the weak openness of the operation $\Phi \colon C(X) \times C(X) \to C(X)$ determined by a function φ where X is a connected Hausdorff compact space.

18.1.3 Opennes of multiplication in other function spaces

Consider the Banach algebra BV of all real-valued functions on $[-1,1]$ with the bounded variation (interval $[-1,1]$ is used only by technical reasons). The norm on BV is given by $||f|| := |f(0)| + V_{-1}^1(f)$ where $V_{-1}^1(f)$ denotes the variation of f on $[-1,1]$. The following proposition shows that the argument used in Example 18.7 does not work in BV.

Proposition 18.23 ([16]). *Let $f_0(x) := x$ for $x \in [-1,1]$. There exists $\delta > 0$ such that $f_0^2 + \delta$ can be expressed in the form fg where $f,g \in B(f_0,1)$ in* BV.

Proof. Fix $c \in (0,1/3)$ and choose δ such that $0 < \delta < (c-c^2)/(5+3c)$. Then let
$$\alpha(x) := \begin{cases} -c & \text{if } -1 \le x < 0 \\ c & \text{if } 0 \le x < 1. \end{cases}$$
We have $\alpha \in \mathrm{BV}$ and $||\alpha|| = |\alpha(0)| + V_{-1}^1(\alpha) = c + 2c = 3c < 1$. We set $f := f_0 + \alpha$. Hence $f \in B(f_0,1)$ in BV. We want to obtain $f_0^2 + \delta = fg$ for some $g \in B(f_0,1)$. This g will be of the form $f_0 + \beta$ with $\beta \in \mathrm{BV}$ and $||\beta|| < 1$. Thus $x^2 + \delta = (x + \alpha(x))(x + \beta(x))$ for $x \in [-1,1]$. Hence let
$$\beta(x) := \frac{\delta - x\alpha(x)}{x + \alpha(x)} \text{ for } x \in [-1,1].$$
For $0 \le x \le 1$, we have $\beta(x) = (\delta - xc)/(x+c)$ and $\beta'(x) = -(c^2+\delta)/(x+c)^2 < 0$. Hence
$$V_0^1(\beta) = \beta(0) - \beta(1) = (c^2+\delta)/(c+c^2). \tag{18.2}$$

For $x \in [-1,0)$, we have $\beta(x) = (\delta + xc)(x-c)$ and $\beta'(x) = -(c^2+\delta)/(x-c)^2 < 0$. Hence

$$V_{-1}^{0}(\beta) = (\beta(-1) - \beta(0^-)) + |\beta(0) - \beta(0^-)| = \frac{c^2+\delta}{c(1+c)} + \frac{2\delta}{c}. \tag{18.3}$$

Now, by (18.2) and (18.3), we have $\beta \in \mathrm{BV}$ and

$$||\beta|| = |\beta(0)| + V_{-1}^{1}(\beta) = 2\frac{c^2+\delta}{c(1+c)} + 3\frac{\delta}{c} = \frac{2c^2+5\delta+3\delta c}{c(1+c)}.$$

Then we calculate that $||\beta|| < 1$ is equivalent to $\delta < (c-c^2)/(5+3c)$ which is true by our choice of δ. $\square$

Question 18.24. Is multiplication in BV *open?*

Let us finish with some other questions and remarks.

Question 18.25. Consider the subspace CBV of BV consisting of continuous functions with a bounded variation. Note that multiplication in CBV is not open since Fremlin's example works. Is multiplication in CBV weakly open?

It might be interesting to consider the question on the weak openness of multiplication in CBV when the Adams metric is considered in these spaces:

$$d(f,g) := \int_{-1}^{1} |f-g| + |V(f) - V(g)|.$$

Observe that Fremlin's example, in spite of changing a metric, is still valid. Indeed, let id be the identity function on $[-1,1]$. If we suppose that $\mathrm{id}^2 \in \mathrm{int}(B(\mathrm{id},1/2)\cdot B(\mathrm{id},1/2))$, then for some $\varepsilon > 0$ we have $\mathrm{id}^2 + \varepsilon \in \mathrm{int}(B(\mathrm{id},1/2)\cdot B(\mathrm{id},1/2))$. So, there are functions $f,g \in B(\mathrm{id},1/2)$ such that $fg = \mathrm{id}^2 + \varepsilon$. However, note that since f,g are continuous, both these functions are either positive or negative on $[-1,1]$. Assume for instance that both f and g are positive. Then $d(f,\mathrm{id}) \geq \int_{-1}^{0} |f - \mathrm{id}| \geq \int_{-1}^{0} |\mathrm{id}| = 1/2$ which yields $f \notin B(\mathrm{id},1/2)$, a contradiction. Observe that the same example works for the space of all continuous functions on $[-1,1]$ with the integral norm, so we can also ask about the weak openness of multiplication in this space.

Recall that multiplication is not continuous in *CBV* under the Adams metric – this is a consequence of the example given by Adams and Clarkson in [3] where it is shown that even the operation of addition in *CBV* is not continuous. Using the composition with the exponential function one can prove that it is the case for multiplication, too. For more details on the Adams metric, see also [2] and [4].

18.2 Dichotomies for bilinear mappings

18.2.1 Introduction

Let (X,Σ,μ) be a measure space. If $p,q,r \in (0,\infty]$ satisfy $1/p+1/q=1/r$, then by a general version of the Hölder inequality [21], Exercise 1.1.2, we see that $\Phi(f,g)=fg\in L^r$ for any $f\in L^p$ and $g\in L^q$ (later we will write fg instead of $\Phi(f,g)$). Hence in this case,

$$\{(f,g)\in L^p\times L^q : fg\in L^r\} = L^p\times L^q.$$

On the other hand, Balcerzak and Wachowicz in [11] proved the following theorem (we consider the Lebesgue measure on the interval $[0,1]$ and c_0 denotes the space of all real sequences which converge to 0, endowed with the supremum norm; we abbreviate $x=(x_k)$, $y=(y_k)$, etc.).

Theorem 18.26.

(i) The set $\left\{(f,g)\in L^1[0,1]\times L^1[0,1] : fg\in L^1[0,1]\right\}$ *is a meager subset of* $L^1[0,1]\times L^1[0,1]$.
(ii) The set $\left\{(x,y)\in c_0\times c_0 : (\sum_{k=1}^{n} x_k y_k)_{n=1}^{\infty} \text{ is bounded}\right\}$ *is a meager subset of* $c_0\times c_0$.

Note that Wachowicz in [44] extended part (i) by considering $L^p[0,1]$ space for $p\in[1,\infty)$.

This result suggests the following general problem: given spaces of real functions X,Y,Z and a bilinear map T defined on $X\times Y$, investigate the size of the set

$$\{(f,g)\in X\times Y : T(f,g)\in Z\}.$$

In this section we present several results which deal with such a problem, mainly in the case when T is multiplication or convolution. In most situations an interesting phenomenon is observed – either such sets are equal $X\times Y$, or they are very small (σ-porous and, in particular, meager).

Now we describe the notions of smallness that our study relies on.

If (X,τ) is a Baire topological space, then it is known that *meager* sets (or sets of the first category, that is sets which are countable unions of nowhere dense sets) can be considered as small sets. It turns out that within complete metric spaces we can define a notion of smallness more restrictive than meagerness – the *σ-porosity*. The main idea is that we can modify the “ball" definition of nowhere density by the requirement that the sizes of “pores" are somehow estimated. This idea can be formalized in many ways, so there are many notions of porosity (cf. the survey papers of Zajíček [45] and [46]). Note that σ-porous sets (that is, countable unions of porous sets) are meager, but the really interesting fact is that in every "reasonable" complete metric space there are sets which are meager but not σ-porous. Hence the fact that a particular set is not only meager but also σ-porous, means that it is even smaller.

Let X be a metric space and $M \subset X$. We say that M is *α-lower porous* (for $\alpha > 0$), if

$$\forall_{x\in M}\ \forall_{\beta\in\left(0,\frac{\alpha}{2}\right)}\ \exists_{R_0>0}\ \forall_{R\in(0,R_0)}\ \exists_{z\in X}\ B(z,\beta R) \subset B(x,R)\backslash M.$$

If M is a countable union of α-lower porous sets (for the same constant $\alpha > 0$), then we say that M is *σ-α-lower porous*. The notion of α-lower porosity is commonly known – it appears at the beginning of the mentioned survey [46] (in fact, this notion is defined in [46] in a bit different but equivalent way). It turns out that there are sets which are meager and are not σ-α-lower porous for any $\alpha > 0$. For more information on porosity, we refer the reader to [45] and [46].

18.2.2 Bilinear mappings and the Banach-Steinhaus principle

Recall the well known Banach-Steinhaus principle:

Theorem 18.27. *Let X, Z be Banach spaces and let $\{T_n : n \in \mathbb{N}\} \subset L(X,Z)$. Put*

$$E := \{x \in X : (T_n(x)) \text{ is bounded}\}.$$

The following conditions are equivalent:

(i) E is meager in X;
(ii) $E \neq X$;
(iii) $\sup\{||T_n|| : n \in \mathbb{N}\} = \infty$.

In fact, implication (iii)⇒(i) is a formulation of the Banach-Steinhaus principle – the rest are trivial. Jachymski showed in [28] that an analogous fact holds for bilinear mappings:

Theorem 18.28 ([28]). *Let X, Y, Z be Banach spaces and let $\{T_n : n \in \mathbb{N}\} \subset B(X \times Y, Z)$. Put*

$$E := \{(x,y) \in X \times Y : (T_n(x,y)) \text{ is bounded}\}.$$

The following conditions are equivalent:

(i) E is meager in $X \times Y$;
(ii) $E \neq X \times Y$;
(iii) $\sup\{||T_n|| : n \in \mathbb{N}\} = \infty$.

Proof. Implications (i)⇒(ii) and (ii)⇒(iii) are trivial. We prove (iii)⇒(i). We will show that for any $M > 0$, the set

$$E_M := \{(x,y) \in X \times Y : ||T_n(x,y)|| \leq M \text{ for every } n \in \mathbb{N}\}$$

is nowhere dense. Assume on the contrary that E_M is not nowhere dense. Since it is also closed, E_M has nonempty interior. Then for some $(x_0, y_0) \in X \times Y$ and $r > 0$, $B((x_0,y_0),r) \subset E_M$. Now let $x \in X$ and $y \in Y$ be such that $||x|| \leq 1$ and $||y|| \leq 1$. Then for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
||T_n(x,y)|| &= \frac{4}{r^2}\left|\left|T_n\left(\frac{r}{2}x, \frac{r}{2}y\right)\right|\right| = \frac{4}{r^2}\left|\left|T_n\left(\frac{r}{2}x, y_0 + \frac{r}{2}y\right) - T_n\left(\frac{r}{2}x, y_0\right)\right|\right| = \\
&= \frac{4}{r^2}\left|\left|T_n\left(x_0 + \frac{r}{2}x, y_0 + \frac{r}{2}y\right) - T_n\left(x_0, y_0 + \frac{r}{2}y\right) - T_n\left(x_0 + \frac{r}{2}x, y_0\right) + T_n(x_0,y_0)\right|\right| \leq \\
&\leq \frac{4}{r^2}\left|\left|T_n\left(x_0 + \frac{r}{2}x, y_0 + \frac{r}{2}y\right)\right|\right| + \frac{4}{r^2}\left|\left|T_n\left(x_0, y_0 + \frac{r}{2}y\right)\right|\right| + \frac{4}{r^2}\left|\left|T_n\left(x_0 + \frac{r}{2}x, y_0\right)\right|\right| + \\
&+ \frac{4}{r^2}||T_n(x_0,y_0)|| \leq \frac{4}{r^2}(M+M+M+M) = \frac{16M}{r^2}.
\end{aligned}$$

Hence $||T_n|| \leq \frac{16M}{r^2}$ for any $n \in \mathbb{N}$. This is a contradiction. □

As immediate corollaries, we get the following results. The first one is an extension of Theorem 18.26, part (i), proved in [28]. We will give a proof of the second one.

Theorem 18.29 ([28]). *Assume that z is any sequence of reals and let*

$$E_z := \left\{(x,y) \in c_0 \times c_0 : \left(\sum_{k=1}^{n} z_k x_k y_k\right)_{n=1}^{\infty} \text{ is bounded}\right\}.$$

Then the following statements are equivalent:

(i) E_z is meager in $c_0 \times c_0$;
(ii) $E_z \neq c_0 \times c_0$;
(iii) $z \notin l^1$, that is $\sum_{n=1}^{\infty} |z_n| = \infty$.

Theorem 18.30 ([28]). *Assume that z is any sequence of reals, $p \in [1,\infty]$, and let*

$$E_z := \left\{(x,y) \in c_0 \times l^p : \left(\sum_{k=1}^{n} z_k x_k y_k\right)_{n=1}^{\infty} \text{ is bounded}\right\}.$$

Then the following statements are equivalent:

(i) E_z *is meager in* $c_0 \times l^p$*;*
(ii) $E_z \neq c_0 \times l^p$*;*
(iii) $z \notin l^q$*, where* q *is such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Note that in the above formulation we use the abbreviation $\frac{1}{\infty} := 0$.

Proof. For every $n \in \mathbb{N}$, set $T_n(x,y) := \sum_{k=1}^{n} x_k y_k z_k$. Then every T_n is bilinear, continuous and

$$E_z = \{(x,y) \in c_0 \times l^p : (T_n(x,y)) \text{ is bounded}\}.$$

Using the fact that $(l^p)^* = l^q$, we get $||T_n|| = (\sum_{k=1}^{n} |z_k|^q)^{\frac{1}{q}}$ (in the case $q < \infty$) or $||T_n|| = \max\{|z_1|, \dots, |z_n|\}$ (in the case $q = \infty$). Hence $\sup\{||T_n|| : n \in \mathbb{N}\} = \infty$ if and only if $z \notin l^q$. The result follows then from Theorem 18.28. □

Using Theorem 18.28, we could try to strengthen, in the same way, part (ii) of Theorem 18.26. However, the problem would appear with the bilinearity of an appropriate functional. To overcome these difficulties, Jachymski proved a nonlinear version of the Banach-Steinhaus principle which is an extension of Theorem 18.28. We will present its particular version.

A function $\varphi\colon X \to \mathbb{R}_+$ is *L-subadditive* for some $L \geq 1$, if $\varphi(x+y) \leq L(\varphi(x) + \varphi(y))$ for $x, y \in X$, and is *even*, if $\varphi(-x) = \varphi(x)$ for $x \in X$.

Theorem 18.31 ([28]). *Given* $k \in \mathbb{N}$*, let* $X_1, \dots, X_k$ *be Banach spaces,* $X = X_1 \times \dots \times X_k$*. Assume that* $L \geq 1$*,* $F_n\colon X \to \mathbb{R}_+$ $(n \in \mathbb{N})$ *are lower semicontinuous and such that all functions* $x_i \mapsto F_n(x_1, \dots, x_k)$ $(i = 1, \dots, k)$ *are L-subadditive and even. Let*

$$E := \{x \in X : (F_n(x))_{n=1}^{\infty} \textit{ is bounded}\}.$$

Then the following statements are equivalent:

(i) E *is meager;*
(ii) $E \neq X$*;*
(iii) $\sup\{F_n(x) : n \in \mathbb{N},\ ||x|| \leq 1\} = \infty$.

Note that Theorem 18.28 is given in [28] as a corollary of the above theorem; we gave here an alternative, straightforward proof. As a corollary, Jachymski obtained a strengthening of part (ii) of Theorem 18.26. Put

$$\Sigma_+ := \{A \in \Sigma : 0 < \mu(A) < \infty\}.$$

Theorem 18.32. *Assume that* $p \in [1, \infty)$ *and* (X, Σ, μ) *is a measure space such that* $\Sigma_+ \neq \emptyset$*. Let*

$$E_p := \{(f,g) \in L^p \times L^p : fg \in L^p\}.$$

Then the following statements are equivalent:

(i) E_p *is meager in* $L^p \times L^p$*;*
(ii) $E_p \neq L^p \times L^p$*;*
(iii) $\inf\{\mu(A) : A \in \Sigma_+\} = 0$.

Proof. The equivalence of (i) and (ii) is stated in [28], Proposition 2. The equivalence of (ii) and (iii) follows from [28, Proposition 3, (i)⇔(ii)] and [28, Lemma 4, (i)⇔(iii)]. □

Note that the condition $\inf\{\mu(A) : A \in \Sigma_+\} = 0$ means that there are sets with positive measure which is as small as we want. In particular, the Lebesgue measure on $[0,1]$ satisfies this condition.

Theorem 18.31 is a nice and deep tool, and it seems that it could be used to study similar problems for other bilinear mappings. A question arises whether, in this result, we can replace meagerness by (any known notion of) σ-porosity. It turns out that in general this is not possible, cf. [22], p. 2. However, the following question is open:

Question 18.33. Can we replace meagerness by (any notion of) σ*-porosity in the condition* (i) *of Theorem 18.28? Note that in the case of classical Banach-Steinhaus principle the answer is positive, cf. [34] and [40].*

In the next subsections we will prove, however, that Theorem 18.32 can be further strengthened. It is possible to replace meagerness by the appropriate notion of σ-porosity but also it is possible to consider more general spaces (not necessarily Banach spaces). Note that in [23] and [42] there are results which generalize and strengthen Theorem 18.29.

18.2.3 Dichotomies for multiplication in L^p spaces

Let (X,Σ,μ) be a measure space. We will consider here L^p spaces also for $p \in (0,1)$. In this case, L^p is the space of all measurable real functions such that

$$||f||_p := \int_X |f|^p \, d\mu < \infty.$$

It is well known that for $p \in (0,1)$, $||\cdot||_p$ is not a norm, but the function d defined by $d(f,g) := ||f-g||_p$ is a complete metric. For any $p,q,r \in (0,\infty]$, define

$$E_{p,q}^r := \{(f,g) \in L^p \times L^q : fg \in L^r\}.$$

By the mentioned Hölder inequality, $E_{p,q}^r = L^p \times L^q$ if $p,q,r \in (0,\infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Theorem 18.32 discusses the size of the set $E_{p,p}^p$. In this section we present results which solve completely the problem of the size of sets $E_{p,q}^r$.

The next two results follow from the main results of [22]. The first one is an extension of Theorem 18.32.

Theorem 18.34. *Let (X,Σ,μ) be such that $\Sigma_+ \neq \emptyset$ and $p,q,r \in (0,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} > \frac{1}{r}$. Then the following conditions are equivalent:*

(i) $E_{p,q}^r$ *is a* σ-$\frac{2}{3}$-*lower porous subset of* $L^p \times L^q$;
(ii) $E_{p,q}^r \neq L^p \times L^q$;
(iii) $\inf\{\mu(A) : A \in \Sigma_+\} = 0$.

Proof. Implication (i)⇒(ii) is trivial, implication (ii)⇒(iii) follows from [22], Proposition 8, and implication (iii)⇒(i) is an immediate consequence of [22], Theorem 6. □

The next result shows what happens if $\frac{1}{p} + \frac{1}{q} < \frac{1}{r}$. Note that the condition $\sup\{\mu(A)\colon A \in \Sigma_+\} = \infty$ means that there are sets with finite measure which is as big as we want. In particular, the counting measure on $\mathbb{N}$ satisfies this condition, hence the space l^p is an example of a space which satisfies the assumption (iii) in the next result. The proof goes the same way as for Theorem 18.34, so we omit it.

Theorem 18.35. *Let (X,Σ,μ) be such that $\Sigma_+ \neq \emptyset$ and $p,q,r \in (0,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} < \frac{1}{r}$ and $p < \infty$ or $q < \infty$. Then the following conditions are equivalent:*

(i) $E_{p,q}^r$ *is a* σ-$\frac{2}{3}$-*lower porous subset of* $L^p \times L^q$;
(ii) $E_{p,q}^r \neq L^p \times L^q$;
(iii) $\sup\{\mu(A)\colon A \in \Sigma_+\} = \infty$.

The implication (iii)⇒(i) in Theorems 18.34 and 18.35 can be strengthened in classical cases as follows:

Corollary 18.36. *Let $p,q \in (0,\infty]$ be such that $p < \infty$ or $q < \infty$, and put $r_{p,q} := \frac{1}{1/p+1/q}$. Then the sets*

$$A := \left\{(x,y) \in l^p \times l^q \colon \exists_{0<r<r_{p,q}}\, xy \in l^r\right\} = \bigcup_{0<r<r_{p,q}} E_{p,q}^r$$

and

$$B := \left\{(f,g) \in L^p[0,1] \times L^q[0,1] \colon \exists_{r_{p,q} < r \le \infty}\, fg \in L^r[0,1]\right\} = \bigcup_{r_{p,q} < r \le \infty} E^r_{p,q}$$

are σ-$\frac{2}{3}$-lower porous in $l^p \times ^q$ and $L^p[0,1] \times L^q[0,1]$, respectively.

Proof. For $0 < r < r' \le \infty$, we have $l^r \subset l^{r'}$ and $L^{r'}[0,1] \subset L^r[0,1]$. Hence $A = \bigcup_{r \in (0,r_{p,q}) \cap \mathbb{Q}} E^r_{p,q}$ and $B = \bigcup_{r \in (r_{p,q},\infty] \cap \mathbb{Q}} E^r_{p,q}$. The result follows from implication (iii)$\Rightarrow$(i) in Theorems 18.34 and 18.35. □

Remark 18.37. It can easily be seen that the proof of the above corollary works in a more general case. Namely, instead of l^p spaces we can take any L^p space with $\inf\{\mu(A) \colon A \in \Sigma_+\} > 0$ and $\sup\{\mu(A) \colon A \in \Sigma_+\} = \infty$, and instead of $L^p[0,1]$ – any L^p space with $\inf\{\mu(A) \colon A \in \Sigma_+\} = 0$ and $\sup\{\mu(A) \colon A \in \Sigma_+\} < \infty$. It seems that, if we modify a bit the proof of [22], Theorem 6, we would see that in the first case it is enough to assume that $\sup\{\mu(A) \colon A \in \Sigma_+\} = \infty$, and in the second case, that $\inf\{\mu(A) \colon A \in \Sigma_+\} = 0$.

Now we will show that for the remaining cases ($\Sigma_+ = \emptyset$ or ($p = q = \infty$ and $r < \infty$)) one can observe similar dichotomies.

Proposition 18.38. *Let (X, Σ, μ) be a measurable space and $r \in (0,\infty]$. The following conditions are equivalent:*

(i) $E^r_{\infty,\infty}$ is 1-lower porous in $L^\infty \times L^\infty$;
(ii) $E^r_{\infty,\infty} \ne L^\infty \times L^\infty$;
(iii) $\mu(X) = \infty$ and $r < \infty$.

Proof. Implication (i)$\Rightarrow$(ii) is trivial. If $\mu(X) < \infty$ and $r < \infty$, then for every $f,g \in L^\infty$, we have $||fg||_r \le ||f||_\infty ||g||_\infty \mu(X)^{\frac{1}{r}} < \infty$, and if $r = \infty$, then for every $f,g \in L^\infty$, we have $||fg||_r \le ||f||_\infty ||g||_\infty$. This gives (ii)$\Rightarrow$(iii).

Now we show (iii)$\Rightarrow$(i). Let $R > 0$, $\alpha \in \left(0, \frac{1}{2}\right)$ and $f,g \in L^\infty$. Define $\tilde{f}$ by:

$$\tilde{f}(x) = \begin{cases} f(x) + \frac{1}{2}R, & f(x) \ge 0; \\ f(x) - \frac{1}{2}R, & f(x) < 0, \end{cases}$$

and $\tilde{g}$ in an analogous way. Then $||f - \tilde{f}||_\infty = ||g - \tilde{g}||_\infty = \frac{1}{2}R$. Now take $h,s \in L^\infty$ such that $||\tilde{f} - h||_\infty < \alpha R$ and $||\tilde{g} - s||_\infty < \alpha R$. Then for μ-a.e. $x \in X$, $|s(x)|, |h(x)| \ge R\left(\frac{1}{2} - \alpha\right)$, hence $|s(x)h(x)| \ge R^2\left(\frac{1}{2} - \alpha\right)^2$ for μ-a.e. $x \in X$. Since $\mu(X) = \infty$ and $r < \infty$, $sh \notin L^r$. Thus $B\left(\left(\tilde{f}, \tilde{g}\right), \alpha R\right) \subset B((f,g),R) \setminus E^r_{\infty,\infty}$, and the proof is completed. □

Proposition 18.39. *Let (X, Σ, μ) be a measure space such that $\Sigma_+ = \emptyset$, and $p,q,r \in (0,\infty]$ be such that $p < \infty$ or $q < \infty$. Then $E^r_{p,q} = L^p \times L^q$.*

Proof. The result follows from the fact that if $p < \infty$ and $\Sigma_+ = \emptyset$, then $L^p = \{0\}$. □

Remark 18.40. In the papers [25], [41], [6] some extensions of presented results are obtained – instead of L^p spaces there are considered Orlicz spaces and Lorentz spaces. Also, multiplication of n functions is considered, for any fixed $n \in \mathbb{N}$ (instead of just two). It turns out that in these general cases, similar dichotomies hold (and the proofs are based on the same ideas).

18.2.4 Dichotomies for convolution in L^p spaces

Let G be a locally compact topological group and let μ be a left-invariant Haar measure on G (cf. [26] for details). If f, g are two measurable functions on G, $x \in G$, and an appropriate function is integrable, then we define the convolution of f and g at the point x by:

$$(f \star g)(x) := \int_G f(y)g(y^{-1}x)\, d\mu(y).$$

Clearly, the mapping $(f,g) \to f \star g$ is bilinear, provided it is well defined (i.e., for every f, g, $f \star g$ is defined μ-a.e.).

Recall that G is called *unimodular*, if for every $x \in G$ and a measurable $A \subset G$, $\mu(Ax) = \mu(A)$, and G is called *discrete* or *compact*, if the topology on G is so.

The Young inequality [21], Theorem 1.2.12, states that if G is unimodular and $p,q,r \in [0,\infty]$ are such that $1/p + 1/q = 1 + 1/r$, then for every $f \in L^p$ and $g \in L^q$, $||f \star g||_r \le ||f||_p ||g||_q$ and, in particular, $f \star g \in L^r$. The Minkowski inequality [21], Theorem 1.2.10, states that if $f \in L^1$ and $g \in L^q$ (for some $q \in [1,\infty]$), then $||f \star g||_q \le ||f||_1 ||g||_q$ and, in particular, $f \star g \in L^q$. (The Minkowski inequality is the Young inequality for $p = 1$ and $r = q$, but it works in all locally compact groups.)

At first we will discuss the size of the sets

$$E_{p,q} := \{(f,g) \in L^p \times L^q : |(f \star g)(x)| < \infty \text{ for } \mu\text{-a.e. } x \in G\}.$$

By the Minkowski inequality, we have $E_{1,q} = L^1 \times L^q$ for any $q \in [1,\infty]$. The next results show that in many other cases we observe a dichotomy. Note that the presented results generalize some earlier ones (cf. [1], [36]).

The first result is connected with the famous L^p-conjecture. Assume that $p > 1$. The L^p-conejcture, stated by Żelazko and Rajagopalan in 1960's, asserts that, if for all $f,g \in L^p$, $f \star g \in L^p$, then G is compact. During the next 30 years many partial confirmations of this conjecture had been established, and, finally, in 1990 Saeki [38] proved the L^p-conjecture in its general form.

By the main result of [24], we have the following strengthening of the L^p-conjecture for $p > 2$.

Theorem 18.41. *Assume that G is a locally compact topological group and $p,q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} < 1$. The following conditions are equivalent:*

(i) $E_{p,q}$ is σ-α-lower porous for some $\alpha > 0$;
(ii) $E_{p,q} \neq L^p \times L^q$;
(iii) G is not compact.

Proof. Implication (i)⇒(ii) is trivial. Now we prove (ii)⇒(iii). Assume that G is compact and take any $f \in L^p$ and $g \in L^q$. Since G is compact, it is unimodular, and since G is compact and $p,q \geq 1$, $L^p \subset L^1$ and $L^q \subset L^1$. In particular, $f,g \in L^1$ and by the Young inequality, $f \star g \in L^1$. In particular, we get (iii). Now assume (iii). By [24], Theorem 1, for any compact subset $K \subset G$, the set

$$\{(f,g) \in L^p \times L^q \colon (f \star g)(x) \text{ is well defined for some } x \in K\}$$

is σ-α-lower porous. By taking K with nonempty interior, we get (i). □

The main result of the paper [5] of Akbarbaglu and Maghsoudi implies the following result which together with Theorem 18.41 show that the assumptions on p and q in the Minkowski inequality cannot be relaxed.

Theorem 18.42. *Let G be a locally compact group, $p \in (1,\infty)$, $q \in [1,\infty)$ and $\frac{1}{p} + \frac{1}{q} \geq 1$. Then the following conditions are equivalent:*

(i) $E_{p,q}$ is σ-α-lower porous for some $\alpha > 0$;
(ii) $E_{p,q} \neq L^p \times L^q$;
(iii) G is not unimodular.

Proof. Implication (i)⇒(ii) is trivial. Implication (ii)⇒(iii) follows from the Young inequality (note that $1/p + 1/q \leq 2$), and implication (iii)⇒(i) follows immediately from [5], Theorem 2.4 (in the same way as this implication in the proof of Theorem 18.41). □

The next result follows from the main result of the paper [7] of Akbarbaglu and Maghsoudi.

Theorem 18.43. *Let G be a locally compact group, $p \in (0,1)$ and $q \in (0,\infty]$. Then the following conditions are equivalent:*

(i) $E_{p,q}$ is σ-α-lower porous for some $\alpha > 0$;
(ii) $E_{p,q} \neq L^p \times L^q$;
(iii) G is not discrete.

Proof. The implication (i)⇒(ii) is trivial. If G is discrete, then $L^r \subset L^{r'}$ for r, r' such that $0 < r \leq r' \leq \infty$. Hence if $(f,g) \in L^p \times L^q$, then $(f,g) \in L^1 \times L^{q+1}$. Thus the Minkowski inequality implies the implication (ii)⇒(iii). The remaining implication follows from [7], Theorem 2.1. □

Question 18.44. It is natural to ask what happens in the remaining cases, i.e.,

* $p \in (1,\infty)$ *and* $q \in (0,1) \cup \{\infty\}$;
* $p = 1$ *and* $q \in (0,1)$;
* $p = \infty$.

It seems that the proof of [5], Theorem 2.4, can easily be rewritten so that the assertion of Theorem 18.42 holds also for the case $p \in (1,\infty)$ and $q \in (0,1)$. Also, the case $q = \infty$ seems to be easy to handle.

Question 18.45. We can ask similar (but it seems that more difficult) questions concerning sets

$$E^r_{p,q} := \{(f,g) \in L^p \times L^q : f \star g \in L^r\}.$$

Clearly, $E^r_{p,q} \subset E_{p,q}$ for every r. It is easy to observe that using the Young and the Minkowski inequalities we can strengthen a bit Theorems 18.41, 18.42 and 18.43 by adding another condition

(iv) $E^r_{p,q} \neq L^p \times L^q$,

for $r := 1$ in Theorem 18.41, $r := r_{(p,q)}$ in Theorem 18.42, and $r := q+1$ in Theorem 18.43.

Some results and questions can also be found in [7], [35] and [38]. In particular, it is natural to ask if the L^p-conjecture can be strengthened by proving that in the case $p > 1$ and G noncompact, the set $E^p_{p,p}$ is meager (or σ-porous) in $L^p \times L^p$ (cf. [38], Problem 2).

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