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Chapter 24

Density type topologies generated by functions. Properties of f-density

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Notions of f-density operators, f-density topologies and their basic properties were described in the previous chapter. Recall that by $\mathcal{A}$ we denote the family of all nondecreasing functions $f:(0,\infty)\to(0,\infty)$ with $\lim_{x\to 0+} f(x)=0$ and $\liminf_{x\to 0+}\frac{f(x)}{x}<\infty$. We say that $x\in\mathbb{R}$ is a *right-hand f-density point* of a measurable set A for a fixed $f\in\mathcal{A}$ if

$$\lim_{h\to 0+}\frac{\lambda\left([x,x+h]\setminus A\right)}{f\left(h\right)}=0.$$

By $\Phi_f^+(A)$ we denote the set of all right-hand f-density points of A, and in analogous way we define a *left-hand f-density point* and the set $\Phi_f^-(A)$. Finally, if $x\in\Phi_f(A):=\Phi_f^+(A)\cap\Phi_f^-(A)$ then we say that x is an *f-density point* of A. The family $\mathcal{T}_f=\left\{A\in\mathcal{L}:A\subset\Phi_f(A)\right\}$ forms a topology called *f-density topology*.

In chapter 23 f-density is treated mainly as a generalization of $\langle s\rangle$-density and ψ-density. Now we will focus our attention on the more advanced properties, which are generally more difficult to prove. All presented results are known but proofs contained in this chapter are considerably shortened and simplified.

24.1 Comparison of f-density topologies

A simple sufficient condition for the inclusion $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ is presented in Theorem 23.30. There is also formulated a necessary and sufficient condition to distinguish $\mathcal{T}_f$ from $\mathcal{T}_d$. Theorem 23.32 says that $\mathcal{T}_d \subset \mathcal{T}_f$ ($\mathcal{T}_f \subset \mathcal{T}_d$) if and only if $\liminf_{x\to 0+} \frac{f(x)}{x} > 0$ ($\limsup_{x\to 0+} \frac{f(x)}{x} < \infty$). Consequently, we divide the family $\mathcal{A}$ into two subfamilies:

$$\mathcal{A}^1 := \left\{ f \in \mathcal{A} : \liminf_{x\to 0+} \frac{f(x)}{x} > 0 \right\} \text{ and } \mathcal{A}^0 := \left\{ f \in \mathcal{A} : \liminf_{x\to 0+} \frac{f(x)}{x} = 0 \right\}.$$

Topologies $\mathcal{T}_f$ generated by functions from the family $\mathcal{A}^1$ are bigger then the density topology, and any $\langle s \rangle$-density topology is an f-density topology generated by some $f \in \mathcal{A}^1$ (compare Theorem 23.34). Topologies generated by $f \in \mathcal{A}^0$ are smaller then $\mathcal{T}_d$ or incomparable with $\mathcal{T}_d$. Any ψ-density topology is an f-density topology for some $f \in \mathcal{A}^0$ (compare Proposition 23.44 and [5]).

Now we will formulate the necessary and sufficient condition for the inclusion $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$. The analogous condition for ψ-density topology was formulated in [16]. However, the proof for f-density is much shorter and simpler than that for ψ-density.

In further considerations we will use the observation that to prove $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ it suffices to show that, for any measurable set A, $0 \in \Phi^+_{f_1}(A)$ implies $0 \in \Phi^+_{f_2}(A)$ (see Theorem 23.29). We will also need the following Lemma (compare [6]).

Lemma 24.1. *Let $f \in \mathcal{A}$, $t, h \in (0,\infty)$ and A be a measurable set satisfying $\limsup_{x\to 0+} \frac{\lambda([0,x]\cap A)}{f(x)} > t$. There is an interval $[a,b] \subset (0,h)$ such that*

$$\frac{\lambda([a,b]\cap A)}{f(b)} \geq t \text{ and } \frac{\lambda([a,x]\cap A)}{f(x)} \leq t \text{ for } x \in (a,b].$$

Proof. Since $\limsup_{x\to 0+} \frac{\lambda([0,x]\cap A)}{f(x)} > t$, there is a number $y \in (0,h)$ such that $\frac{\lambda([0,y]\cap A)}{f(y)} > t$. From the continuity of Lebesgue measure, it follows that $\frac{\lambda([a,y]\cap A)}{f(y)} = t$ for some $a \in (0,y)$. Let

$$b := \inf\left\{ x \in [a,y] : \frac{\lambda([a,x]\cap A)}{f(x)} \geq t \right\}.$$

Obviously, $a < b \le y$. To finish the proof it remains to check that $\frac{\lambda([a,b]\cap A)}{f(b)} \ge t$. Suppose that $\frac{\lambda([a,b]\cap A)}{f(b)} < t$. Thus there is $b' > b$ such that $\frac{\lambda([a,b']\cap A)}{f(b)} < t$, and consequently

$$\frac{\lambda([a,x]\cap A)}{f(x)} \le \frac{\lambda([a,b']\cap A)}{f(b)} < t$$

for any $x \in [b,b']$, which gives a contradiction with the definition of b. $\square$

Let $f_1, f_2 \in \mathcal{A}$. We define sequences

$$A_{nf_1f_2} := \left\{ x \in (0,\infty) : f_1(x) < \frac{1}{n} f_2(x) \right\},$$
$$\varepsilon_{nf_1f_2} := \limsup_{x\to 0+} \frac{\lambda(A_{nf_1f_2} \cap [0,x])}{f_1(x)}.$$

Of course, these sequences are decreasing, so $(\varepsilon_{nf_1f_2})_{n\in\mathbb{N}}$ is convergent.

Theorem 24.2 ([6]). *$\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$ if and only if* $\lim_{n\to\infty} \varepsilon_{nf_1f_2} = 0$.

Proof. Let us denote $A_{nf_1f_2}$ and $\varepsilon_{nf_1f_2}$ briefly by A_n and ε_n.

"$\Leftarrow$" Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and $0 \in \Phi^+_{f_2}(E)$ (i.e. $\lim_{x\to 0+} \frac{\lambda([0,x]\setminus E)}{f_2(x)} = 0$). We should to prove that $0 \in \Phi^+_{f_1}(E)$. Since

$$\limsup_{x\to 0+} \frac{\lambda([0,x]\setminus E)}{f_1(x)} \le \limsup_{x\to 0+} \frac{\lambda([0,x]\setminus E\setminus A_n)}{f_1(x)} + \limsup_{x\to 0+} \frac{\lambda([0,x]\cap A_n)}{f_1(x)} =$$
$$= \limsup_{x\to 0+} \frac{\lambda([0,x]\setminus E\setminus A_n)}{f_1(x)} + \varepsilon_n,$$

it suffices to show that

$$\lim_{x\to 0+} \frac{\lambda([0,x]\setminus E\setminus A_n)}{f_1(x)} = 0. \tag{24.1}$$

Let us fix a positive integer n and a positive x with $f_1(x) < 1$. If $x \notin A_n$ then $f_1(x) \ge \frac{1}{n} f_2(x)$, and consequently $\frac{\lambda([0,x]\setminus E\setminus A_n)}{f_1(x)} \le n\frac{\lambda([0,x]\setminus E)}{f_2(x)}$. If $(0,x] \subset A_n$ then $\frac{\lambda([0,x]\setminus E\setminus A_n)}{f_1(x)} = 0$. Finally, if $x \in A_n$ and $(0,x]\setminus A_n \ne \emptyset$, then for $a := \sup((0,x]\setminus A_n)$ and for any y from $[a - af_1(a), a]\setminus A_n$ we have $f_1(y) \ge \frac{1}{n} f_2(y)$. Hence

$$\frac{\lambda([0,x]\setminus E\setminus A_n)}{f_1(x)} \le \frac{\lambda([0,y]\setminus E\setminus A_n)}{f_1(y)} + \frac{a-y}{f_1(a)} \le n\frac{\lambda([0,y]\setminus E)}{f_2(y)} + x,$$

which implies (24.1).

"⇒" Suppose that $\lim_{n\to\infty}\varepsilon_n > 0$. We look for a measurable set E such that $0 \in \Phi^+_{f_2}(E) \setminus \Phi^+_{f_1}(E)$. There is a positive number t such that $\limsup_{x\to 0+} \frac{\lambda(A_n\cap[0,x])}{f_1(x)} > t$ for sufficiently large n. We can assume that this inequality holds for every n. Applying Lemma 24.1, we can define intervals $[a_n,b_n]$ such that $b_{n+1} < \min\left\{a_n, \frac{1}{n} f_2(a_n)\right\}$,

$$\frac{\lambda([a_n,b_n]\cap A_n)}{f_1(b_n)} \geq t \text{ and } \frac{\lambda([a_n,x]\cap A_n)}{f_1(x)} \leq t \text{ for } x\in(a_n,b_n].$$

Set

$$E := \mathbb{R} \setminus \bigcup_{n=1}^{\infty} (A_n \cap [a_n,b_n]).$$

Since $\frac{\lambda([0,b_n]\setminus E)}{f_1(b_n)} \geq \frac{\lambda([a_n,b_n]\cap A_n)}{f_1(b_n)} \geq t > 0$, $0 \notin \Phi_{f_1}(E)$.
Let us consider $x \in (0,b_1]$. We first assume that $x\in(a_n,b_n]$ for some n. If $\lambda([a_n,x]\cap A_n) = 0$ then

$$\frac{\lambda([0,x]\setminus E)}{f_2(x)} \leq \frac{b_{n+1}}{f_2(a_n)} < \frac{1}{n}.$$

If $\lambda([a_n,x]\cap A_n) > 0$ then one can find $y,z\in[a_n,x]$ such that $z<y\leq x$, $\lambda(A_n\cap[y,x]) = 0$, $z\in A_n$ and $y-z<f_1(a_n)$. Thus

$$\frac{\lambda([0,x]\setminus E)}{f_2(x)} \leq \frac{\lambda([a_n,y]\cap A_n)+b_{n+1}}{f_2(z)} \leq \frac{\lambda([a_n,z]\cap A_n)}{f_2(z)} + \frac{y-z}{nf_1(z)} + \frac{b_{n+1}}{f_2(a_n)} <$$
$$< \frac{t}{n} + \frac{1}{n} + \frac{1}{n} = \frac{t+2}{n}.$$

Assume now that $x\in(b_{n+1},a_n]$. From the previous case we obtain

$$\frac{\lambda([0,x]\setminus E)}{f_2(x)} \leq \frac{\lambda([0,b_{n+1}]\setminus E)}{f_2(b_{n+1})} < \frac{t+2}{n+1}.$$

This gives $0\in\Phi_{f_2}(E)$, which completes the proof. □

As a straightforward consequence we obtain:

Theorem 24.3 ([6]). *Let* f_1, $f_2 \in \mathcal{A}$. *If* $\lim_{x\to 0+}\frac{f_1(x)}{f_2(x)} = 0$ *then* $\mathcal{T}_{f_1} \subsetneq \mathcal{T}_{f_2}$.

Proof. There are $\delta > 0$ and $n_0\in\mathbb{N}$ such that $A_{nf_1f_2} \subset (0,\delta)$ and $A_{nf_2f_1} \cap (0,\delta) = \emptyset$ for $n\geq n_0$. Clearly, $\lim_{n\to\infty}\varepsilon_{nf_2f_1} = 0$, so $\mathcal{T}_{f_1}\subset\mathcal{T}_{f_2}$.
Since $A_{nf_1f_2}\cap(0,x) = (0,x)$ for $x\in(0,\delta)$ and $n\geq n_0$, we have $\varepsilon_{nf_1f_2} = \limsup_{x\to 0+}\frac{x}{f_1(x)}$. By the definition of the family $\mathcal{A}$, we have $\limsup_{x\to 0+}\frac{x}{f_1(x)} > 0$, and consequently $\lim_{n\to\infty}\varepsilon_{nf_1f_2} > 0$. Therefore $\mathcal{T}_{f_1} \neq \mathcal{T}_{f_2}$. □

Theorem 23.32 shows that the condition $\lim_{x\to 0+}\frac{f_1(x)}{f_2(x)}=0$ is not necessary for $\mathcal{T}_{f_1}\subsetneqq\mathcal{T}_{f_2}$.

24.2 Properties of f-density for $f\in\mathcal{A}^0$

In Theorem 23.38 it is proved that $\lambda\left(\Phi_f(A)\triangle A\right)=0$ for $f\in\mathcal{A}^1$ and $A\in\mathcal{L}$. Thus for any f from $\mathcal{A}^1$ an operator Φ_f and a topology $\mathcal{T}_f$ have properties similar to the properties of "classical" density operator Φ_d and the density topology $\mathcal{T}_d$ (compare Theorem 23.39). Now we will study properties of Φ_f and $\mathcal{T}_f$ for $f\in\mathcal{A}^0$. The essential role in these considerations is played by the result analogous to The Second Taylor's Theorem (compare [14] and chapter 22) for f-density.

We begin by defining a Cantor-type set generating by two sequences. In n-th step of the construction of the Cantor ternary set, two subintervals of the length $\frac{1}{3^n}$ are chosen from any component. In our construction, we will choose k_n subintervals of the length r_n each.

Let $(r_n)_{n=0,1,\ldots}$ be a sequence of positive numbers and $(k_n)_{n=0,1,\ldots}$ be a sequence of positive integers such that $k_0=r_0=1$, $k_n\geq 2$ and $k_nr_n<r_{n-1}$ for $n\geq 1$. We define inductively a decreasing sequence $(F_n)_{n=0,1\ldots}$ of closed sets consisting of $p_n:=k_0\cdot\ldots\cdot k_n$ pairwise disjoint closed intervals I_i^n of the length r_n.

For $n=0$ we put $F_0:=I_1^0:=[0,1]$. Suppose that we have defined disjoint closed intervals $I_1^n,\ldots,I_{p_n}^n$ and the set $F_n:=I_1^n\cup\ldots\cup I_{p_n}^n$ for some $n\geq 0$. For any $i\in\{1,\ldots,p_n\}$ we define k_{n+1} pairwise disjoint closed subintervals $I_{(i-1)k_{n+1}+1}^{n+1},\ldots,I_{ik_{n+1}}^{n+1}$ of the interval I_i^n, of the length r_{n+1} each. We choose them in such a way that the left endpoint of $I_{(i-1)k_{n+1}+1}^{n+1}$ is the left enpoint of I_i^n, the right endpoint of $I_{ik_{n+1}}^{n+1}$ is the right endpoint of I_i^n and distances between subintervals are the same. Let $F_{n+1}:=I_1^{n+1}\cup\ldots\cup I_{p_{n+1}}^{n+1}$. Thus we have defined the sequence $(F_n)_{n=0,1\ldots}$. Put

$$F:=F\left((r_n),(k_n)\right):=\bigcap_{n=0}^{\infty}F_n.$$

Remark 24.4. From now on we will assume that $F_0=I_1^0:=[0,1]$ and will define sequences (r_n) and (k_n) for $n\geq 1$.

Lemma 24.5. *For any $\varepsilon\in(0,1)$ and any tending to zero sequence (x_n) of positive numbers there exists a subsequence (r_n) of the sequence (x_n) and a se-*

quence (k_n) *of positive integers such that the set* $F := F((r_n),(k_n))$ *satisfies* $\lambda(F) > 1-\varepsilon$ *and*

$$\frac{\varepsilon}{p_n 2^{n+2}} < \lambda\left(I_i^n \setminus F_{n+1}\right) < \frac{\varepsilon}{p_n 2^{n+1}} \text{ for } n = 0,1,\ldots,\ i = 1,\ldots,p_n. \qquad (24.2)$$

Proof. Fix a natural number m and suppose that we have defined $r_j = x_{t_j}$ and k_j for $j = 1,\ldots,m$. As r_{m+1} we choose any element x_t from the sequence (x_n) such that $t > t_m$ and $x_t < \frac{\varepsilon}{p_m 2^{m+2}}$. Now we put

$$k_{m+1} := \max\left\{k \in \mathbb{N} : r_m - k r_{m+1} > \frac{\varepsilon}{p_m 2^{m+2}}\right\}.$$

From the definition of k_{m+1} it follows that

$$\frac{\varepsilon}{p_m 2^{m+2}} < r_m - k_{m+1} r_{m+1} \le \frac{\varepsilon}{p_m 2^{m+2}} + r_{m+1} < \frac{\varepsilon}{p_m 2^{m+1}}.$$

Since $\lambda\left(I_i^m \setminus F_{m+1}\right) = r_m - k_{m+1} r_{m+1}$, we obtain (24.2). Moreover

$$\lambda([0,1] \setminus F) = \sum_{n=0}^{\infty} \lambda\left(F_n \setminus F_{n+1}\right) = \sum_{n=0}^{\infty} \sum_{i=1}^{p_n} \lambda\left(I_i^n \setminus F_{n+1}\right) < \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon.$$

□

Theorem 24.6 ([9], [2]). *For any function* $f \in \mathcal{A}^0$ *and any real number* $c \in [0,1)$ *there exists a closed set* $F \subset [0,1]$ *such that* $\lambda(F) \ge c$ *and* $\Phi_f(F) = \emptyset$.

Proof. Let $\varepsilon := 1-c$. Since $f \in \mathcal{A}^0$, there is a decreasing sequence (x_n) tending to zero and such that

$$\frac{f(x_n)}{x_n} < \frac{1}{n 2^n}$$

for every n. Clearly, this inequality holds also for any subsequence of (x_n). Let (r_n), (k_n) and $F := F((r_n),(k_n))$ satisfy the assertion of the latter Lemma. Since F is closed, $\Phi_f(F) \subset F$, so it remains to prove that $F \cap \Phi_f(F) = \emptyset$. Let us fix $x \in F$. There exists a sequence $\left(I_{i_n}^n\right)$ of closed intervals such that $\lambda\left(I_{i_n}^n\right) = r_n$ and $\bigcap_{n=1}^{\infty} I_{i_n}^n = \{x\}$. As $\lambda\left(I_{i_n}^n\right) < \frac{1}{p_n}$ we have

$$\frac{\lambda\left(I_{i_n}^n \setminus F\right)}{f\left(\lambda\left(I_{i_n}^n\right)\right)} > \frac{\lambda\left(I_{i_n}^n \setminus F_{n+1}\right)}{f(r_n)} > \frac{\frac{\varepsilon}{p_n 2^{n+2}}}{f(r_n)} > \frac{r_n \varepsilon}{f(r_n) 2^{n+2}} > \frac{\varepsilon n}{4}.$$

Hence $\lim_{n\to\infty} \frac{\lambda\left(I_{i_n}^n \setminus F\right)}{f\left(\lambda\left(I_{i_n}^n\right)\right)} = \infty$, and consequently $x \notin \Phi_f(F)$ (compare Proposition 23.21). □

Using the set F we will construct a closed set with exactly one f-density point.

Theorem 24.7 ([9]). *For any function $f \in \mathcal{A}^0$ there exists a closed set $F_1 \subset [0,1]$ such that $\lambda(F_1) > 0$ and $\Phi_f(F_1) = \{0\}$.*

Proof. Since $f \in \mathcal{A}^0$, there exists a sequence (x_n) such that $nx_{n+1} < f(x_n) < x_n$ for $n \geq 2$. Let $J_n := \left[x_{n+1} + \frac{1}{n} f(x_{n+1}), x_n\right]$. According to Theorem 24.6 there is a closed set $E_n \subset J_n$ with $\Phi_f(E_n) = \emptyset$ and $\lambda(J_n \setminus E_n) < \frac{1}{n} f(x_{n+1})$. Put $E := \bigcup_{n=2}^{\infty} E_n$. Of course, $\Phi_f(E) = \emptyset$. It remains to check that $0 \in \Phi_f^+(E)$. For $x \in [x_{n+1}, x_n]$ we have

$$[0,x] \setminus E \subset \left[0, x_{n+2} + \frac{1}{n+1} f(x_{n+2})\right] \cup (J_{n+1} \setminus E_{n+1}) \cup \\ \cup \left[x_{n+1}, x_{n+1} + \frac{1}{n} f(x_{n+1})\right] \cup (J_n \setminus E_n).$$

Therefore

$$\frac{\lambda([0,x] \setminus E)}{f(x)} < \frac{x_{n+2} + \frac{1}{n+1} f(x_{n+2}) + \frac{1}{n+1} f(x_{n+2}) + \frac{1}{n} f(x_{n+1}) + \frac{1}{n} f(x_{n+1})}{f(x_{n+1})} < \frac{5}{n},$$

and consequently $0 \in \Phi_f^+(E)$. Thus $F_1 := E \cup (-E) \cup \{0\}$ is the desired set. □

The sets constructed in Theorem 24.6 and Theorem 24.7 show that properties of f-density operators for $f \in \mathcal{A}^0$ differ considerably from the properties of the classical density operator Φ_d. Theorems 23.6 and 23.8 contain the list of differences. In particular we have.

Proposition 24.8. *If $f \in \mathcal{A}^0$ then the space $(\mathbb{R}, \mathcal{T}_f)$ is of the first category.*

In chapter 23 it is proved that $\operatorname{int}_{\mathcal{T}_f}(A) = A \cap \Phi_f(A)$ for $f \in \mathcal{A}^1$ and $A \in \mathcal{L}$. This equality need not be true for $f \in \mathcal{A}^0$. If F_1 is the set constructed in Theorem 24.7 then $F_1 \cap \Phi_f(F_1) = \{0\}$ but $\operatorname{int}_{\mathcal{T}_f}(F_1) = \emptyset$. Now we will describe an interior operation in an f-density topology (in the general case).

Let $f \in \mathcal{A}$ and $A \in \mathcal{L}$. By induction we define $\Phi_f^{\alpha}(A)$ for $1 \leq \alpha < \omega_1$:

$$\Phi_f^1(A) := \Phi_f(A), \; \Phi_f^{\alpha+1}(A) := \Phi_f\left(\Phi_f^{\alpha}(A)\right),$$
$$\Phi_f^{\alpha}(A) := \bigcap_{1 \leq \beta < \alpha} \Phi_f^{\beta}(A) \text{ if } \alpha \text{ is a limit number.}$$

Obviously, $\Phi_f^{\alpha} = \Phi_f$ for $f \in \mathcal{A}^1$ and $1 \leq \alpha < \omega_1$.

We will use some properties of operators Φ_f described in Theorem 23.24. Note that for any $f \in \mathcal{A}$ and any measurable sets A, B, $\lambda(\Phi_f(A) \setminus A) = 0$ and the condition $\lambda(A \setminus B) = 0$ implies $\Phi_f(A) \subset \Phi_f(B)$. Therefore $\Phi_f(\Phi_f(A)) \subset \Phi_f(A)$. Recall that by K_A we denote a measurable kernel of $A \subset \mathbb{R}$.

Proposition 24.9 ([9]). *Let $f \in \mathcal{A}$, $E \subset \mathbb{R}$, $A, B \in \mathcal{L}$ and $A \subset B$. If $1 \le \beta \le \alpha < \omega_1$ then*

(1) $\Phi_f^{\alpha}(A) \subset \Phi_f^{\alpha}(B)$,
(2) $\Phi_f^{\alpha}(A) \subset \Phi_f^{\beta}(A)$,
(3) $\mathrm{int}_{\mathcal{T}_f}(E) \subset E \cap \Phi_f^{\alpha}(K_E)$.

Proof. (1) follows from the monotonicity of Φ_f. Suppose that (2) is true for ordinal numbers less than α. If α is a limit number then

$$\Phi_f^{\alpha}(A) = \bigcap_{1 \le \gamma < \alpha} \Phi_f^{\gamma}(A) \subset \Phi_f^{\beta}(A)$$

for any $\beta < \alpha$. If $\alpha = \gamma + 1$ then $\Phi_f^{\alpha}(A) = \Phi_f\left(\Phi_f^{\gamma}(A)\right) \subset \Phi_f^{\gamma}(A)$, so $\Phi_f^{\alpha}(A) \subset \Phi_f^{\beta}(A)$ for $\beta < \alpha$, which ends the proof of (2). By Theorem 23.25 we have $\mathrm{int}_{\mathcal{T}_f}(E) \subset \Phi_f(K_E)$. Suppose that $1 \le \alpha < \omega_1$ and $\mathrm{int}_{\mathcal{T}_f}(E) \subset \Phi_f^{\gamma}(K_E)$ for $\gamma < \alpha$. If $\alpha = \beta + 1$ then

$$\mathrm{int}_{\mathcal{T}_f}(E) \subset \Phi_f\left(\mathrm{int}_{\mathcal{T}_f}(E)\right) \subset \Phi_f\left(\Phi_f^{\beta}(K_E)\right) = \Phi_f^{\alpha}(K_E).$$

If α is a limit number then $\mathrm{int}_{\mathcal{T}_f}(E) \subset \bigcap_{1 \le \gamma < \alpha} \Phi_f^{\gamma}(K_E) = \Phi_f^{\alpha}(K_E)$, which ends the proof of (3). □

Theorem 24.10 ([9]). *For any function $f \in \mathcal{A}$ and any set $E \subset \mathbb{R}$ there is $1 \le \alpha < \omega_1$ such that* $\mathrm{int}_{\mathcal{T}_f}(E) = E \cap \Phi_f^{\alpha}(K_E)$.

Proof. By the latter proposition, the sequence $\left(\Phi_f^{\alpha}(K_E)\right)_{\alpha < \omega_1}$ is decreasing. Since $(\mathcal{L}, \mathcal{N})$ fulfils countable chain condition, there is $\alpha < \omega_1$ such that $\lambda\left(\Phi_f^{\alpha}(K_E) \setminus \Phi_f^{\alpha+1}(K_E)\right) = 0$, and hence $\Phi_f^{\alpha+1}(K_E) = \Phi_f^{\alpha}(K_E)$. We shall show that $\mathrm{int}_{\mathcal{T}_f}(E) = E \cap \Phi_f^{\alpha}(K_E)$. Proposition 24.9 implies $\mathrm{int}_{\mathcal{T}_f}(E) \subset E \cap \Phi_f^{\alpha}(K_E)$. From $\lambda\left(\Phi_f^{\alpha}(K_E) \setminus K_E\right) \le \lambda(\Phi_f(K_E) \setminus K_E) = 0$ it follows that

$$\Phi_f\left(E \cap \Phi_f^{\alpha}(K_E)\right) = \Phi_f\left(\Phi_f^{\alpha}(K_E)\right) = \Phi_f^{\alpha}(K_E) \supset E \cap \Phi_f^{\alpha}(K_E).$$

Thus $E \cap \Phi_f^{\alpha}(K_E)$ is $\mathcal{T}_f$-open, and consequently

$$E \cap \Phi_f^{\alpha}(K_E) = \mathrm{int}_{\mathcal{T}_f}\left(E \cap \Phi_f^{\alpha}(K_E)\right) \subset \mathrm{int}_{\mathcal{T}_f}(E).$$

□

The following theorem is an interesting strengthening of Theorem 24.7. In [17] it was proved for ψ-density and in [9] for f-density.

Theorem 24.11 ([9]). *Let $f \in \mathcal{A}^0$. For an arbitrary perfect nowhere dense set A there exists a perfect nowhere dense set B such that $\Phi_f(B) = A$.*

Theorem 24.12 ([9]). *Let $f \in \mathcal{A}^0$. For each $n > 1$ there exists a closed nowhere dense set A such that* $\mathrm{int}_{\mathcal{T}_f}(A) = \Phi_f^n(A) \subsetneqq \Phi_f^{n-1}(A)$.

Proof. For $n = 2$ it suffices to take the set F_1 from Theorem 24.7. Let $n \geq 2$. Suppose that there exists a closed nowhere dense set A such that $\mathrm{int}_{\mathcal{T}_f}(A) = \Phi_f^n(A) \subsetneqq \Phi_f^{n-1}(A)$. Let B be a closed set such that $\Phi_f(B) = A$. Obviously, B is nowhere dense, $A \subset B$ and

$$\Phi_f^{n+1}(B) = \Phi_f^n(A) = \mathrm{int}_{\mathcal{T}_f}(A) \subset \mathrm{int}_{\mathcal{T}_f}(B) \subset \Phi_f^{n+1}(B).$$

Hence $\mathrm{int}_{\mathcal{T}_f}(B) = \Phi_f^{n+1}(B)$ and $\mathrm{int}_{\mathcal{T}_f}(B) = \mathrm{int}_{\mathcal{T}_f}(A) \subsetneqq \Phi_f^{n-1}(A) = \Phi_f^n(B)$, which completes the proof. □

24.3 Separating axioms for f-density topologies

Recall that any f-density topology is Hausdorff but not normal (see Theorem 23.25). We will prove that $\mathcal{T}_f$ is completely regular for $f \in \mathcal{A}^1$. The proof is similar to the proof for classical density topology $\mathcal{T}_d$ (compare [1]). We will write $B \subset_f A$ instead of $B \subset A \cap \Phi_f(A)$.

Theorem 24.13 ([3]). *Let $f \in \mathcal{A}^1$, $A \in \mathcal{L}$ and F be a closed set such that $F \subset_f A$. There exists a closed set P such that $F \subset_f P \subset_f A$.*

Proof. Without loss of generality we can assume that $\mathrm{dist}(x,F) < 1$ for $x \in A$. Let $B := A \cap \Phi_f(A)$ (i.e. $B = \mathrm{int}_{\mathcal{T}_f}(A)$). Of course, $F \subset_f B$. Define $B_n := \left\{x \in B : \frac{1}{n+1} < \mathrm{dist}(x,F) \leq \frac{1}{n}\right\}$. For any $n \in \mathbb{N}$ there is a closed set $P_n \subset B_n$ such that $\lambda(B_n \setminus P_n) < \frac{1}{2^n} f\left(\frac{1}{n+1}\right)$. Put $P := F \cup \bigcup_{n=1}^{\infty} P_n$. Obviously, P is closed and $F \subset P \subset B \subset_f A$. It suffices to show that $F \subset \Phi_f(P)$. For any $x \in F$ and $h \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ we have

$$\lambda\left((B \setminus P) \cap [x-h, x+h]\right) \leq \lambda\left(\bigcup_{k=n}^{\infty} (B_k \setminus P_k)\right) \leq f\left(\frac{1}{n+1}\right)\frac{1}{2^{n-1}},$$

and consequently

$$\frac{\lambda([x-h,x+h]\setminus P)}{f(h)} \le \frac{\lambda([x-h,x+h]\setminus B)}{f(h)} + \frac{\lambda([x-h,x+h]\cap(B\setminus P))}{f(h)} \le$$
$$\le \frac{\lambda([x-h,x]\setminus B)}{f(h)} + \frac{\lambda([x,x+h]\setminus B)}{f(h)} + \frac{1}{2^{n-1}}.$$

Since $x \in \Phi_f(B)$, the last inequality implies $x \in \Phi_f(P)$. □

Remark 24.14. Repeating the proof of Theorem 24.13, one can show that if $f \in \mathcal{A}$, $A \in \mathcal{L}$ and F is a closed set such that $F \subset_f A$, then there exists a closed set P such that $F \subset_f P \subset A$. It means that an f-density topology fulfils Lusin-Menchoff condition for each $f \in \mathcal{A}$ (not only from $\mathcal{A}^1$) (compare [3]).

Theorem 24.15 ([3]). *If $f \in \mathcal{A}^1$ then $\mathcal{T}_f$ is completely regular.*

Proof. Let G be a $\mathcal{T}_f$-open set and $x_0 \in G$. There is an F_σ set F such that $x_0 \in F \subset G$ and $\lambda(G\setminus F) = 0$. Then $F \in \mathcal{T}_f$ and there are closed sets F_n such that $F = \bigcup_{n=1}^{\infty} F_n$. We first construct a family $\{P_\alpha : \alpha \in [1,\infty)\}$ of closed sets such that

$$P_\alpha \subset_f P_\beta \subset F \text{ for } \beta > \alpha \ge 1. \tag{24.3}$$

Let $P_1 := F_1$. Since $P_1 \subset_f F$, Theorem 24.13 shows that there is a closed set K_2 such that $P_1 \subset_f K_2 \subset F$. Thus the set $P_2 := F_2 \cup K_2$ fulfils $P_1 \subset_f P_2 \subset F$. Proceeding inductively we define sets P_n such that

$$F_n \subset P_n \subset_f P_{n+1} \subset F.$$

Similarly, for every γ from the set $\mathbb{Q}_2 := \left\{\frac{n}{2^m} : m \in \mathbb{N} \text{ and } n \ge 2^m\right\}$ we can define P_γ, in such a way that $P_\alpha \subset_f P_\beta$ for $\alpha,\beta \in \mathbb{Q}_2$, $\alpha < \beta$. Finally, for any $\alpha \in [1,\infty)$ we put $P_\alpha := \bigcap_{\beta\in\mathbb{Q}_2,\beta\ge\alpha} P_\beta$. It is easy to check that sets P_α satisfy condition (24.3). Write

$$g(x) := \begin{cases} \frac{1}{\inf\{\alpha : x\in P_\alpha\}} & \text{for } x \in F, \\ 0 & \text{for } x \notin F. \end{cases}$$

It is clear that $0 < g(x) \le 1$ for $x \in F$. We show that g is $\mathcal{T}_f$-continuous. If $x \notin F$ then $g(x) = 0$ and for any $\varepsilon \in (0,1)$ we have $x \in \mathbb{R}\setminus P_{2/\varepsilon} \subset \{t : g(t) < \varepsilon\}$. Since $\mathbb{R}\setminus P_{1/\varepsilon}$ is open, g is continuous at x (even in a usual sense). Assume now that $x \in F$. Let $\alpha := \frac{1}{g(x)}$, $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$A := \left\{t : g(t) < \frac{1}{\alpha - 2\varepsilon}\right\} \text{ and } B := \left\{t : g(t) > \frac{1}{\alpha + 3\varepsilon}\right\}.$$

We have $x \in \mathbb{R} \setminus P_{\alpha-\varepsilon} \subset A$ and $x \in P_{\alpha+\varepsilon} \subset_f P_{\alpha+2\varepsilon} \subset B$. Hence

$$x \in \operatorname{int}_{\mathcal{T}_{nat}}(A) \cap \operatorname{int}_{\mathcal{T}_f}(P_{\alpha+2\varepsilon}) \subset \operatorname{int}_{\mathcal{T}_f}(A) \cap \operatorname{int}_{\mathcal{T}_f}(B) = \operatorname{int}_{\mathcal{T}_f}(A \cap B),$$

and consequently g is $\mathcal{T}_f$-continuous at x. Since g is $\mathcal{T}_f$-continuous, the function

$$h(x) := \frac{g(x)}{g(x) + |x - x_0|}.$$

is $\mathcal{T}_f$-continuous, $h(x_0) = 1$ and $h(x) = 0$ for $x \notin G$. Thus $\mathcal{T}_f$ is completely regular. □

Theorem 24.16 ([3]). *If $f \in \mathcal{A}^0$ then $\mathcal{T}_f$ is not regular.*

Proof. Since $f \in \mathcal{A}^0$, there is a nowhere dense closed set $F \subset [0,1]$ of a positive measure and such that $\Phi_f(F) = \emptyset$. Let $(q_j)_{j\in\mathbb{N}}$ be a sequence of all rational numbers. For any natural j, the set $F_j := F + q_j$ is $\mathcal{T}_f$-closed and $\mathcal{T}_f$-nowhere dense. However, the set $B := \bigcup_{j=1}^{\infty} F_j$ is $\mathcal{T}_f$-open because it is a set of a full measure (compare Smital's Lemma [10, p. 65]). We will show that no point from B can be separated from $\mathbb{R} \setminus B$ by $\mathcal{T}_f$-open sets. Let A be a nonempty $\mathcal{T}_f$-open subset of B. The proof will be completed by showing that

$$\operatorname{cl}_{\mathcal{T}_f}(A) \setminus B \neq \emptyset. \tag{24.4}$$

Since $f \in \mathcal{A}^0$, there exists a decreasing sequence (t_n) of positive numbers such that $f(2t_n) < 2t_n$ for every n. As $A \setminus F_1$ is $\mathcal{T}_f$-open and nonempty we have $\lambda(A \setminus F_1) > 0$, and the set

$$A_1 := (A \setminus F_1) \cap F_{i_1}$$

has positive measure for some $i_1 > 1$. Let $a_1 \in A_1 \cap \Phi_d(A_1)$. There is a natural number n_1 such that

$$[a_1 - t_{n_1}, a_1 + t_{n_1}] \cap F_1 = \emptyset \text{ and } \frac{\lambda(A_1 \cap [a_1 - t_{n_1}, a_1 + t_{n_1}])}{2t_{n_1}} > \frac{1}{2}.$$

Let us denote $x_1 := a_1 - t_{n_1}$ and $y_1 := a_1 + t_{n_1}$.
Note that $\lambda\left(A \cap (x_1, y_1) \setminus \bigcup_{j=1}^{i_1} F_j\right) > 0$ and the set

$$A_2 := \left(A \cap (x_1, y_1) \setminus \bigcup_{j=1}^{i_1} F_j\right) \cap F_{i_2}$$

is of a positive measure for some $i_2 > i_1$. Fix $a_2 \in A_2 \cap \Phi_d(A_2)$. There is a natural number $n_2 > n_1$ such that the points $x_2 := a_2 - t_{n_2}$ and $y_2 := a_2 + t_{n_2}$ satisfy

$$[x_2, y_2] \subset (x_1, y_1), \ [x_2, y_2] \cap \bigcup_{j=1}^{i_1} F_j = \emptyset \text{ and } \frac{\lambda(A_2 \cap [x_2, y_2])}{y_2 - x_2} > \frac{1}{2}.$$

Proceeding by induction we define increasing sequences (i_k), (n_k) of natural numbers and a decreasing sequence of closed intervals $([x_k, y_k])$ such that

$$y_k - x_k = 2t_{n_k}, \ [x_k, y_k] \cap \bigcup_{j=1}^{i_{k-1}} F_j = \emptyset \text{ and } \frac{\lambda(A \cap [x_k, y_k])}{y_k - x_k} > \frac{1}{2}.$$

Let

$$\{x\} = \bigcap_{k=1}^{\infty} [x_k, y_k].$$

Then $x \notin \bigcup_{j=1}^{\infty} F_j = B$, but for every k we have

$$\frac{\lambda(A \cap [x_k, y_k])}{f(y_k - x_k)} = \frac{\lambda(A \cap [x_k, y_k])}{2t_{n_k}} \cdot \frac{2t_{n_k}}{f(2t_{n_k})} > \frac{1}{2},$$

which implies that x is not a $\mathcal{T}_f$-interior point of $\mathbb{R} \setminus A$, and consequently $x \in \mathrm{cl}_{\mathcal{T}_f}(A)$. This establishes (24.4) and completes the proof. □

24.4 Homeomorphisms of f-density topologies

Theorem 24.17 ([4]). *If $f_1, f_2 \in \mathcal{A}$ and $h : (\mathbb{R}, \mathcal{T}_{f_1}) \to (\mathbb{R}, \mathcal{T}_{f_2})$ is a homeomorphism, then*

(1) h and h^{-1} are continuous (in a usual sense), strictly monotonic and satisfy Lusin's condition (N),
(2) the sets

$$A := \{x : \text{there exists derivative } h'(x)\},$$
$$B := \left\{x : \text{there exists derivative } \left(h^{-1}\right)'(h(x))\right\}$$

have full measure,
(3) if $h'(x) = 1$ for every $x \in A \cap B$, then $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$.

Proof. Let I be an open interval. By Theorem 23.37,

$$\{E : E \text{ is } \mathcal{T}_{f_1}\text{-connected}\} = \{E : E \text{ is } \mathcal{T}_{f_2}\text{-connected}\} = \{E : E \text{ is connected}\},$$

so $h^{-1}(I)$ is an interval, too. Since no end of an interval can be its f_1-density point, the interval $h^{-1}(I)$ has to be open. Thus h is continuous. Since h is also an injection, it is strictly monotonic. Let P be a null set. Then P and all subsets of P are closed in $\mathcal{T}_{f_1}$. Consequently, $h(P)$ and all its subsets are closed in $\mathcal{T}_{f_2}$, and so they are measurable. Hence $h(P)$ is of measure zero, which finishes the proof of (1).

Any monotonic function is almost everywhere differentiable, so A has full measure. Observe that $B = h^{-1}\left(\left\{y;\text{ there exists } \left(h^{-1}\right)'(y)\right\}\right)$. Using Lusin's condition (N) for h^{-1}, we conclude that B has full measure too. Suppose that $h'(x) = 1$ for $x \in A \cap B$. By (1) and Banach-Zarecki theorem, we deduce that $h(x)$ is absolutely continuous on any interval $[a,b]$ (see [11]). Since $h'(x) = 1$ almost everywhere, $h(x) = x$, which gives $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$. □

Theorem 24.18 ([4]). *Let $f_1,\ f_2 \in \mathcal{A}^1$. If topological spaces $(\mathbb{R}, \mathcal{T}_{f_1})$ and $(\mathbb{R}, \mathcal{T}_{f_2})$ are homeomorphic, then topologies $\mathcal{T}_{f_1}$ and $\mathcal{T}_{f_2}$ are comparable i.e. $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ or $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$.*

Proof. Suppose, contrary to our claim, that $\mathcal{T}_{f_1}$ and $\mathcal{T}_{f_2}$ are not comparable. Let h be a homeomorphism from $(\mathbb{R}, \mathcal{T}_{f_1})$ onto $(\mathbb{R}, \mathcal{T}_{f_2})$. By Theorem 24.17, h is strictly monotonic and for some x_0 there exist derivatives $h'(x_0)$, $\left(h^{-1}\right)'(h(x_0))$ with $h'(x_0) = c \neq 1$. Since f-density topologies are invariant with respect to translations and symmetries, we can assume that h is increasing and $h(x_0) = x_0 = 0$. We can also assume that $0 < c < 1$ (we replace h by h^{-1}, if necessary).

Since $\mathcal{T}_{f_1} \setminus \mathcal{T}_{f_2} \neq \emptyset$, Theorem 23.29 shows that there is a measurable set A such that $0 \in \Phi^+_{f_1}(A) \setminus \Phi^+_{f_2}(A)$. Thus there exist a positive number η and a decreasing and tending to 0 sequence (h_n) such that $\frac{\lambda([0,h_n]\setminus A)}{f_2(h_n)} > \eta$ for every n. It is not difficult to define sequences (b_n) and (c_n) satisfying $b_{n+1} < c_n < b_n \leq h_n$ and

$$\frac{\lambda\left([c_n, b_n] \setminus A\right)}{f_2(b_n)} > \eta.$$

Let us define $a_n := b_n - \lambda\left([c_n, b_n] \setminus A\right)$ and

$$B := \bigcup_{n=1}^{\infty} (b_{n+1}, a_n) \cup \{0\} \cup \bigcup_{n=1}^{\infty} (-a_n, -b_{n+1}).$$

Of course, $c_n \le a_n \le b_n$. An easy computation shows that $\lambda([0,x]\setminus B) \le \lambda([0,x]\setminus A)$ for $x \le b_1$. Hence $0 \in \Phi^+_{f_1}(B)$ and $B \in \mathcal{T}_{f_1}$.
The proof will be completed by showing that $0 \notin \Phi_{f_2}(h(B))$. Since $f_2 \in \mathcal{A}^1$, there exists $\alpha > 0$ such that $\frac{f_2(b_n)}{b_n} > 4\alpha$ for almost all n. Let $\varepsilon := c\eta\alpha$. Since $0 < h'(0) = c < 1$, we have

$$\left|\frac{h(a_n)}{a_n} - c\right| < \varepsilon, \quad \left|\frac{h(b_n)}{b_n} - c\right| < \varepsilon \quad \text{and} \quad h(b_n) < b_n$$

for sufficiently large n. Hence

$$h(b_n) - h(a_n) > (c-\varepsilon)b_n - (c+\varepsilon)a_n > c(b_n - a_n) - 2\varepsilon b_n.$$

Therefore

$$\frac{\lambda([0,b_n]\setminus h(B))}{f_2(b_n)} \ge \frac{h(b_n)-h(a_n)}{f_2(b_n)} \ge c\frac{(b_n-a_n)}{f_2(b_n)} - 2\varepsilon\frac{b_n}{f_2(b_n)} >$$
$$> c\eta - \frac{2\varepsilon}{4\alpha} = \frac{c\eta}{2} > 0,$$

which gives $0 \notin \Phi_{f_2}(h(B))$. □

Theorem 24.19 ([4]). *The density topology $\mathcal{T}_d$ is not homeomorphic to any topology $\mathcal{T}_f \neq \mathcal{T}_d$.*

Proof. If $f \in \mathcal{A}^0$ then, by Theorem 24.16, $\mathcal{T}_f$ is not regular, so $\mathcal{T}_f$ and $\mathcal{T}_d$ are not homeomorphic. Assume that $f \in \mathcal{A}^1$ and $\mathcal{T}_f \neq \mathcal{T}_d$. Of course, $\mathcal{T}_f \setminus \mathcal{T}_d \neq \emptyset$. Suppose, contrary to our claim, that there is a homeomorphism $h : (\mathbb{R}, \mathcal{T}_f) \to (\mathbb{R}, \mathcal{T}_d)$. We can choose $x_0 \in \mathbb{R}$ such that $h'(x_0) = c \neq 1$. We can also assume that h is increasing and $h(x_0) = x_0 = 0$. If $c < 1$ then repeating the proof of Theorem 24.18 we obtain a contradiction. Suppose that $c > 1$ and set $g(x) := \frac{h(x)}{2c}$. Since $\mathcal{T}_d$ is invariant under multiplication by nonzero numbers, for any $U \in \mathcal{T}_f$ we have $g(U) = \frac{1}{2c}h(U) \in \mathcal{T}_d$, and for any $V \in \mathcal{T}_d$ we have $g^{-1}(V) = h^{-1}(2cV) \in \mathcal{T}_f$. Hence g is a homeomorphism. As $g'(x_0) = \frac{1}{2}$, we can repeat the proof of Theorem 24.18 for g. □

24.5 f-density topologies and (Δ_2) condition

In the theory of Orlicz spaces there is often use of the condition called (Δ_2). W. Orlicz says that a continuous, nondecreasing and unbounded function

$\varphi : [0,\infty) \to [0,\infty)$, with $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$, satisfies (Δ_2) condition if $\limsup_{x\to\infty} \frac{\varphi(2x)}{\varphi(x)} < \infty$ (see [12] or [13]). In the consideration of ψ-density, the analogous condition is used for functions belonging to the family $\widehat{C}$ (compare [8] and chapter 22). We will consider this condition for functions from the family $\mathcal{A}$.

We say that a function $f \in \mathcal{A}$ *fulfils* (Δ_2) *condition* $(f \in \Delta_2)$ if

$$\limsup_{x\to 0+} \frac{f(2x)}{f(x)} < \infty.$$

We will use (Δ_2) condition to compare f-density topologies and to study their algebraic properties. It is useful to observe:

Proposition 24.20 ([7]). *For any $f \in \mathcal{A}$ the following conditions are equivalent:*

(1) $f \in \Delta_2$,
(2) for any positive number β, $\limsup_{x\to 0+} \frac{f(\beta x)}{f(x)} < \infty$,
(3) there exists $\beta > 1$ such that $\limsup_{x\to 0+} \frac{f(\beta x)}{f(x)} < \infty$.

Proof. (1)⇒(2). There is $n \in \mathbb{N}$ such that $2^n \geqslant \beta$. From

$$\frac{f(\beta x)}{f(x)} \leqslant \frac{f(2^n x)}{f(x)} = \frac{f(2^n x)}{f(2^{n-1}x)} \cdot \frac{f\left(2^{n-1}x\right)}{f(2^{n-2}x)} \cdot \ldots \cdot \frac{f(2x)}{f(x)}$$

we obtain $\limsup_{x\to 0+} \frac{f(\beta x)}{f(x)} \leqslant \left(\limsup_{x\to 0+} \frac{f(2x)}{f(x)}\right)^n < \infty$. The implication (2)⇒(3) is obvious. The proof of (3)⇒(1) is analogous to the first one. □

Note that for any $\alpha \geqslant 1$, the function x^α fulfils (Δ_2) condition and $\mathcal{T}_{x^\alpha} \subset \mathcal{T}_d$. If $\alpha \in (0,1)$ then the function x^α does not belong to $\mathcal{A}$, because $\lim_{x\to 0+} \frac{x^\alpha}{x} = \infty$. To obtain a function $f \in \Delta_2$, generating topology $\mathcal{T}_f$ bigger than $\mathcal{T}_d$ (or incomparable with $\mathcal{T}_d$), we will "glue together" square functions with various coefficients and constant ones. Such a construction is presented in the following lemma. It is worth to observe that the construction works not only for square functions. We can use for example x^α with $\alpha > 1$.

Lemma 24.21 ([7]). *If $(a_n)_{n\geq 0}$ is a decreasing sequence tending to zero and $b_n := \sqrt{a_n a_{n-1}}$ for $n \in \mathbb{N}$, then the functions*

$$f(x) := \begin{cases} \frac{x^2}{a_n} & \text{for } x \in [a_n, b_n], \\ a_{n-1} & \text{for } x \in [b_n, a_{n-1}], \\ a_0 & \text{for } x \geqslant a_0, \end{cases} \qquad g(x) := \begin{cases} \frac{x^2}{a_{2n-1}} & \text{for } x \in [b_{2n}, b_{2n-1}], \\ a_{2n} & \text{for } x \in [b_{2n+1}, b_{2n}], \\ a_0 & \text{for } x \geqslant b_1 \end{cases}$$

are continuous and fulfil (Δ_2) *condition.*

Proof. Obviously, the functions f and g are continuous and belong to $\mathcal{A}$. Observe that

$$f(x) \leq \frac{x^2}{a_n} \text{ for } x \geq a_n.$$

Indeed, if $x \in [a_k, b_k]$ for some $k \leq n$, then $f(x) = \frac{x^2}{a_k} \leq \frac{x^2}{a_n}$, whereas for $x \in [b_k, a_{k-1}]$, $k \leq n$, we have $f(x) = f(b_k) = \frac{b_k^2}{a_k} \leq \frac{x^2}{a_n}$.
Let $x > 0$. If $x \in [a_n, b_n]$ for some n, then $\frac{f(2x)}{f(x)} \leq \frac{(2x)^2/a_n}{x^2/a_n} = 4$. If x and $2x$ belong to $[b_n, a_{n-1})$ then $\frac{f(2x)}{f(x)} = 1$. Finally, if $x \in [b_n, a_{n-1}]$ and $2x > a_{n-1}$ then $\frac{f(2x)}{f(x)} = \frac{f(2x)}{f(a_{n-1})} \leq \frac{f(2a_{n-1})}{f(a_{n-1})} \leq 4$. Thus f fulfils (Δ_2) condition.
Similarly we show that $g(x) \leq \frac{x^2}{a_{2n-1}}$ for $x \geq b_{2n}$, and hence $\frac{g(2x)}{g(x)} \leq 4$. □

Example 24.22 ([7]). Let $a_n := \frac{1}{(n+1)!}$, $n = 0, 1, \ldots$ and let f, g be the functions defined in Lemma 24.21. Then $f, g \in \Delta_2$. Since $\liminf_{x\to 0+} \frac{f(x)}{x} = 1$ and $\limsup_{x\to 0+} \frac{f(x)}{x} = \infty$, Theorem 23.32 implies $\mathcal{T}_d \subsetneqq \mathcal{T}_f$. Similarly, from $\liminf_{x\to 0+} \frac{g(x)}{x} = 0$ and $\limsup_{x\to 0+} \frac{g(x)}{x} = \infty$, we conclude that the topologies $\mathcal{T}_g$ and $\mathcal{T}_d$ are not comparable ($\mathcal{T}_g \nsubseteq \mathcal{T}_d$ and $\mathcal{T}_d \nsubseteq \mathcal{T}_g$).

Recall that the density topology $\mathcal{T}_d$ is invariant under multiplication by nonzero numbers and any f-density topology is invariant under multiplication by numbers $\alpha \geq 1$ and $\alpha \leq -1$ (compare Theorem 23.40). Theorem 23.12 states that $\langle s \rangle$-density topologies different from $\mathcal{T}_d$ are not invariant under multiplication by numbers $\alpha \in (-1, 1)$. Since any $\langle s \rangle$-density topology is equal to the topology $\mathcal{T}_f$ for some $f \in \mathcal{A}^1$, there are f-density topologies which are not invariant under multiplication by numbers $\alpha \in (-1, 1)$. There is a natural question if there exists an f-density topology $\mathcal{T}_f \supsetneqq \mathcal{T}_d$ which is invariant under multiplication by nonzero numbers. The following theorem gives a straightforward answer.

Theorem 24.23 ([7]). *If* $f \in \Delta_2$ *then* $\mathcal{T}_f$ *is invariant under multiplication by nonzero numbers.*

Proof. According to Theorem 23.29, it is enough to show that if $\alpha \in (0, 1)$ and $0 \in \Phi_f^+(A)$, then $0 \in \Phi_f^+(\alpha A)$. If $f \in \Delta_2$ then we have

$$\limsup_{x\to0+}\frac{\lambda([0,x]\setminus\alpha A)}{f(x)}=\limsup_{x\to0+}\frac{\alpha\lambda\left(\left[0,\frac{x}{\alpha}\right]\setminus A\right)}{f\left(\frac{x}{\alpha}\right)}\cdot\frac{f\left(\frac{x}{\alpha}\right)}{f(x)}\le$$
$$\le\alpha\cdot\limsup_{x\to0+}\frac{\lambda\left(\left[0,\frac{x}{\alpha}\right]\setminus A\right)}{f\left(\frac{x}{\alpha}\right)}\cdot\limsup_{x\to0+}\frac{f\left(\frac{x}{\alpha}\right)}{f(x)}=0.$$

□

Now we are in the position to give a simple example of a function $f\in\mathcal{A}^1$ such that $\mathcal{T}_f\neq\mathcal{T}_{\langle s\rangle}$ for $\langle s\rangle\in\widetilde{\mathcal{S}}$ (much nicer than Example 23.41).

Example 24.24. Let f be the function from Example 24.22. Then $f\in\Delta_2$ and, by Theorem 24.23, topology $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers. Thus Theorem 23.12 implies $\mathcal{T}_f\neq\mathcal{T}_{\langle s\rangle}$ for $\langle s\rangle\in\widetilde{\mathcal{S}}$.

In the paper [15] it was shown that, for ψ-density topologies, the invariantness of $\mathcal{T}_\psi$ under multiplication by nonzero numbers implies $\limsup_{x\to0+}\frac{\psi(2x)}{\psi(x)}<\infty$. This result can be generalized to f-density topologies contained in $\mathcal{T}_d$, i.e. the invariantness of $\mathcal{T}_f$ under multiplication by nonzero numbers implies $f\in\Delta_2$. Moreover, if $\mathcal{T}_f\nsubseteq\mathcal{T}_d$ then the considered implication is untrue.

Theorem 24.25 ([7]). *Let $f\in\mathcal{A}$. The following conditions are equivalent.*

(1) $\mathcal{T}_f\subset\mathcal{T}_d$.
(2) The topology $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers if and only if $f\in\Delta_2$.

Proof. (1)⇒(2). By Theorem 24.23, if $f\in\Delta_2$ then $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers. Suppose that $f\notin\Delta_2$. There exists a decreasing sequence (b_n) such that $b_{n+1}<\frac{1}{2}\min\{b_n,f(b_n)\}$ and $\frac{f(2b_n)}{f(b_n)}>n$ for $n\in\mathbb{N}$. Let $a_n:=\max\left\{\frac{b_n}{2},b_n-\frac{f(b_n)}{2}\right\}$ and $A:=\mathbb{R}\setminus\bigcup_{n=1}^{\infty}[a_n,b_n]$. We will show that $A\notin\mathcal{T}_f$ but $4\cdot A\in\mathcal{T}_f$. Since $\mathcal{T}_f\subset\mathcal{T}_d$, Theorem 23.32 implies $\frac{f(x)}{x}\le M$ for some positive M and any x from some interval $(0,h)$. We can assume that this inequality holds for all $x>0$. Thus

$$\frac{\lambda([0,b_n]\setminus A)}{f(b_n)}>\frac{b_n-a_n}{f(b_n)}\geqslant\min\left\{\frac{b_n}{2f(b_n)},\frac{1}{2}\right\}\ge\min\left\{\frac{1}{2M},\frac{1}{2}\right\}>0.$$

Consequently, $0\notin\Phi_f^+(A)$ and $A\notin\mathcal{T}_f$.
Observe that $4a_n\geqslant2b_n$. For $x\in[4a_n,4b_n]$ we have

$$\frac{\lambda([0,x]\setminus4A)}{f(x)}\le\frac{4(b_{n+1}+(b_n-a_n))}{f(4a_n)}<\frac{4\left(\frac{f(b_n)}{2}+\frac{f(b_n)}{2}\right)}{f(2b_n)}<\frac{4}{n}.$$

Using the preceding inequality, for $x \in [4b_{n+1}, 4a_n]$ we obtain

$$\frac{\lambda\left([0,x]\setminus 4A\right)}{f(x)} = \frac{\lambda\left([0,4b_{n+1}]\setminus 4A\right)}{f(x)} \leqslant \frac{\lambda\left([0,4b_{n+1}]\setminus 4A\right)}{f(4b_{n+1})} < \frac{4}{n+1}.$$

Hence $0 \in \Phi_f^+(4A)$, and consequently $4A \in \mathcal{T}_f$.

(2)⇒(1). It is sufficient to show that for any function $f \in \Delta_2$ such that $\mathcal{T}_f \not\subseteq \mathcal{T}_d$ there is a function $g \notin \Delta_2$ for which $\mathcal{T}_f = \mathcal{T}_g$. Since $\mathcal{T}_f \not\subseteq \mathcal{T}_d$, Theorem 23.32 implies $\limsup_{x\to 0+} \frac{f(x)}{x} = \infty$. From $f \in \Delta_2$ it follows that $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$ and $\limsup_{x\to 0+} \frac{f(2x)}{f(x)} < \infty$. Thus there are a positive number M and sequences (a_n), (b_n) such that

$$b_{n+1} < a_n < b_n, \quad \frac{f(a_n)}{a_n} < M, \quad \frac{f(b_n)}{b_n} > n^2 \text{ and } \frac{f(2x)}{x} < M$$

for $n \in \mathbb{N}$ and $x \in (0, b_1]$. Write $c_n := \sup\{x \in [a_n, b_n] : f(x) \le nb_n\}$ for $n > M^2$. Then

$$nb_n \ge f\left(\frac{c_n}{2}\right) \ge \frac{f(2c_n)}{M^2} > \frac{nb_n}{M^2} \text{ and } 2a_n \le c_n,$$

because $f(2a_n) \le Mf(a_n) < M^2 a_n < na_n < nb_n$. Let us define

$$g(x) := \begin{cases} f(b_n) \text{ for } x \in \left(\frac{c_n}{2}, b_n\right], n > M^2, \\ f(x) \quad \text{for } x \in (0,\infty) \setminus \bigcup_{n>M^2} \left(\frac{c_n}{2}, b_n\right]. \end{cases}$$

From $\frac{g(c_n)}{g\left(\frac{c_n}{2}\right)} = \frac{f(b_n)}{f\left(\frac{c_n}{2}\right)} > \frac{n^2 b_n}{nb_n} = n$ we obtain $g \notin \Delta_2$. Since $f \le g$, $\mathcal{T}_f \subset \mathcal{T}_g$. To prove the inverse inclusion, we use Theorem 24.2. Let

$$A := A_{1fg} = \{x > 0 : f(x) < g(x)\} \text{ and } \varepsilon := \varepsilon_{1fg} = \limsup_{x\to 0+} \frac{\lambda(A \cap [0,x])}{f(x)}.$$

Since $A \subset \bigcup_{n>M^2} \left(\frac{c_n}{2}, b_n\right]$, for any $x \in \left(\frac{c_n}{2}, \frac{c_{n-1}}{2}\right]$ we have

$$\frac{\lambda(A \cap [0,x])}{f(x)} < \frac{b_n}{f\left(\frac{c_n}{2}\right)} < \frac{M^2}{n},$$

and consequently $\varepsilon = 0$. Thus $\mathcal{T}_g \subset \mathcal{T}_f$. □

In Theorem 23.30 it is shown that $\limsup_{x\to 0+} \frac{f(x)}{g(x)} < \infty$ implies $\mathcal{T}_f \subset \mathcal{T}_g$. Example 23.31 asserts that the functions $f(x) = \frac{1}{n!}$, $x \in \left[\frac{1}{(n+1)!}, \frac{1}{n!}\right)$ and $g(x) = \frac{1}{n!}$, $x \in \left(\frac{1}{(n+1)!}, \frac{1}{n!}\right]$ generates the same topology, although $\limsup_{x\to 0+} \frac{f(x)}{g(x)} =$

∞. It turns out that, if we additionally assume that $f \in \Delta_2$ and $\mathcal{T}_f \subset \mathcal{T}_d$, then the condition $\mathcal{T}_f \subset \mathcal{T}_g$ is equivalent to $\limsup_{x\to 0+} \frac{f(x)}{g(x)} < \infty$.

Theorem 24.26 ([7], Th. 7). *Suppose that $f \in \Delta_2$, $g \in \mathcal{A}$ and $\mathcal{T}_f \subset \mathcal{T}_d$. If $\mathcal{T}_f \subset \mathcal{T}_g$ then* $\limsup_{x\to 0+} \frac{f(x)}{g(x)} < \infty$.

Note that the assumption $\mathcal{T}_f \subset \mathcal{T}_d$ cannot be omitted. In [7, Ex. 3] there are constructed the functions $f, g \in \Delta_2$ such that $\limsup_{x\to 0+} \frac{f(x)}{g(x)} = \infty$ and $\mathcal{T}_f = \mathcal{T}_g$.

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