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# Chapter 23

# Density type topologies generated by functions. $f$-density as a generalization of $\langle s \rangle$-density and $\psi$-density

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The notions of a density point and an approximately continuous function have been defined at the beginning of XX century. The density topology was defined by Haupt and Pauc in the fifties ([17]) and re-invented by Goffman and Waterman in 1961 ([16]). Let us recall the basic notions. We write $A \sim B$ instead of $\lambda\left(A \triangle B\right) = 0$. A point $x \in \mathbb{R}$ is called a *density point* of a set $A \in \mathcal{L}$ if

$$\lim_{h \to 0+} \frac{\lambda\left(A \cap [x-h, x+h]\right)}{2h} = 1.$$

The set of all density points of a set $A \in \mathcal{L}$ we denote by $\Phi_d(A)$. The operator $\Phi_d$ is a lower density operator and a family

$$\mathcal{T}_d := \{A \in \mathcal{L} : A \subset \Phi_d\left(A\right)\}$$

is a topology called the *density topology*.

Over the last thirty years several density-type topologies has been studied by many mathematicians. All such topologies are generated by operators called lower density operators or slight different operators (studied by Hejduk in [19] and called almost density operators).

We start by describing the properties of density-type topologies that arise from the definition of a lower density operator and an almost lower density operator in the sense of Hejduk. We recall definitions of an $\langle s\rangle$-density topology and a $\psi$-density topology (described in the previous chapter). Then we define an $f$-density topology which is a generalization of several density-type topologies. In particular, $\langle s\rangle$-density topologies and $\psi$-density topologies are $f$-density topologies. At first sight these topologies have quite different properties. However, they can be studied as special cases of the more general concept.

For any $A\subset\mathbb{R}$ we denote by $K_A$ a measurable kernel of $A$, and we set $-A:=\{-a:a\in A\}$ and $A+x:=\{a+x:a\in A\}$, $x\in\mathbb{R}$.

## 23.1 Density-type topologies consisting of measurable sets

We will study operators $\Phi:\mathcal{L}\to\mathcal{P}(\mathbb{R})$. Consider the following properties:

(D0) if $A\in\mathcal{T}_{nat}$ then $A\subset\Phi(A)$,
(D1) $\Phi(\emptyset)=\emptyset$ and $\Phi(\mathbb{R})=\mathbb{R}$,
(D2) $\Phi(A\cap B)=\Phi(A)\cap\Phi(B)$,
(D3) if $A\sim B$ then $\Phi(A)=\Phi(B)$,
(D4) $\Phi(A)\sim A$

and

(D4’) $\lambda(\Phi(A)\setminus A)=0$

for measurable $A$ and $B$.

If $\Phi$ fulfils (D1)-(D4) then it is called a *lower density operator*. An operator fulfilling (D1)-(D3) and (D4’) has been named by Hejduk an *almost lower density operator*.

**Remark 23.1.** If $\Phi$ fulfils (D2) then $\Phi$ is monotonic i.e.

(D2’) if $A\subset B$ then $\Phi(A)\subset\Phi(B)$.

**Remark 23.2.** From (D2’) and (D3) it follows that for any $A,B\in\mathcal{L}$ such that $\lambda(A\setminus B)=0$ we have $\Phi(A)\subset\Phi(B)$.

We define

$$\mathcal{T}_\Phi:=\{A\in\mathcal{L}:A\subset\Phi(A)\}.$$

**Remark 23.3.** If $\Phi_1(A)\subset\Phi_2(A)$ for $A\in\mathcal{L}$ then $\mathcal{T}_{\Phi_1}\subset\mathcal{T}_{\Phi_2}$. Consequently, if $\Phi_1=\Phi_2$ then $\mathcal{T}_{\Phi_1}=\mathcal{T}_{\Phi_2}$.

We will prove that if an operator $\Phi$ satisfies conditions (D0)-(D3) and (D4') then $\mathcal{T}_\Phi$ forms a topology. We will examine properties of such operators and topologies generated by them. We will also consider operators satisfying the condition (D4), stronger than (D4'). To make the text self-contained we shortly repeat some proofs known in the literature. A lot of proofs are based on considerations for density topology (compare [23] and [29]). Some properties can be proved under the weaker assumptions (see [22] and [19]). Recall that $\mathcal{N}$ stands for the $\sigma$-ideal of null sets on the real line.

**Theorem 23.4.** *If $\Phi : \mathcal{L} \to \mathcal{P}(\mathbb{R})$ fulfils (D0)-(D3) and (D4') then*

*(1) $\mathcal{T}_\Phi$ forms a topology on $\mathbb{R}$,*
*(2) $\text{int}_{\mathcal{T}_\Phi} E \subset E \cap \Phi(K_E)$ for $E \subset \mathbb{R}$,*
*(3) if $A \in \mathcal{N}$ then $A$ is $\mathcal{T}_\Phi$-closed, $\mathcal{T}_\Phi$-nowhere dense and $\mathcal{T}_\Phi$-discrete,*
*(4) $A \in \mathcal{N}$ if and only if $A$ is $\mathcal{T}_\Phi$-closed and $\mathcal{T}_\Phi$-discrete,*
*(5) $A$ is a $\mathcal{T}_\Phi$-compact set if and only if $A$ is finite,*
*(6) the space $(\mathbb{R}, \mathcal{T}_\Phi)$ is neither first countable, nor Lindelöf, nor separable,*
*(7) $\text{card}\,\mathcal{T}_\Phi = 2^{\mathfrak{c}}$,*
*(8) $\mathcal{T}_{nat} \subsetneqq \mathcal{T}_\Phi$,*
*(9) $A \in \mathcal{L}$ if and only if $A$ is a $\mathcal{T}_\Phi$-Borel set,*
*(10) $(\mathbb{R}, \mathcal{T}_\Phi)$ is a Hausdorff space,*
*(11) $(\mathbb{R}, \mathcal{T}_\Phi)$ is not a normal space.*

*Proof.* (1) Of course $\emptyset \in \mathcal{T}_\Phi$. If $A, B \in \mathcal{T}_\Phi$ then by (D2), $A \cap B \subset \Phi(A) \cap \Phi(B) = \Phi(A \cap B)$, which implies $A \cap B \in \mathcal{T}_\Phi$. Assume that $A_t \in \mathcal{T}_\Phi$ for $t \in T$ and $A := \bigcup_{t \in T} A_t$. Since $A_t \setminus K_A \in \mathcal{N}$, Remark 23.2 shows that $\Phi(A_t) \subset \Phi(K_A)$, and consequently

$$K_A \subset A = \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \Phi(A_t) \subset \Phi(K_A).$$

From (D4') it follows that $A \sim K_A$, so $A \in \mathcal{L}$. Using (D2') again, we conclude that $A \subset \Phi(K_A) \subset \Phi(A)$, which gives $A \in \mathcal{T}_\Phi$.
(2) Since $\text{int}_{\mathcal{T}_\Phi} E$ is a measurable subset of $E$, $\text{int}_{\mathcal{T}_\Phi} E \setminus K_E \in \mathcal{N}$. From Remark 23.2 we obtain $\text{int}_{\mathcal{T}_\Phi} E \subset \Phi(\text{int}_{\mathcal{T}_\Phi} E) \subset \Phi(K_E)$.
(3) Let $A \in \mathcal{N}$. By (D3) and (D1), $\mathbb{R} \setminus A \subset \mathbb{R} = \Phi(\mathbb{R}) = \Phi(\mathbb{R} \setminus A)$, hence $A$ is $\mathcal{T}_\Phi$-closed. The set $A$ is nowhere dense because from (2), (D1) and (D3) we have $\text{int}_{\mathcal{T}_\Phi} A \subset \Phi(A) = \emptyset$. If $x \in A$ then $(\mathbb{R} \setminus A) \cup \{x\} \in \mathcal{T}_\Phi$, and consequently $\{x\}$ is $\mathcal{T}_\Phi$-open in $A$. Thus $A$ is $\mathcal{T}_\Phi$-discrete.
(4) Suppose that $A$ is $\mathcal{T}_\Phi$-closed and $\mathcal{T}_\Phi$-discrete. For any $x \in A$ there exists a $\mathcal{T}_\Phi$-open set $U_x$ such that $A \cap U_x = \{x\}$. Thus we have

$$x \in U_x \subset \Phi(U_x) = \Phi(U_x \setminus \{x\}) \subset \Phi(\mathbb{R} \setminus A),$$

and so $A \subset \Phi(\mathbb{R} \setminus A)$. Therefore $A \subset \Phi(\mathbb{R} \setminus A) \setminus (\mathbb{R} \setminus A) \in \mathcal{N}$, by (D4').
(5) Suppose that $A$ is infinite. Let $B \subset A$ be a countable infinite subset of $A$. Then $\{(\mathbb{R} \setminus B) \cup \{x\}\}_{x \in B}$ is a $\mathcal{T}_\Phi$-open cover of $A$ without finite subcover.
(6) If $A_n$, $n \in \mathbb{N}$, are $\mathcal{T}_\Phi$-open neighbourhoods of $x$ and $x_n \in A_n \setminus \{x\}$ then the set $A_1 \setminus \{x_n : n \in \mathbb{N}\}$ is a $\mathcal{T}_\Phi$-open neighbourhood of $x$ which does not include any $A_n$. Thus $(\mathbb{R}, \mathcal{T}_\Phi)$ is not first countable.
Let $C$ be the Cantor ternary set. Then $\{(\mathbb{R} \setminus C) \cup \{x\}\}_{x \in C}$ is a $\mathcal{T}_\Phi$-open cover of $A$ without countable subcover, and consequently $(\mathbb{R}, \mathcal{T}_\Phi)$ is not Lindelöf. Since any countable set is $\mathcal{T}_\Phi$-closed, $(\mathbb{R}, \mathcal{T}_\Phi)$ is not separable.
(7) Any subset of the Cantor set $C$ is $\mathcal{T}_\Phi$-closed. Hence $\operatorname{card}\mathcal{T}_\Phi \geq \operatorname{card}\mathcal{P}(C) = 2^{\mathfrak{c}}$.
(8) By (D0), $\mathcal{T}_{nat} \subset \mathcal{T}_\Phi$. Moreover $(\mathbb{R} \setminus \mathbb{Q}) \in \mathcal{T}_\Phi \setminus \mathcal{T}_{nat}$.
(9) The inclusion $\mathcal{B}or_{\mathcal{T}_\Phi} \subset \mathcal{L}$ follows from $\mathcal{T}_\Phi \subset \mathcal{L}$. From (3) and (8) we conclude that $\mathcal{N} \subset \mathcal{B}or_{\mathcal{T}_\Phi}$ and $\mathcal{B}or_{\mathcal{T}_{nat}} \subset \mathcal{B}or_{\mathcal{T}_\Phi}$. Thus we have $\mathcal{L} \subset \mathcal{B}or_{\mathcal{T}_\Phi}$, because every measurable set is a sum of a Borel set and a null set.
(10) It follows from (8).
(11) (compare [22, Prop. 7.17]) Suppose, contrary to our claim, that $(\mathbb{R}, \mathcal{T}_\Phi)$ is a normal space. Write $\mathcal{F} := \{C \cap \Phi(A) : A \in \mathcal{L}\}$, where $C$ is the Cantor set. For any measurable set $A$ there is a Borel set $A'$ such that $A \sim A'$, so $\operatorname{card}\{\Phi(A) : A \in \mathcal{L}\} \leq \mathfrak{c}$. Therefore, one can find a set $H \subset C$ such that $H \notin \mathcal{F}$. Since $H$ and $F := C \setminus H$ are $\mathcal{T}_\Phi$-closed, there are disjoint $\mathcal{T}_\Phi$-open sets $A_H, A_F$ containing $H$ and $F$. Thus $\Phi(A_H) \cap \Phi(A_F) = \Phi(A_H \cap A_F) = \emptyset$ and

$$H \subset C \cap A_H \subset C \cap \Phi(A_H) \subset C \setminus \Phi(A_F) \subset C \setminus A_F \subset C \setminus F = H.$$

Hence $H = C \cap \Phi(A_H)$, which gives a contradiction, because $C \cap \Phi(A_H) \in \mathcal{F}$.
□

**Remark 23.5.** The assumption (D0) has not been used in the proof of conditions (1)-(7) and (11).

If $\Phi$ fulfils additionally the condition (D4), we obtain stronger results.

**Theorem 23.6.** *If $\Phi : \mathcal{L} \to \mathcal{P}(\mathbb{R})$ fulfils (D0)-(D4) then*

*(1) $\Phi(A) \in \mathcal{L}$ for $A \in \mathcal{L}$,*
*(2) $\Phi(\Phi(A)) = \Phi(A)$ for $A \in \mathcal{L}$,*
*(3) $\mathcal{T}_\Phi = \{\Phi(A) \setminus N : A \in \mathcal{L},\ N \in \mathcal{N}\}$,*
*(4) $\operatorname{int}_{\mathcal{T}_\Phi} E = E \cap \Phi(K_E)$ for $E \subset \mathbb{R}$,*
*(5) $\operatorname{int}_{\mathcal{T}_\Phi} A \sim A \sim \operatorname{cl}_{\mathcal{T}_\Phi} A$ for $A \in \mathcal{L}$,*

*(6)* $\Phi(A) = \mathrm{int}_{\mathcal{T}_\Phi}(\mathrm{cl}_{\mathcal{T}_\Phi} A)$ *for* $A \in \mathcal{L}$*,*

*(7)* $A \in \mathcal{N} \Leftrightarrow A$ *is* $\mathcal{T}_\Phi$*-nowhere dense* $\Leftrightarrow A$ *is* $\mathcal{T}_\Phi$*-meager,*

*(8)* $A \in \mathcal{L} \Leftrightarrow A$ *has* $\mathcal{T}_\Phi$*-Baire property* $\Leftrightarrow A$ *is a sum of a* $\mathcal{T}_\Phi$*-open set and a* $\mathcal{T}_\Phi$*-closed set,*

*(9)* $(\mathbb{R}, \mathcal{T}_\Phi)$ *is a Baire space.*

*Proof.* (1)-(2) From (D4) we have $\Phi(A) \sim A$. Hence $\Phi(A) \in \mathcal{L}$ and $\Phi(\Phi(A)) = \Phi(A)$, by (D3).
(3) If $A \in \mathcal{T}_\Phi$ then $A = \Phi(A) \setminus (\Phi(A) \setminus A)$, and by (D4), $\Phi(A) \setminus A \in \mathcal{N}$. Let $A \in \mathcal{L}$ and $N \in \mathcal{N}$. From (D3) and (2) it follows that $\Phi(\Phi(A) \setminus N) = \Phi(\Phi(A)) = \Phi(A) \supset \Phi(A) \setminus N$, which gives $\Phi(A) \setminus N \in \mathcal{T}_\Phi$.
(4) The inclusion $\mathrm{int}_{\mathcal{T}_\Phi} E \subset E \cap \Phi(K_E)$ follows from Theorem 23.4. Let $x \in E \cap \Phi(K_E)$ and $A := K_E \cup \{x\}$. The set $A \cap \Phi(A)$ is a $\mathcal{T}_\Phi$-open neighbourhood of $x$, because

$$\Phi(A \cap \Phi(A)) = \Phi(A) \cap \Phi(\Phi(A)) = \Phi(A) \supset A \cap \Phi(A).$$

Since $A \cap \Phi(A) \subset A \subset E$, we have $x \in \mathrm{int}_{\mathcal{T}_\Phi} E$.
(5) From (D4) and (4) we conclude that $\mathrm{int}_{\mathcal{T}_\Phi} A = A \cap \Phi(A) \sim A$ and $\mathrm{cl}_{\mathcal{T}_\Phi} A = \mathbb{R} \setminus \mathrm{int}_{\mathcal{T}_\Phi}(\mathbb{R} \setminus A) \sim \mathbb{R} \setminus (\mathbb{R} \setminus A) = A$.
(6) For any measurable set $A$ we have $A \sim \mathrm{cl}_{\mathcal{T}_\Phi} A$ and $\mathrm{cl}_{\mathcal{T}_\Phi} A = \mathbb{R} \setminus \mathrm{int}_{\mathcal{T}_\Phi}(\mathbb{R} \setminus A) \supset \mathbb{R} \setminus \Phi(\mathbb{R} \setminus A) \supset \Phi(A)$. Hence

$$\mathrm{int}_{\mathcal{T}_\Phi}(\mathrm{cl}_{\mathcal{T}_\Phi} A) = \mathrm{cl}_{\mathcal{T}_\Phi} A \cap \Phi(\mathrm{cl}_{\mathcal{T}_\Phi} A) = \mathrm{cl}_{\mathcal{T}_\Phi} A \cap \Phi(A) = \Phi(A).$$

(7) According to Theorem 23.4, it is sufficient to prove that each $\mathcal{T}_\Phi$-nowhere dense set is a null set. Let $A$ be a $\mathcal{T}_\Phi$-nowhere dense set. Then $\mathrm{cl}_{\mathcal{T}_\Phi} A$ is also $\mathcal{T}_\Phi$-nowhere dense. Using (6) we get $\Phi(A) = \mathrm{int}_{\mathcal{T}_\Phi}(\mathrm{cl}_{\mathcal{T}_\Phi} A) = \emptyset$, which yields $A \sim \emptyset$.
(8) If $A \in \mathcal{L}$ then $A = (A \cap \Phi(A)) \cup (A \setminus \Phi(A))$, $A \cap \Phi(A)$ is $\mathcal{T}_\Phi$-open and $A \setminus \Phi(A)$ is $\mathcal{T}_\Phi$-closed (because it is a null set). Of course, a sum of a $\mathcal{T}_\Phi$-open set and a $\mathcal{T}_\Phi$-closed set has $\mathcal{T}_\Phi$-Baire property. Finally, if $A = U \triangle N$ where $U$ is $\mathcal{T}_\Phi$-open and $N$ is $\mathcal{T}_\Phi$-meager, then $U \in \mathcal{L}$ and $N \in \mathcal{N}$, by (7). Consequently, $A$ is measurable.
(9) If $A$ is $\mathcal{T}_\Phi$-open and $\mathcal{T}_\Phi$-meager, then $A \subset \Phi(A)$ and $A \in \mathcal{N}$, so $A = \Phi(A) = \emptyset$. □

**Remark 23.7.** From (6) it follows that $A$ is $\mathcal{T}_\Phi$-regular open if and only if $\Phi(A) = A$.

In the following chapter there are constructed an operator $\Phi$, satisfying (D0)-(D3) and (D4'), and closed sets of positive measure $F_0$, $F_1$ such that

$\Phi(F_0) = \emptyset$ and $\Phi(F_1)$ is a singleton. In fact, it is proved that for any function $f$ with $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$ there are closed sets of positive measure such that $\Phi_f(F_0) = \emptyset$ and $\Phi_f(F_1) = \{0\}$, where $\Phi_f$ is an $f$-density operator defined in the section 5 (see Theorem 24.6 and Theorem 24.7 in chapter 24). Using this result one can easy check that properties (2)-(9) from Theorem 23.6 can be false if we replace (D4) by (D4').

Note that in the following theorem items start from (2) to stress the similarity to properties of Theorem 23.6.

**Theorem 23.8.** *If $\Phi : \mathcal{L} \to \mathcal{P}(\mathbb{R})$ fulfils (D0)-(D3) and (D4') and $F_0$, $F_1$ are closed sets of positive measure such that $\Phi(F_0) = \emptyset$, $\Phi(F_1) = \{0\}$ then*

*(2)* $\Phi(\Phi(F_1)) = \emptyset \neq \Phi(F_1)$,
*(3)* $\Phi(F_1) \notin \mathcal{T}_\Phi$,
*(4)* $\mathrm{int}_{\mathcal{T}_\Phi} F_1 = \emptyset \neq F_1 \cap \Phi(F_1)$,
*(5)* $\lambda(F_0 \setminus \mathrm{int}_{\mathcal{T}_\Phi} F_0) = \lambda(F_0) > 0$ *and* $\lambda(\mathrm{cl}_{\mathcal{T}_\Phi}(\mathbb{R}\setminus F_0) \setminus (\mathbb{R}\setminus F_0)) = \lambda(F_0) > 0$,
*(6)* $\Phi(F_1) = \{0\} \neq \mathrm{int}_{\mathcal{T}_\Phi} \mathrm{cl}_{\mathcal{T}_\Phi} F_1$,
*(7) $F_0$ is $\mathcal{T}_\Phi$-nowhere dense, but is not a null set; $\mathbb{R}$ is $\mathcal{T}_\Phi$-meager, but is not $\mathcal{T}_\Phi$-nowhere dense,*
*(8) each nonmeasurable set is $\mathcal{T}_\Phi$-meager, and consequently has $\mathcal{T}_\Phi$-Baire property,*
*(9) $(\mathbb{R}, \mathcal{T}_\Phi)$ is not a Baire space.*

*Proof.* Conditions (2)-(6) are clear. Obviously, $F_0$ is $\mathcal{T}_\Phi$-nowhere dense. By Smital's Lemma, the set $E := \bigcup_{q\in\mathbb{Q}} (F_0 + q)$ has a full measure (compare [20], p. 65). Thus $\mathbb{R}\setminus E$ is $\mathcal{T}_\Phi$-nowhere dense. Since every $F_0 + q$ is $\mathcal{T}_\Phi$-nowhere dense too, $\mathbb{R}$ is $\mathcal{T}_\Phi$-meager. This implies (8)-(9). □

Now we construct an easy example of an operator $\Phi : \mathcal{L} \to \mathcal{P}(\mathbb{R})$, satisfying (D0)-(D3) and (D4'), such that the set $\Phi(A)$ need not be measurable, for measurable $A$.

*Example 23.9.* Suppose that $\Phi : \mathcal{L} \to \mathcal{P}(\mathbb{R})$ fulfils (D0)-(D3) and (D4'), $F_0$ is a closed set of positive measure such that $\Phi(F_0) = \emptyset$ and $D$ is a nonmeasurable subset of $F_0$. Write

$$\widehat{\Phi}(A) := (\Phi(A) \setminus D) \cup (\Phi_d(A) \cap D)$$

for $A \in \mathcal{L}$. It is easy to check that $\widehat{\Phi}$ fulfils (D0)-(D3) and (D4'). But $\widehat{\Phi}(F_0) = \Phi_d(F_0) \cap D \notin \mathcal{L}$.

## 23.2 $\langle s\rangle$-density

Reminding the concept of ordinary density point it is worth observing that $x \in \Phi(A)$ if and only if

$$\lim_{n\to\infty} \frac{\lambda\left(A\cap\left[x-\frac{1}{n},x+\frac{1}{n}\right]\right)}{\frac{2}{n}} = 1.$$

Replacing the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ by a fixed sequence $(s_n)_{n\in\mathbb{N}}$ decreasingly tending to zero, we obtain the notion of a density generated by the sequence $(s_n)_{n\in\mathbb{N}}$.

Denote by $\widetilde{\mathcal{S}}$ the family of all nonincreasing and tending to zero sequences of positive numbers and fix $\langle s\rangle = (s_n)_{n\in\mathbb{N}} \in \widetilde{\mathcal{S}}$ and $A \in \mathcal{L}$. If

$$\lim_{n\to\infty} \frac{\lambda\left(A\cap[x-s_n,x+s_n]\right)}{2s_n} = 1$$

then $x$ is called an $\langle s\rangle$-*density point* of a set $A$. Analogously, if

$$\lim_{n\to\infty} \frac{\lambda\left(A\cap[x,x+s_n]\right)}{s_n} = 1 \quad \left(\lim_{n\to\infty} \frac{\lambda\left(A\cap[x-s_n,x]\right)}{s_n} = 1\right)$$

then we say that $x$ is a *right-hand (left-hand)* $\langle s\rangle$-*density* point of $A$. The set of all $\langle s\rangle$-density points (right-hand, left-hand $\langle s\rangle$-density points) of a set $A$ we denote by $\Phi_{\langle s\rangle}(A)$ ($\Phi^+_{\langle s\rangle}(A)$, $\Phi^-_{\langle s\rangle}(A)$, respectively). Clearly, $\Phi_{\langle s\rangle}(A) = \Phi^+_{\langle s\rangle}(A) \cap \Phi^-_{\langle s\rangle}(A)$. We will write $\mathcal{T}_{\langle s\rangle}$ instead of $\mathcal{T}_{\Phi_{\langle s\rangle}}$, i.e.

$$\mathcal{T}_{\langle s\rangle} = \left\{A \in \mathcal{L} : A \subset \Phi_{\langle s\rangle}(A)\right\}.$$

Obviously, $\mathcal{T}_d = \mathcal{T}_{\left\langle \frac{1}{n}\right\rangle}$ and $\mathcal{T}_d \subset \mathcal{T}_{\langle s\rangle}$ for any $\langle s\rangle \in \widetilde{\mathcal{S}}$.

The notion of an $\langle s\rangle$-density point has been defined in [12]. Properties of an operator $\Phi_{\langle s\rangle}$ and a topology $\mathcal{T}_{\langle s\rangle}$ have been studied also in [11], [13] and [21]. Note that the authors used nondecreasing sequences tending to $\infty$ instead of nonincreasing sequences tending to zero, and considered $\frac{1}{s_n}$ instead of $s_n$. Let us recall the basic properties of $\Phi_{\langle s\rangle}$ and $\mathcal{T}_{\langle s\rangle}$.

**Theorem 23.10** ([12]). *For any sequence* $\langle s\rangle \in \widetilde{\mathcal{S}}$

*(1)* $\Phi_{\langle s\rangle}$ *satisfies (D0)-(D4),*
*(2)* $\mathcal{T}_{\langle s\rangle}$ *is a topology,*
*(3)* $\Phi_{\langle s\rangle}$ *and* $\mathcal{T}_{\langle s\rangle}$ *satisfy all conditions of theorems 23.4 and 23.6.*

**Theorem 23.11** ([12]). *Let $\langle s\rangle \in \widetilde{\mathcal{S}}$.*

*(1) If* $\liminf_{n\to\infty} \frac{s_{n+1}}{s_n} > 0$ *then* $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_d$.
*(2) If* $\liminf_{n\to\infty} \frac{s_{n+1}}{s_n} = 0$ *then* $\mathcal{T}_{\langle s\rangle} \supsetneqq \mathcal{T}_d$.

The density topology is invariant under translation and under multiplication by nonzero numbers. However, $\langle s\rangle$-density topologies bigger than $\mathcal{T}_d$ are not invariant under multiplication.

**Theorem 23.12** ([12]). *Let $\langle s\rangle \in \widetilde{\mathcal{S}}$ and $m \in \mathbb{R}$.*

*(1) The topology $\mathcal{T}_{\langle s\rangle}$ is invariant under translation.*
*(2) If $|m| \geq 1$ then $\mathcal{T}_{\langle s\rangle}$ is invariant under multiplication by m.*
*(3) If $\mathcal{T}_{\langle s\rangle} \supsetneqq \mathcal{T}_d$ and $|m| < 1$ then $\mathcal{T}_{\langle s\rangle}$ is not invariant under multiplication by m.*

All $\langle s\rangle$-density topologies fulfil the same separating axioms as $\mathcal{T}_d$.

**Theorem 23.13** ([21]). *For any $\langle s\rangle \in \widetilde{\mathcal{S}}$ the space $\left(\mathbb{R}, \mathcal{T}_{\langle s\rangle}\right)$ is completely regular but not normal.*

**Theorem 23.14** ([13]). *Let $\langle s\rangle \in \widetilde{\mathcal{S}}$. A set A is connected in $\left(\mathbb{R}, \mathcal{T}_{\langle s\rangle}\right)$ if and only if A is connected in $(\mathbb{R}, \mathcal{T}_{nat})$.*

Despite the fact that $\langle s\rangle$-density topologies have very similar properties, there are a lot of nonhomeomorphic $\langle s\rangle$-density topologies.

**Theorem 23.15** ([13]). *Suppose that $\mathcal{T}_{\langle s\rangle} \neq \mathcal{T}_d \neq \mathcal{T}_{\langle t\rangle}$.*

*(1) The spaces $(\mathbb{R}, \mathcal{T}_d)$ and $\left(\mathbb{R}, \mathcal{T}_{\langle s\rangle}\right)$ are not homeomorphic.*
*(2) If $\mathcal{T}_{\langle s\rangle} \not\subseteq \mathcal{T}_{\langle t\rangle}$ and $\mathcal{T}_{\langle t\rangle} \not\subseteq \mathcal{T}_{\langle s\rangle}$ then the spaces $\left(\mathbb{R}, \mathcal{T}_{\langle s\rangle}\right)$ and $\left(\mathbb{R}, \mathcal{T}_{\langle t\rangle}\right)$ are not homeomorphic.*
*(3) For any $m > 1$, $\mathcal{T}_{\left\langle \frac{1}{m}s\right\rangle} \subsetneqq \mathcal{T}_{\langle s\rangle} \subsetneqq \mathcal{T}_{\langle ms\rangle}$ and the spaces $\left(\mathbb{R}, \mathcal{T}_{\left\langle \frac{1}{m}s\right\rangle}\right)$, $\left(\mathbb{R}, \mathcal{T}_{\langle s\rangle}\right)$, $\left(\mathbb{R}, \mathcal{T}_{\langle ms\rangle}\right)$ are homeomorphic.*

## 23.3 $\psi$-density

In [25] Terepeta and Wagner-Bojakowska, based on the results of Taylor ([24]), introduced the notion of a $\psi$-density point. They defined a $\psi$-density operator and a $\psi$-density topology, and studied the basic properties of these notions. The other interesting results can be found in [27], [28], [1], [26] and [14]. Some of them are presented in the previous chapter of this book.

Recall that $\widehat{\mathcal{C}}$ denotes the family of all continuous and nondecreasing functions $\psi:(0,\infty)\to(0,\infty)$ with $\lim_{x\to 0+}\psi(x)=0$. We say that $x$ is a *$\psi$-density point* of a measurable set $A$ if

$$\lim_{h\to 0+}\frac{\lambda([x-h,x+h]\setminus A)}{2h\psi(2h)}=0.$$

The set of all $\psi$-density points of $A$ is denoted by $\Phi_\psi(A)$. We write $\mathcal{T}_\psi$ instead of $\mathcal{T}_{\Phi_\psi}$ i.e.

$$\mathcal{T}_\psi=\left\{A\in\mathcal{L}:A\subset\Phi_\psi(A)\right\}.$$

The Second Taylor's Theorem (see section 22.2 in the previous chapter) implies:

**Theorem 23.16.** *For each $\psi\in\widehat{\mathcal{C}}$ there exists a set $E$ such that $\lambda\left(E\setminus\Phi_\psi(E)\right)$ is positive.*

Therefore, the Lebesgue Density Theorem does not hold for $\psi$-density, and no $\Phi_\psi$ is a lower density operator. However, $\Phi_\psi(A)\subset\Phi_d(A)$, so

$$\lambda\left(\Phi_\psi(A)\setminus A\right)=0$$

for $A\in\mathcal{L}$. We also have:

**Proposition 23.17** ([25]). *For any $\psi\in\widehat{\mathcal{C}}$ and $A\in\mathcal{L}$, $\Phi_\psi(A)$ is the set of type $F_{\sigma\delta}$.*

**Theorem 23.18** ([25]). *For any $\psi\in\widehat{\mathcal{C}}$*

*(1) $\Phi_\psi$ satisfies (D0)-(D3) and (D4'),*
*(2) $\mathcal{T}_\psi$ is a topology,*
*(3) $\Phi_\psi$ and $\mathcal{T}_\psi$ satisfy all conditions of Theorem 23.4.*

Moreover, for any $\psi\in\widehat{\mathcal{C}}$, $\mathcal{T}_{nat}\subsetneqq\mathcal{T}_\psi\subsetneqq\mathcal{T}_d$ and the family of $\mathcal{T}_\psi$-connected sets is the same as in $(\mathbb{R},\mathcal{T}_{nat})$ (compare [25]). Since $\Phi_\psi$ does not satisfy (D4), there are several differences between the density topology and $\psi$-density topologies. For example, the space $\left(\mathbb{R},\mathcal{T}_\psi\right)$ is not regular ([2]) and it is not a Baire space ([26]). More properties of $\psi$-density topologies are described in section 22.3 in the previous chapter.

## 23.4 The definition and basic properties of $f$-density

By $\mathcal{A}$ we denote the family of all functions $f:(0,\infty)\to(0,\infty)$ such that

(A1) $\lim_{x\to 0+} f(x) = 0$,
(A2) $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$,
(A3) $f$ is nondecreasing.

Let $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$. We say that $x$ is a *right-hand* (*left-hand*) *f-density point* of $A$ if

$$\lim_{h\to 0+} \frac{\lambda\left([x,x+h]\setminus A\right)}{f(h)} = 0 \quad \left(\lim_{h\to 0+} \frac{\lambda\left([x-h,x]\setminus A\right)}{f(h)} = 0\right).$$

By $\Phi_f^+(A)$ ($\Phi_f^-(A)$) we denote the set of all right-hand (left-hand) $f$-density points of $A$. If $x \in \Phi_f(A) := \Phi_f^+(A) \cap \Phi_f^-(A)$ then we say that $x$ is an *f-density point* of $A$.

**Remark 23.19.** Condition (A2) is essential because $\lim_{h\to 0+} \frac{f(h)}{h} = \infty$ implies $\frac{\lambda([x,x+h]\setminus A)}{f(h)} \le \frac{h}{f(h)} \underset{h\to 0+}{\to} 0$.

The definition of $f$-density was introduced in [3]. In this paper continuity of functions from the family $\mathcal{A}$ was assumed. In subsequent papers this condition was omitted (compare Theorem 23.34). Considering functions from the family $\mathcal{A}$, we often define $f(x)$ only for $x \in (0,\delta)$, for some $\delta > 0$. To distinguish the notion of $f$-density from $\psi$-density (considered in the previous section) we will use Latin letters $(f,g)$ defining density generated by function and Greek letters $(\psi)$ in the second case.

Straightforward from the properties of Lebesgue measure it follows:

**Proposition 23.20.** *$\Phi_f(A+x) = \Phi_f(A) + x$ for $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$.*

Since functions from the family $\mathcal{A}$ are monotonic, we can describe an $f$-density point in equivalent way.

**Proposition 23.21** ([3]). *Let $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$. Then $x$ is an $f$-density point of $A$ if and only if*

$$\lim_{\substack{h\to 0, k\to 0 \\ h\ge 0, k\ge 0, h+k>0}} \frac{\lambda\left([x-h,x+k]\setminus A\right)}{f(h+k)} = 0.$$

*Proof.* Sufficiency is evident. Since

$$\frac{\lambda\left([x-h,x+k]\setminus A\right)}{f(h+k)} \le \frac{\lambda\left([x-h,x]\setminus A\right)}{f(h)} + \frac{\lambda\left([x,x+k]\setminus A\right)}{f(k)}$$

for $h,k > 0$, we obtain necessity. □

**Theorem 23.22** ([3]). *If $f \in \mathcal{A}$ and $A \in \mathcal{L}$ then $\Phi_f(A)$, $\Phi_f^+(A)$ and $\Phi_f^-(A)$ are the sets of type $F_{\sigma\delta}$.*

*Proof.* For any $h > 0$ the function $F_h(x) = \frac{\lambda([x-h,x]\setminus A)}{f(h)}$ is continuous. Thus theorem follows from the equality

$$\Phi_f^+(A) = \bigcap_{n\in\mathbb{N}} \bigcup_{\delta\in\mathbb{Q}_+} \bigcap_{h\in(0,\delta)} F_h^{-1}\left(\left[0,\frac{1}{n}\right]\right).$$

□

The following theorem states that an $f$-density point of a set $A$ can not be a dispersion point of $A$. Thus an operator $\Phi_f$ satisfies the condition (D4').

**Theorem 23.23** ([3]). *Let $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$.*

*(1) If $x \in \Phi_f^+(A)$ then $\liminf_{h\to 0+} \frac{\lambda([x,x+h]\setminus A)}{h} = 0$.*
*(2) $\Phi_f(A) \subset \mathbb{R}\setminus \Phi_d(\mathbb{R}\setminus A)$.*

*Proof.* Suppose that $\liminf_{h\to 0+} \frac{\lambda([x,x+h]\setminus A)}{h} > 0$. From (A2) it follows that

$$\limsup_{h\to 0+} \frac{\lambda([x,x+h]\setminus A)}{f(h)} \geq \liminf_{h\to 0+} \frac{\lambda([x,x+h]\setminus A)}{h} \cdot \limsup_{h\to 0+} \frac{h}{f(h)} > 0$$

which gives $x \notin \Phi_f^+(A)$. The second condition is a consequence of the first.

□

**Theorem 23.24** ([3]). *Let $f \in \mathcal{A}$. The operator $\Phi_f$ fulfils (D0)-(D3) and (D4') i.e. for any $A, B \in \mathcal{L}$ we have:*

*(0) if $A \in \mathcal{T}_{nat}$ then $A \subset \Phi_f(A)$,*
*(1) $\Phi_f(\emptyset) = \emptyset$ and $\Phi_f(\mathbb{R}) = \mathbb{R}$,*
*(2) $\Phi_f(A\cap B) = \Phi_f(A)\cap\Phi_f(B)$,*
*(3) if $A \sim B$ then $\Phi_f(A) = \Phi_f(B)$,*
*(4') $\lambda(\Phi_f(A)\setminus A) = 0$.*

*Proof.* Conditions (0), (3), $\Phi_f(\mathbb{R}) = \mathbb{R}$ and $\Phi_f(A\cap B) \subset \Phi_f(A)\cap\Phi_f(B)$ are obvious. The equality $\Phi_f(\emptyset) = \emptyset$ follows from (A2). Since

$$\lambda(I\setminus(A\cap B)) \leq \lambda(I\setminus A) + \lambda(I\setminus B)$$

for every interval $I$, we have $\Phi_f(A)\cap\Phi_f(B) \subset \Phi_f(A\cap B)$. From Theorem 23.23 we conclude that

$$\Phi_f(A)\setminus A \subset (\mathbb{R}\setminus A)\setminus \Phi_d(\mathbb{R}\setminus A).$$

Thus the Lebesgue Density Theorem implies (4'). □

The family $\mathcal{T}_{\Phi_f}$ will be denoted by $\mathcal{T}_f$, i.e.

$$\mathcal{T}_f = \left\{A \in \mathcal{L} : A \subset \Phi_f(A)\right\}.$$

According to Theorem 23.24, $\Phi_f$ and $\mathcal{T}_f$ fulfil all conditions of Theorem 23.4:

**Theorem 23.25.** *For any $f \in \mathcal{A}$*

*(1) $\mathcal{T}_f$ forms a topology on $\mathbb{R}$,*
*(2) $\mathrm{int}_{\mathcal{T}_f} E \subset E \cap \Phi_f(K_E)$ for $E \subset \mathbb{R}$,*
*(3) if $A \in \mathcal{N}$ then $A$ is $\mathcal{T}_f$-closed, $\mathcal{T}_f$-nowhere dense and $\mathcal{T}_f$-discrete,*
*(4) $A \in \mathcal{N}$ if and only if $A$ is $\mathcal{T}_f$-closed and $\mathcal{T}_f$-discrete,*
*(5) $A$ is a $\mathcal{T}_f$-compact set if and only if $A$ is finite,*
*(6) the space $(\mathbb{R}, \mathcal{T}_f)$ is neither first countable, nor Lindelöf, nor separable,*
*(7) $\mathrm{card}\mathcal{T}_f = 2^{\mathfrak{c}}$,*
*(8) $\mathcal{T}_{nat} \subsetneqq \mathcal{T}_f$,*
*(9) $A \in \mathcal{L}$ if and only if $A$ is a $\mathcal{T}_f$-Borel set,*
*(10) $(\mathbb{R}, \mathcal{T}_f)$ is a Hausdorff space,*
*(11) $(\mathbb{R}, \mathcal{T}_f)$ is not a normal space.*

In general, there are two possibilities to define density points. We can do it "in a symmetric way", examining the set on intervals centered at $x$, or "in one-sided way" - on intervals $[x-h, x]$ and $[x, x+h]$. In this sense, the definition of $f$-density is "one-sided". However, "symmetric" definition leads us to the same notion (using, if necessary, another function).

Let $f \in \mathcal{A}$, $A \in \mathcal{L}$ and $x \in \mathbb{R}$. We say that $x$ is a *symmetric $f$-density point* of $A$ if

$$\lim_{h \to 0+} \frac{\lambda([x-h, x+h] \setminus A)}{f(2h)} = 0.$$

By $\Phi_f^s(A)$ we denote the set of all symmetric $f$-density points of $A$. For any function $f$ defined on $(0, \infty)$ we set $f^*(x) := f(2x)$. An easy verification shows that:

**Proposition 23.26** ([15]).

*(1) $f \in \mathcal{A}$ if and only if $f^* \in \mathcal{A}$,*
*(2) $\Phi_{f^*} = \Phi_f^s$ for $f \in \mathcal{A}$,*
*(3) $\left\{\Phi_f : f \in \mathcal{A}\right\} = \left\{\Phi_f^s : f \in \mathcal{A}\right\}$.*

Of course, the equality $\Phi_f = \Phi_f^s$ does not have to happen.

*Example 23.27.* Let $f(x) := \frac{1}{n!}$ for $x \in \left(\frac{1}{(n+1)!}, \frac{1}{n!}\right]$ and

$$A := (-\infty, 0] \cup \bigcup_{n=2}^{\infty} \left[\frac{1}{(n+1)!}, \frac{1}{2\cdot n!}\right].$$

Clearly, $f \in \mathcal{A}$. It is easy to check that $\frac{\lambda([-h,h]\setminus A)}{f(2h)} \leq \frac{1}{n}$ for $h \in \left(\frac{1}{(n+1)!}, \frac{1}{n!}\right]$, and consequently, $0 \in \Phi_f^s(A)$. But $\frac{\lambda\left(\left[0,\frac{1}{n!}\right]\setminus A\right)}{f\left(\frac{1}{n!}\right)} > \frac{1}{2}$, which gives $0 \notin \Phi_f(A)$.

It should be mentioned that the notion of $f$-density could be defined even more generally. In [18] Hejduk considered $f$-density points (and symmetric $f$-density points) assuming only condition (A2).

**Theorem 23.28** ([18]). *Let $f : (0,\infty) \to (0,\infty)$ be a function satisfying*

$$\liminf_{x\to 0+} \frac{f(x)}{x} < \infty.$$

*Then*

*(1) $\Phi_f$ and $\Phi_f^s$ satisfy conditions (D0)-(D3) and (D4'),*
*(2) $\mathcal{T}_f$ and $\mathcal{T}_f^s$ form topologies such that $\mathcal{T}_{nat} \subset \mathcal{T}_f^s \subset \mathcal{T}_f$.*

## 23.5 The comparison of $f$-density topologies

Let $f, g \in \mathcal{A}$. We ask for conditions under which the inclusion $\mathcal{T}_f \subset \mathcal{T}_g$ holds. The necessary and sufficient condition is presented in the following chapter (Theorem 24.2). Now, we show an easy sufficient condition. We also formulate a necessary and sufficient condition to compare $\mathcal{T}_f$ with $\mathcal{T}_d$. Using it, we divide the family of all $f$-density topologies into two parts. The first consists of topologies similar to $\langle s\rangle$-density topologies and the second - similar to $\psi$-density topologies.

Obviously, different functions can generate the same operator (for example $\Phi_f = \Phi_{2f}$). Fortunately, different operators generate different topologies. Moreover, to prove the inclusion $\mathcal{T}_f \subset \mathcal{T}_g$, it is enough to show that the condition $0 \in \Phi_f^+(A)$ implies $0 \in \Phi_g^+(A)$.

**Theorem 23.29** ([5], [9]). *For each $f, g \in \mathcal{A}$ the following conditions are equivalent*

*(1) $\forall_{A\in\mathcal{L}} \left(0 \in \Phi_f^+(A) \Rightarrow 0 \in \Phi_g^+(A)\right)$,*

*(2)* $\forall_{A\in\mathcal{L}}\ (0 \in \Phi_f(A) \Rightarrow 0 \in \Phi_g(A))$,
*(3)* $\forall_{A\in\mathcal{L}}\ \Phi_f(A) \subset \Phi_g(A)$,
*(4)* $\mathcal{T}_f \subset \mathcal{T}_g$.

*Proof.* Implication (1)⇒(2) follows from $0 \in \Phi_f^-(A) \Leftrightarrow 0 \in \Phi_f^+(-A)$. By $x \in \Phi_f(A) \Leftrightarrow 0 \in \Phi_f(A-x)$, we obtain (2)⇒(3). Implication (3)⇒(1) is clear. Assume now that $\mathcal{T}_f \subset \mathcal{T}_g$, $0 \in \Phi_f^+(A)$ but $0 \notin \Phi_g^+(A)$. We can find $\varepsilon > 0$ and a decreasing sequence $(x_n)$ tending to zero such that

$$\frac{\lambda([0,x_n]\setminus A)}{g(x_n)} \geq \varepsilon$$

for every $n$. Defining $a_n = x_n - \lambda((x_{n+1},x_n)\setminus A)$ and

$$B := (-\infty,0] \cup \bigcup_{n=1}^{\infty} (x_{n+1},a_n)$$

we obtain that $0 \notin \Phi_g^+(B)$. On the other hand $0 \in \Phi_f^+(B)$, because

$$\lambda([0,x]\setminus B) \leq \lambda([0,x]\setminus A)$$

for any $x > 0$. Consequently, $B \in \mathcal{T}_f \setminus \mathcal{T}_g$, which gives a contradiction. □

According to the latter theorem, we will usually formulate conditions for topologies and operators, and will prove only the condition 23.29 from Theorem 23.29.

Let $f,g \in \mathcal{A}$. We say that $f$ *precedes* $g$ if $\limsup_{x\to 0+} \frac{f(x)}{g(x)} < \infty$. We denote it by $f \prec g$. If $\limsup_{x\to 0+} \frac{f(x)}{g(x)} = \infty$ we write $f \not\prec g$ (compare [25]).

**Theorem 23.30** ([3]). *Let $f,g \in \mathcal{A}$. If $f \prec g$ then $\mathcal{T}_f \subset \mathcal{T}_g$.*

*Proof.* Suppose that $0 \in \Phi_f^+(A)$. Then

$$\limsup_{h\to 0+} \frac{\lambda([0,h]\setminus A)}{g(h)} \leq \limsup_{h\to 0+} \frac{\lambda([0,h]\setminus A)}{f(h)} \cdot \limsup_{h\to 0+} \frac{f(h)}{g(h)} = 0 \cdot \limsup_{h\to 0+} \frac{f(h)}{g(h)} = 0,$$

which means that $0 \in \Phi_g^+(A)$. □

*Example 23.31* ([5]). There exist $f,g \in \mathcal{A}$ such that $f \not\prec g$ and $\mathcal{T}_f \subset \mathcal{T}_g$. Let $f(x) := \frac{1}{n!}$ for $x \in \left[\frac{1}{(n+1)!};\frac{1}{n!}\right)$ and $g(x) := \frac{1}{n!}$ for $x \in \left(\frac{1}{(n+1)!};\frac{1}{n!}\right]$. Then $f \geq g$ and $f \not\prec g$ because $\frac{f\left(\frac{1}{n!}\right)}{g\left(\frac{1}{n!}\right)} = n$. Suppose that $0 \in \Phi_f^+(A)$. Since $f(x) = g(x)$ for

$x \neq \frac{1}{n!}$, to prove $0 \in \Phi_g^+(A)$ it is enough to show that $\lim_{n\to\infty} \frac{\lambda\left(\left[0,\frac{1}{n!}\right]\setminus A\right)}{g\left(\frac{1}{n!}\right)} = 0$. For $n \geq 2$ we have

$$0 \leq \frac{\lambda\left(\left[0,\frac{1}{n!}\right]\setminus A\right)}{g\left(\frac{1}{n!}\right)} \leq \frac{\lambda\left(\left[0,\frac{1}{1+n!}\right]\setminus A\right)+\frac{1}{(n!)^2}}{f\left(\frac{1}{1+n!}\right)} = \frac{\lambda\left(\left[0,\frac{1}{1+n!}\right]\setminus A\right)}{f\left(\frac{1}{1+n!}\right)} + \frac{1}{n!} \underset{n\to\infty}{\longrightarrow} 0,$$

which completes the proof.

Example 23.31 shows that Theorem 23.30 can not be reversed. Fortunately, if one of considered topologies is $\mathcal{T}_d$, the opposite implication also holds.

**Theorem 23.32** ([5], [10]). *For any $f \in \mathcal{A}$ we have*

*(1) $\mathcal{T}_f \subset \mathcal{T}_d$ if and only if $\limsup_{x\to 0+} \frac{f(x)}{x} < \infty$.*
*(2) $\mathcal{T}_d \subset \mathcal{T}_f$ if and only if $\liminf_{x\to 0+} \frac{f(x)}{x} > 0$.*

*Proof.* By Theorem 23.30, inequalities under consideration are sufficient conditions for inclusions of topologies. Suppose that $\limsup_{x\to 0+} \frac{f(x)}{x} = \infty$. Let $A := \bigcup_{n=1}^{\infty} (2x_{n+1}, x_n)$, where $(x_n)$ is a decreasing sequence of positive numbers such that $2x_{n+1} < x_n < \frac{f(x_n)}{n}$. Since $\frac{\lambda(A\cap[0,2x_n])}{2x_n} < \frac{1}{2}$, $0 \notin \Phi_d^+(A)$. But $0 \in \Phi_f^+(A)$, because for $h \in (x_{n+1}, x_n]$ we have $\frac{\lambda([0,h]\setminus A)}{f(h)} \leq \frac{2x_{n+1}}{f(x_{n+1})} < \frac{2}{n+1}$. Suppose now that $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$. Let $A := \bigcup_{n=1}^{\infty} (x_{n+1}, x_n - f(x_n))$ where $x_{n+1} < f(x_n) < \frac{1}{n} x_n$. Then

$$\frac{\lambda(A\cap[0,x_n])}{x_n} > \frac{x_n - f(x_n) - x_{n+1}}{x_n} > 1 - \frac{2}{n},$$

and we easily obtain $0 \in \Phi_d^+(A)$. But $0 \notin \Phi_f^+(A)$ since $\frac{\lambda([0,x_n]\setminus A)}{f(x_n)} \geq 1$. □

**Corollary 23.33.** *Let $f \in \mathcal{A}$.*

*(1) $\mathcal{T}_f = \mathcal{T}_d$ if and only if $0 < \liminf_{x\to 0+} \frac{f(x)}{x} \leq \limsup_{x\to 0+} \frac{f(x)}{x} < \infty$.*
*(2) $\mathcal{T}_f \subsetneq \mathcal{T}_d$ if and only if $0 = \liminf_{x\to 0+} \frac{f(x)}{x} \leq \limsup_{x\to 0+} \frac{f(x)}{x} < \infty$.*
*(3) $\mathcal{T}_d \subsetneq \mathcal{T}_f$ if and only if $0 < \liminf_{x\to 0+} \frac{f(x)}{x} < \limsup_{x\to 0+} \frac{f(x)}{x} = \infty$.*
*(4) $\mathcal{T}_f \not\subseteq \mathcal{T}_d$ and $\mathcal{T}_d \not\subseteq \mathcal{T}_f$ if and only if $0 = \liminf \frac{f(x)}{x} < \limsup_{x\to 0+} \frac{f(x)}{x} = \infty$.*

Thus the family $\mathcal{A}$ splits into four subfamilies. However, it turns out that properties of the topology $\mathcal{T}_f$ depend mainly on whether $\mathrm{id} \prec f$. Therefore we define two subfamilies of $\mathcal{A}$ as follows:

$$\mathcal{A}^1 := \{f \in \mathcal{A} : \mathrm{id} \prec f\} = \left\{ f \in \mathcal{A} : \liminf_{x \to 0+} \frac{f(x)}{x} > 0 \right\},$$
$$\mathcal{A}^0 := \{f \in \mathcal{A} : \mathrm{id} \not\prec f\} = \left\{ f \in \mathcal{A} : \liminf_{x \to 0+} \frac{f(x)}{x} = 0 \right\}.$$

## 23.6 $f$-density and $\langle s \rangle$-density

In this section we will describe a relation between $\langle s \rangle$-density and $f$-density for $f \in \mathcal{A}^1$. First of all, we will show that any operator $\Phi_f$ generated by $f \in \mathcal{A}$ is generated by some continuous function $f_1 \in \mathcal{A}$ and by some function $f_2 \in \mathcal{A}$ constant on intervals. By $\mathcal{A}_c$ we denote the family of continuous functions from $\mathcal{A}$, and by $\mathcal{A}_s$ the family of functions from $\mathcal{A}$ for which there exist decreasing and tending to 0 sequences $(x_n)$ and $(a_n)$ such that $f(x) = a_n$ for $x \in (x_{n+1}, x_n]$.

**Theorem 23.34** ([5], [6]). $\left\{\Phi_f : f \in \mathcal{A}\right\} = \left\{\Phi_f : f \in \mathcal{A}_c\right\} = \left\{\Phi_f : f \in \mathcal{A}_s\right\}$.

*Proof.* Let $f \in \mathcal{A}$ and $a := f(1)$. As $f$ is nondecreasing, the set $f^{-1}\left(\left(\frac{a}{2^{n+1}}, \frac{a}{2^n}\right]\right)$ is either empty or is an interval (may be degenerated). Let $(k_n)$ be an increasing sequence of all numbers $n$ for which $I_n := f^{-1}\left(\left(\frac{a}{2^{k_n+1}}, \frac{a}{2^{k_n}}\right]\right)$ is a nondegenerated interval, and let $x_n$ be the right endpoint of $I_n$. Evidently, $x_{n+1}$ is the left endpoint of $I_n$. Let us denote

$$g(x) := \frac{a}{2^{k_n}} \text{ for } x \in (x_{n+1}, x_n].$$

Obviously, $g \in \mathcal{A}_s$. It is not difficult to prove that $\Phi_f = \Phi_g$ (see [5], Th. 1).
Assume now that $f \in \mathcal{A}_s$, i.e. $f(x) = a_n$ for $x \in (x_{n+1}, x_n]$, where $(x_n)$ and $(a_n)$ are decreasing sequences tending to 0 and such that $\liminf_{n \to \infty} \frac{a_n}{x_n} < \infty$. Put $\delta_n := \min\left\{\frac{x_n - x_{n+1}}{2}, \frac{a_{n+1}}{n}\right\}$ and

$$g(x) := \begin{cases} a_n \text{ for } x \in [x_{n+1} + \delta_n, x_n], \\ \text{linear on } [x_{n+1}, x_{n+1} + \delta_n]. \end{cases}$$

Of course, $g \in \mathcal{A}_c$. One can check that $\Phi_f = \Phi_g$ (see [5, Th. 1]). □

Now we are in the position to prove that $\langle s \rangle$-density operators and $\langle s \rangle$-density topologies are specific cases of $f$-density operators and $f$-density topologies. Let

$$f_{\langle s \rangle}(x) := \begin{cases} s_n \text{ for } x \in (s_{n+1}, s_n], \\ s_1 \text{ for } x > s_1 \end{cases}$$

for $\langle s\rangle \in \widetilde{\mathcal{S}}$. Obviously, $f_{\langle s\rangle} \in \mathcal{A}$ and $f_{\langle s\rangle}(x) \geq x$ for every $x$.

**Theorem 23.35** ([5]).

*(1) If $\langle s\rangle \in \widetilde{\mathcal{S}}$ then $f_{\langle s\rangle} \in \mathcal{A}_s$ and $\Phi_{\langle s\rangle} = \Phi_{f_{\langle s\rangle}}$.*
*(2) $\left\{\Phi_{\langle s\rangle} : \langle s\rangle \in \widetilde{\mathcal{S}}\right\} \subset \left\{\Phi_f : f \in \mathcal{A}^1\right\}$.*

*Proof.* The condition $f_{\langle s\rangle} \in \mathcal{A}_s$ is obvious. To prove that $\Phi_{\langle s\rangle} = \Phi_{f_{\langle s\rangle}}$ it is sufficient to show that for any measurable $A$

$$0 \in \Phi^{+}_{\langle s\rangle}(A) \Leftrightarrow 0 \in \Phi^{+}_{f_{\langle s\rangle}}(A).$$

The first implication follows from the inequality $\frac{\lambda([0,x]\setminus A)}{f(x)} \leq \frac{\lambda([0,s_n]\setminus A)}{s_n}$ for $x \in (s_{n+1}, s_n]$, and the second from the equality $f(s_n) = s_n$. The inclusion (2) follows immediately from the definition of $f_{\langle s\rangle}$. □

**Proposition 23.36** ([8]). *For any function $f \in \mathcal{A}$, there is a sequence $\langle s\rangle \in \widetilde{\mathcal{S}}$ such that $\mathcal{T}_f \subset \mathcal{T}_{\langle s\rangle}$.*

*Proof.* By (A2), there exist a sequence $\langle s\rangle \in \widetilde{\mathcal{S}}$ and a positive number $M$ such that $\frac{f(s_n)}{s_n} < M$ for every $n$. It is easily seen that $\mathcal{T}_f \subset \mathcal{T}_{\langle s\rangle}$. □

Using this inclusion we can characterize the family of $\mathcal{T}_f$-connected sets.

**Theorem 23.37** ([8]). *For any $f \in \mathcal{A}$, the family of $\mathcal{T}_f$-connected sets is equal to the family of sets connected in the natural topology.*

*Proof.* The latter proposition shows that $\mathcal{T}_{nat} \subset \mathcal{T}_f \subset \mathcal{T}_{\langle s\rangle}$ for some $\langle s\rangle \in \widetilde{\mathcal{S}}$. Thus our claim follows from Theorem 23.14. □

Recall that any $\Phi_{\langle s\rangle}$ is a lower density operator. The same is true for $\Phi_f$ with $f \in \mathcal{A}^1$.

**Theorem 23.38** ([4]). *If $f \in \mathcal{A}^1$ then $\Phi_f$ satisfies (D4).*

*Proof.* By Theorem 23.32, $\mathcal{T}_d \subset \mathcal{T}_f$. Hence $\Phi_d(A) \subset \Phi_f(A)$ for $A \in \mathcal{L}$, and consequently, $A \setminus \Phi_f(A) \subset A \setminus \Phi_d(A)$, which gives $\lambda(A \setminus \Phi_f(A)) = 0$. This completes the proof, because by Theorem 23.24, $\Phi_f$ fulfils (D4'). □

By Theorem 23.22, $\Phi_f(A) \in \mathcal{L}$ for each $f \in \mathcal{A}$ and $A \in \mathcal{L}$. From Theorem 23.6 we obtain:

**Theorem 23.39.** *If $f \in \mathcal{A}^1$ then*

*(1)* $\Phi_f(\Phi_f(A)) = \Phi_f(A)$ *for* $A \in \mathcal{L}$,
*(2)* $\mathcal{T}_f = \{\Phi_f(A) \setminus N : A \in \mathcal{L},\ N \in \mathcal{N}\}$,
*(3)* $\mathrm{int}_{\mathcal{T}_f} E = E \cap \Phi_f(K_E)$ *for* $E \subset \mathbb{R}$,
*(4)* $\mathrm{int}_{\mathcal{T}_f} A \sim A \sim \mathrm{cl}_{\mathcal{T}_f} A$ *for* $A \in \mathcal{L}$,
*(5)* $\Phi_f(A) = \mathrm{int}_{\mathcal{T}_f} \mathrm{cl}_{\mathcal{T}_f} A$ *for* $A \in \mathcal{L}$,
*(6)* $A \in \mathcal{N} \Leftrightarrow A$ *is* $\mathcal{T}_f$*-nowhere dense* $\Leftrightarrow A$ *is* $\mathcal{T}_f$*-meager,*
*(7)* $A \in \mathcal{L} \Leftrightarrow A$ *has* $\mathcal{T}_f$*-Baire property* $\Leftrightarrow A$ *is a sum of a* $\mathcal{T}_f$*-open set and a* $\mathcal{T}_f$*-closed set,*
*(8)* $(\mathbb{R}, \mathcal{T}_f)$ *is a Baire space.*

**Theorem 23.40** ([3]). *If $f \in \mathcal{A}$ and $\alpha \geq 1$ then $\mathcal{T}_f$ is invariant under multiplication by $\alpha$.*

*Proof.* From $\frac{\lambda([0,h]\setminus \alpha A)}{f(h)} = \frac{\alpha\lambda\left(\left[0,\frac{h}{\alpha}\right]\setminus A\right)}{f\left(\alpha\frac{h}{\alpha}\right)} \leq \alpha \frac{\lambda\left(\left[0,\frac{h}{\alpha}\right]\setminus A\right)}{f\left(\frac{h}{\alpha}\right)}$ it follows that $0 \in \Phi_f(A)$ implies $0 \in \Phi_f(\alpha A)$. Let $A \in \mathcal{T}_f$ and $x \in \alpha A$. Then

$$0 \in A - \frac{x}{\alpha} \subset \Phi_f\left(A - \frac{x}{\alpha}\right) \subset \Phi_f(\alpha A - x)$$

and consequently, $0 \in \mathrm{int}_{\mathcal{T}_f}(\alpha A - x) = \mathrm{int}_{\mathcal{T}_f}(\alpha A) - x$, which ends the proof. □

Theorems formulated up to now show that topologies $\mathcal{T}_f$ generated by $f \in \mathcal{A}^1$ have properties similar to the properties of topologies generated by sequences. Any $\langle s\rangle$-density topology is completely regular. In the next chapter it will be proved that $f$-density topologies are completely regular for $f \in \mathcal{A}^1$ (Theorem 24.15). There is a natural question if there exists an $f$-density topology, generated by a function from $\mathcal{A}^1$, which is not generated by a sequence. The positive answer was obtained in [5]. However, the example presented in this paper is quite complicated and the proof of its correctness is rather laborious. The much simpler example is presented in [10]. It is an example of a topology $\mathcal{T}_f$ which is bigger than the density topology and invariant under multiplication by nonzero numbers. By Theorem 23.12, such topology can not be generated by a sequence. The proof is based on the $(\Delta_2)$ condition which is presented in the next chapter (compare Example 24.24).

Now we present the example from the paper [5], omitting technical details. Using this example, it can be shown that the family of topologies generated by functions from $\mathcal{A}^1$ is "much bigger" than the family of topologies generated by sequences.

*Example 23.41* ([5], Theorem 6). Let us define sequences

$$\langle w\rangle := (2,2,3,3,3,4,4,4,4,\dots),\ \langle r\rangle := (1,2,1,2,3,1,2,3,4,\dots),$$
$$a_0 := 1,\ a_n := \frac{a_{n-1}}{w_n^2} \text{ and } b_n := a_n w_n = \frac{a_{n-1}}{w_n}.$$

Of course, $\lim_{n\to\infty} \frac{a_{n-1}}{b_n} = \lim_{n\to\infty} \frac{b_n}{a_n} = \infty$. The function

$$\widehat{f}(x) := \begin{cases} a_{n-1} & \text{for } x \in (b_n, a_{n-1}], \\ b_n r_n & \text{for } x \in (a_n, b_n] \end{cases}$$

belongs to $\mathcal{A}^1$, and $\mathcal{T}_{\widehat{f}} \neq \mathcal{T}_{\langle s\rangle}$ for each $\langle s\rangle \in \widetilde{\mathcal{S}}$.

Slightly modifying the function $\widehat{f}$ one can obtain continuum different topologies from $\{\mathcal{T}_f : f \in \mathcal{A}^1\} \setminus \left\{\mathcal{T}_{\langle s\rangle} : \langle s\rangle \in \widetilde{\mathcal{S}}\right\}$.

*Example 23.42* ([7], Cor. 7 and Th. 8). Let $\widehat{f}$ be a function defined in example 23.41 and $\widehat{f}_\alpha(x) := \widehat{f}\left(\frac{x}{\alpha}\right)$ for $\alpha > 1$. Then $\widehat{f}_\alpha \in \mathcal{A}^1$, $\mathcal{T}_{\widehat{f}_\alpha} \notin \left\{\mathcal{T}_{\langle s\rangle} : \langle s\rangle \in \widetilde{\mathcal{S}}\right\}$ and $\mathcal{T}_{\widehat{f}_\beta} \subsetneqq \mathcal{T}_{\widehat{f}_\alpha}$ for $\beta > \alpha > 1$.

The result from Example 23.41 can be also strengthen in a different way.

**Theorem 23.43** ([7, Th. 11]). *Let $\langle a\rangle, \langle b\rangle \in \widetilde{\mathcal{S}}$. If $\mathcal{T}_{\langle b\rangle} \subsetneqq \mathcal{T}_{\langle a\rangle}$ then there exists $f \in \mathcal{A}^1$ such that $\mathcal{T}_{\langle b\rangle} \subset \mathcal{T}_f \subset \mathcal{T}_{\langle a\rangle}$ and $\mathcal{T}_f \neq \mathcal{T}_{\langle s\rangle}$ for $\langle s\rangle \in \widetilde{\mathcal{S}}$.*

## 23.7 $f$-density and $\psi$-density

By Theorem 23.34, it is clear that a $\psi$-density topology is a specific case of an $f$-density topology. Namely, for $\psi \in \widehat{\mathcal{C}}$, we have $\mathcal{T}_\psi = \mathcal{T}^s_{f_\psi}$, where $f_\psi(x) := x\psi(x)$. Therefore:

**Proposition 23.44.**

$$\left\{\mathcal{T}_\psi : \psi \in \widehat{\mathcal{C}}\right\} \overset{(1)}{\subset} \left\{\mathcal{T}_f : f \in \mathcal{A} \wedge \lim_{x\to 0+} \frac{f(x)}{x} = 0\right\} \overset{(2)}{\subset} \left\{\mathcal{T}_f : f \in \mathcal{A}^0\right\}.$$

Note that both above inclusions are proper. From Theorem 23.32, the equality $\lim_{x\to 0+} \frac{f(x)}{x} = 0$ implies $\mathcal{T}_f \subsetneqq \mathcal{T}_d$. On the other hand, there is $f \in \mathcal{A}^0$ such that topologies $\mathcal{T}_f$ and $\mathcal{T}_d$ are not comparable. Thus the inclusion (2) is proper. A question concerning (1) is more interesting and more difficult. In fact, it is the question whether one can replace monotonicity of $\psi$ by monotonicity of $x\psi(x)$, in the definition of $\psi$-density point. The negative answer was obtained in [15]. The proof is based on the condition which is presented in the next chapter (Theorem 24.2).

**Theorem 23.45** ([15], Theorem 1.3).

$$\left\{\mathcal{T}_\psi : \psi \in \widehat{\mathcal{C}}\right\} \subsetneqq \left\{\mathcal{T}_f : f \in \mathcal{A}_c \wedge \lim_{x\to 0+} \frac{f(x)}{x} = 0\right\}.$$

The family $\left\{\mathcal{T}_f : f \in \mathcal{A}^0\right\}$ is much bigger than the family of all $\psi$-density topologies and it contains topologies incomparable with the density topology. Nevertheless, the properties of $f$-density topologies generated by functions from $\mathcal{A}^0$ are very similar to the properties of $\psi$-density topologies. These topics are more precisely desribed in the next chapter.

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