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# A NOTE ON THE ŁOJASIEWICZ EXPONENT OF NON-DEGENERATE ISOLATED HYPERSURFACE SINGULARITIES

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ABSTRACT. We prove that in order to find the value of the Łojasiewicz exponent $\text{ł}(f)$ of a Kouchnirenko non-degenerate holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singular point at the origin, it is enough to find this value for any other (possibly simpler) function $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, provided this function is also Kouchnirenko non-degenerate and has the same Newton diagram as $f$ does. We also state a more general problem, and then reduce it to a Teissier-like result on (c)-cosecant deformations, for formal power series with coefficients in an algebraically closed field $\mathbb{K}$.

CONTENTS



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## 1. Introduction and statement of the result

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function, defined in a neighborhood of 0, with power series expansion $f = \Sigma_{i\in\mathbb{N}_0^n} a_i z^i$, where $a_0 = 0$ and $z^i := (z_1^{i_1} \cdot \ldots \cdot z_n^{i_n})$. The *support of* $f$ is defined as $\operatorname{Supp} f := \{k \in \mathbb{N}_0^n : a_k \neq 0\}$ and its *Newton polyhedron* is $\Gamma_+(f) := \operatorname{conv}(\operatorname{Supp} f + \mathbb{N}_0^n) \subset \mathbb{R}^n_{\geqslant 0}$. The union of the compact faces of $\Gamma_+(f)$ is called the *Newton diagram of* $f$ and denoted by $\Gamma(f)$. If $\Gamma(f)$ touches all the coordinate axes, we say that $f$ is *convenient*. For a face $\Delta$ of $\Gamma(f)$, we put $f_\Delta := \Sigma_{i\in\mathbb{N}_0^n\cap\Delta} a_i z^i$. We say that $f$ is *(Kouchnirenko) non-degenerate on* $\Delta$ if the system $\{\nabla f_\Delta = 0\}$ has got no solutions in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\nabla$ denotes the gradient vector. If $f$ is non-degenerate on all the faces $\Delta$ of $\Gamma(f)$, then we simply say that $f$ is *(Kouchnirenko) non-degenerate*.

A basic result of A. G. Kouchnirenko ([8]) says: *if $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are two non-degenerate functions with isolated singularities at* 0 *and such that* $\Gamma(f) = \Gamma(g)$, *then their Milnor numbers are equal:* $\mu(f) = \mu(g)$. Moreover, there exists a combinatorial formula (expressed only in terms of the diagram $\Gamma(f)$) for $\mu(f)$.

A fast definition of the *Łojasiewicz exponent* $\text{ł}(f)$ of a function $f$ as above is

$$\text{ł}(f) = \sup_{\varphi} \frac{\operatorname{ord}(\nabla f \circ \varphi)}{\operatorname{ord} \varphi}, \tag{1}$$

where $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, $\varphi \neq 0$, are holomorphic paths through the origin (see [9] or [15]) and, as usually, ord of a mapping is the minimum of ord's of its coordinates.

The main observation of this note is the following Kouchnirenko-like result for the Łojasiewicz exponent:

**Theorem 1.** *For any two Kouchnirenko non-degenerate functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having isolated singularities at* 0 *and the same Newton diagrams their Łojasiewicz exponents are equal:* $\text{ł}(f) = \text{ł}(g)$.

In principle, the proof of Theorem 1 is straightforward and consists of four steps (for $f$ and $g$ as in the above statement):

(1) Reduction to the case of convenient singularities.
(2) Application, to a class of simple linear deformations $f_s$ of the given germ $f$, of topological triviality theorems proved in [4] (or [18]) which imply also so-called Teissier condition (c) for $f_s$.
(3) Application of the results of [17], which can be restated as follows: if a deformation $(f_s)$ of $f$ satisfies condition (c), then $\text{ł}(f_s) = \text{ł}(f)$.
(4) Using the Zariski-openness of the set of Kouchnirenko non-degenerate singularities with a given Newton diagram to join $f$ and $g$ by a family of "linear deformations" of the types considered above.

Although we will mostly stick to the above plan, we want to give a somewhat more direct (and, partly, more general) proof of the main result. Namely, we will avoid falling back on the results of J. Damon and T. Gaffney [4] or E.

Yoshinaga [18] and we will directly prove that the condition (c) holds for the aforementioned class of deformations of a Kouchnirenko non-degenerate function $f$ (even if it has a non-isolated singularity). Moreover, this verification will be valid over any algebraically closed field $\mathbb{K}$ (in the formal category).

## 2. A wider perspective

Although in the previous section we only considered the complex analytic setting, we want to stress that all the definitions and most of the statements given there can be transferred almost verbatim, after obvious changes, to the context of formal power series living in the ring $\mathbb{K}[[z]]$ over an algebraically closed field $\mathbb{K}$ (here $z = (z_1, \ldots, z_n)$ are variables). In particular, an improved version of the Kouchnirenko theorem valid in this context was recently given by P. Mondal (see [11]). Let us remark that for a power series $h \in \mathbb{K}[[z]]$, where $z = (z_1, \ldots, z_n)$ are variables over $\mathbb{K}$, it may sometimes be necessary to introduce $\mathbb{K}$ into the notation, writing e.g. $\Gamma_{\mathbb{K}}(h)$, for otherwise the notation could be misleading if $\mathbb{K}$ happen to contain some symbols that could be treated as variables. Below, we provide the non-so-obvious information in this wider context.

2.1. **The definition of the Łojasiewicz exponent.** The main object of our study requires a modified approach in order to make it more productive. First, let us recall

**Definition 2.** *Let $R$ be a ring (commutative with unity) and $\mathcal{I}$ be an ideal in $R$. We say that $r \in R$ is* integral over $\mathcal{I}$ *if $r$ satisfies an equation of the form*

$$r^n + a_1 r^{n-1} + \ldots + a_n = 0,$$

*where $a_j \in \mathcal{I}^j$ $(j = 1, \ldots, n)$ and $n \in \mathbb{N}$. The set of all the elements of $R$ that are integral over $\mathcal{I}$ is called the* integral closure of $\mathcal{I}$ *and is denoted by $\overline{\mathcal{I}}$.*

We choose [7] as our main source for references for topics concerning integral closure. For now, recall that $\overline{\mathcal{I}}$ is also an ideal in $R$ and that $\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}$. Let us state

**Definition 3.** *Let $\mathbb{K}$ be an algebraically closed field and let $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ be the ring of formal power series with coefficients in $\mathbb{K}$.*

*(1) Let $\mathcal{I}$, $\mathcal{J}$ be two ideals in $\mathbb{K}[[z]]$. We define the* Łojasiewicz exponent $Ł_{\mathbb{K}[[z]],\mathcal{J}}(\mathcal{I})$ of $\mathcal{I}$ relative to $\mathcal{J}$ *as*

$$Ł_{\mathbb{K}[[z]],\mathcal{J}}(\mathcal{I}) := \inf\left\{\tfrac{\alpha}{\beta} : \alpha, \beta \in \mathbb{N} \wedge \mathcal{J}^\alpha \subset \overline{\mathcal{I}^\beta}\right\}.$$

*Usually, we will write just $Ł_{\mathcal{J}}(\mathcal{I})$ in place of $Ł_{\mathbb{K}[[z]],\mathcal{J}}(\mathcal{I})$.*

*(2) Let $h \in \mathbb{K}[[z]]$ be a formal power series. We define the* Łojasiewicz exponent $ł(h)$ of $h$ *to be*

$$ł(h) = ł_{\mathbb{K}[[z]]}(h) := Ł_{\mathbb{K}[[z]],\mathfrak{m}}(\nabla h) = \inf\left\{\tfrac{\alpha}{\beta} : \alpha, \beta \in \mathbb{N} \wedge \mathfrak{m}^\alpha \subset \overline{(\nabla h)^\beta \mathbb{K}[[z]]}\right\},$$

*where $\mathfrak{m} = (z)\mathbb{K}[[z]]$ is the maximal ideal.*

Clearly, in a similar fashion, we may define: $Ł_{\mathbb{C}\{z\},\mathcal{J}}(\mathcal{I})$ for ideals in the convergent power series ring and $ł_{\mathbb{C}\{z\}}(g)$ for holomorphic functions. It was proved in [9] that such definition is in agreement with the one given in the Introduction (cf. formula (1)). This is also an easy consequence of Corollary 7 below, so we prove it in Corollary 5. Still more generally, using Theorem 6 we can show (see [3] for a proof in dimension 2):

**Proposition 4.** *Let $\mathbb{K}$ be an algebraically closed field and $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ be the ring of formal power series with coefficients in $\mathbb{K}$. Let $\mathcal{I}$, $\mathcal{J}$ be two ideals in $\mathbb{K}[[z]]$ with $\mathcal{J}$ being proper. Then*

$$Ł_{\mathcal{J}}(\mathcal{I}) = \sup_{\substack{\varphi \in \mathbb{K}[[t]]^n \\ \varphi(0)=0}} \frac{\operatorname{ord}(\varphi^*\mathcal{I})}{\operatorname{ord}(\varphi^*\mathcal{J})}.$$

In the above statement, as is customary, $\varphi^*\mathcal{K} := (h \circ \varphi : h \in \mathcal{K})\mathbb{K}[[t]]$ and, obviously, $\operatorname{ord}(\varphi^*\mathcal{K}) = \min\{\operatorname{ord}(h \circ \varphi) : h \in \mathcal{K}\}$.

*Proof.* "$\geqslant$" If $\mathcal{J}^\alpha \subset \overline{\mathcal{I}^\beta}$ for some $\alpha, \beta \in \mathbb{N}$, then for any $\varphi \in \mathbb{K}[[t]]^n$ with $\varphi(0) = 0$ we get, by Theorem 6 and properties of order, $\alpha \cdot \operatorname{ord}(\varphi^*\mathcal{J}) \geqslant \beta \cdot \operatorname{ord}(\varphi^*\mathcal{I})$. The ideal $\mathcal{J}$ being proper, $\operatorname{ord}(\varphi^*\mathcal{J}) > 0$ so we infer that $\frac{\alpha}{\beta} \geqslant \frac{\operatorname{ord}(\varphi^*\mathcal{I})}{\operatorname{ord}(\varphi^*\mathcal{J})}$. Passing to the limits with both sides of the last relation, we get the required inequality.

"$\leqslant$" If $Ł_{\mathcal{J}}(\mathcal{I}) > \frac{\alpha}{\beta}$ for some $\alpha, \beta \in \mathbb{N}$, so that $\mathcal{J}^\alpha \not\subset \overline{\mathcal{I}^\beta}$, then Theorem 6 asserts that there exists some $\psi \in \mathbb{K}[[t]]^n$ with $\psi(0) = 0$ such that $\alpha \cdot \operatorname{ord}(\psi^*\mathcal{J}) < \beta \cdot \operatorname{ord}(\psi^*\mathcal{I})$. Hence, $\frac{\alpha}{\beta} < \frac{\operatorname{ord}(\psi^*\mathcal{I})}{\operatorname{ord}(\psi^*\mathcal{J})}$. Similarly as above, this implies the required inequality. □

In the situation of primary interest to us, we can state (see [9])

**Corollary 5.** *Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function and let $\mathbb{C}\{z\}$, with $z = (z_1, \dots, z_n)$, denote the ring of convergent power series. Then*

$$ł_{\mathbb{C}[[z]]}(g) = ł_{\mathbb{C}\{z\}}(g) = \sup_{\varphi:(\mathbb{C},0)\to(\mathbb{C}^n,0)} \frac{\operatorname{ord}(\nabla g \circ \varphi)}{\operatorname{ord} \varphi}.$$

*Proof.* The first equality is a consequence of [7, Proposition 1.6.2] (see the proof of Corollary 7 for details). To justify the second one it is enough to repeat the reasoning from the proof of Proposition 4 with Corollary 7 applied in place of Theorem 6. □

**Remark.** Naturally, in the same way one can show that it holds $Ł_{\mathbb{C}[[z]],\mathcal{J}}(\mathcal{I}) = Ł_{\mathbb{C}\{z\},\mathcal{J}}(\mathcal{I}) = \sup_\varphi \frac{\operatorname{ord}(\varphi^*\mathcal{I})}{\operatorname{ord}(\varphi^*\mathcal{J})}$, where $\mathcal{I}$, $\mathcal{J}$ are ideals in $\mathbb{C}\{z\}$ and $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$. Let us also note that in Proposition 4 and Corollary 5 we may restrict $\varphi$'s to have non-zero components, or even to be polynomials (cf. Theorem 6 and Corollary 7).

2.2. **Testing integral dependence.** Of crucial importance is the following parametric version of the well-known Valuative Criterion of Integral Dependence (see [9] or Corollary 7 for the complex analytic setting; an alternative proof of the theorem stated below, valid in dimension 2 and based on so-called Hamburger-Noether process, can be found in [3, Theorem 21]):

**Theorem 6.** *Let $\mathbb{K}$ be a field and $\mathcal{I}$ be an ideal in the ring $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ of formal power series with coefficients in $\mathbb{K}$. The following conditions are equivalent for an element $g \in \mathbb{K}[[z]]$:*

*(1) $g \in \overline{\mathcal{I}}$,*
*(2) for any formal parametrization $\varphi \in \overline{\mathbb{K}}[[t]]^n$ with $\varphi(0) = 0$, where $\overline{\mathbb{K}}$ denotes the algebraic closure of the field $\mathbb{K}$, it holds*

$$\operatorname{ord}(g \circ \varphi) \geqslant \operatorname{ord}(\varphi^* \mathcal{I}).$$

*Moreover, in item 2 we may restrict ourselves to $\varphi \in \overline{\mathbb{K}}[t]^n$ with non-zero components $\varphi_i$.*

*Proof.* First note that we may assume that $\mathbb{K} = \overline{\mathbb{K}}$, because by [7, Proposition 1.6.1] we have $\overline{\mathcal{I}\,\overline{\mathbb{K}}[[z]]} \cap \mathbb{K}[[z]] = \overline{\mathcal{I}}$. Then, according to [*ibid.*, Proposition 6.8.4], $g \in \overline{\mathcal{I}}$ if, and only if, $g \in \mathcal{I}\,\mathcal{V}$ for all rank one discrete valuation domains $(\mathcal{V}, \mathfrak{m}_\mathcal{V})$ between $\mathbb{K}[[z]]$ and its field of fractions $\mathbb{K}[[z]]_0$ such that $\mathfrak{m}_\mathcal{V} \cap \mathbb{K}[[z]] = \mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{K}[[z]]$. This is the same as saying that $\mathcal{V}$ are regular local rings of Krull dimension 1 (see [*ibid.*, Proposition 6.3.4]). Let $\widehat{\mathcal{V}}$ denote the formal completion of $(\mathcal{V}, \mathfrak{m}_\mathcal{V})$ with respect to the $\mathfrak{m}_\mathcal{V}$-adic topology. Then $\hat{\mathcal{V}}$ is also regular local of the same dimension, hence a valuation domain. Since [*ibid.*, Proposition 6.8.1] asserts that every ideal in a valuation domain is integrally closed, using [*ibid.*, Proposition 1.6.2] we get

$$\mathcal{I}\,\mathcal{V} = \overline{\mathcal{I}\,\mathcal{V}} = \overline{\mathcal{I}\,\hat{\mathcal{V}}} \cap \mathcal{V} = \mathcal{I}\,\hat{\mathcal{V}} \cap \mathcal{V}.$$

From this it follows that checking whether $g \in \overline{\mathcal{I}}$ is equivalent to testing if $g \in \mathcal{I}\,\hat{\mathcal{V}}$ for all rank one complete and discrete valuation domains $\hat{\mathcal{V}}$ over $\mathbb{K}[[z]]$ such that $\mathfrak{m}_{\hat{\mathcal{V}}} \cap \mathbb{K}[[z]] = \mathfrak{m}$. Since $\mathbb{K} \subset \hat{\mathcal{V}}$, by the equicharacteristic case of Cohen Structure Theorem (see e.g. [19, Corollary in Chapter VIII, § 12], we get $\hat{\mathcal{V}} \cong \mathbb{L}[[t]]$ for some field $\mathbb{L} \subset \hat{\mathcal{V}}$. We have $\mathfrak{m}_\mathcal{V} \cap \mathbb{K}[[z]] = \mathfrak{m}$, so $\mathbb{L} = \hat{\mathcal{V}}/\mathfrak{m}_{\hat{\mathcal{V}}} \cong \mathcal{V}/\mathfrak{m}_\mathcal{V} \supset \mathbb{K}[[z]]/\mathfrak{m} = \mathbb{K}$ (see e.g. [10, page 63] for the isomorphism). Thus, we may consider $\mathbb{K}$ as a subfield of $\mathbb{L}$. Let $\psi = (\psi_1, \dots, \psi_n) \in \mathbb{L}[[t]]^n$ be defined by $\psi_i := \iota(z_i)$ $(i = 1, \dots, n)$, where $\iota : \mathbb{K}[[z]] \to \mathbb{L}[[t]]$ is the inclusion. Using $\iota(\mathfrak{m}) \subset (t)\mathbb{L}[[t]]$ we get $\operatorname{ord} \psi > 0$ and then the condition $g \in \mathcal{I}\,\hat{\mathcal{V}}$ may be rewritten as $\iota(g) = g \circ \psi \in \iota(\mathcal{I})\mathbb{L}[[t]] = (\psi^*\mathcal{I})\mathbb{L}[[t]]$. Equivalently, $\operatorname{ord} g \circ \psi \geqslant \operatorname{ord}(\psi^* \mathcal{I})$.

"1⇒2" Since $g$ satisfies an equation of integral dependence, it is easy to directly check that $\operatorname{ord}(g \circ \varphi) \geqslant \operatorname{ord}(\varphi^* \mathcal{I})$, for any $\varphi \in \mathbb{K}[[t]]^n$ with $\varphi(0) = 0$.

"∼1⇒∼2" If $g \notin \overline{\mathcal{I}}$ then, by the above characterization, there exists some $\zeta \in \mathbb{L}[[t]]^n$, for some field $\mathbb{L} \supset \mathbb{K}$, such that $\operatorname{ord} g \circ \zeta < \operatorname{ord}(\zeta^* \mathcal{I})$. Moreover, we may assume e.g. that $g \circ \zeta = t^a + \text{h.o.t.}$. Interpreting the last inequality as a system of algebraic equations over $\mathbb{K}$ with the coefficients of $\zeta$ viewed as unknowns, we infer, by Hilbert's Nullstellensatz, that we can find its solution also over the field $\mathbb{K}$. This delivers $\varphi \in \mathbb{K}[[t]]^n$ (even $\varphi \in \mathbb{K}[t]^n$) such that $\operatorname{ord} g \circ \varphi < \operatorname{ord}(\varphi^* \mathcal{I})$. Changing the components of $\varphi$ by adding to them high enough powers of $t$ we may arrange things so that these components are all non-zero. □

**Corollary 7** ([9]). *Let $\mathcal{I}$ be an ideal in the convergent power series ring $\mathbb{C}\{z\}$ with $z = (z_1, \dots, z_n)$. The following conditions are equivalent for an element $g \in \mathbb{C}\{z\}$:*

*(1) $g \in \overline{\mathcal{I}}$,*
*(2) for any holomorphic curve $\varphi \in \mathbb{C}\{t\}^n$ with $\varphi(0) = 0$ it holds*

$$\operatorname{ord}(g \circ \varphi) \geqslant \operatorname{ord}(\varphi^* \mathcal{I}).$$

*Moreover, in item 2 we may restrict ourselves to $\varphi \in \mathbb{C}[t]^n$ with non-zero components $\varphi_i$.*

*Proof.* We have $g \in \overline{\mathcal{I}} \Leftrightarrow g \in \overline{\mathcal{I}\,\mathbb{C}[[z]]}$, because [7, Proposition 1.6.2] asserts that for any local noetherian ring $(R, \mathfrak{m})$ and an ideal $\mathcal{J} \lhd R$ it holds $\overline{\mathcal{J}\hat{R}} \cap R = \overline{\mathcal{J}}$, where "$\hat{\ }$" denotes the completion of $R$ with respect to the $\mathfrak{m}$-adic topology. The test in item 2 of Theorem 6 can be performed, in particular, for convergent series $\varphi$ and then $\operatorname{ord}(\varphi^* \mathcal{I}) = \operatorname{ord}(\varphi^*(\mathcal{I}\,\mathbb{C}[[z]]))$; on the other hand, as the theorem asserts, it is enough to use polynomials from $\mathbb{C}[t]$ for the test (with non-zero components). □

## 3. The integral closure of the toric gradient ideal of non-degenerate singularities

Here, we prove one of the results given in [18] (see also [16] or [14]) in the more general, formal setting. First, we introduce

**Notation.** *Let $\mathbb{K}$ be a field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a formal power series. Let $l \in \mathbb{N}^n$ be a vector with positive coordinates.*

*(1) If $\varphi \in \mathbb{K}[[t]]^n$, with $\varphi(0) = 0$, is a formal parametrization such that $\operatorname{ord} \varphi_i = l_i$ $(i = 1, \dots, n)$, then we will say that $l$ is the* initial vector of $\varphi$ *and we will write* $\boxed{\operatorname{vord} \varphi = l}$.

*(2) The symbol* $\boxed{\operatorname{ord}_l h}$ *will denote* $\operatorname{ord}(h \circ \psi)$, *where* $\operatorname{vord} \psi = l$ *and $\psi$ is a parametrization with generic (initial) coefficients. In other words, $\operatorname{ord}_l h$ is the minimum value of all scalar products of $l$ and the vectors from $\Gamma(h) = \Gamma_{\mathbb{K}}(h)$. The face* $\boxed{\Delta = \Delta(l)}$ *of $\Gamma(h)$ for which this minimum is attained is said to be* supported by $l$, *and $l$ itself is called a* supporting vector of $\Delta$.

*(3)* $\boxed{\nabla_{\text{tor}} h(z) := \left(z_1 \cdot \frac{\partial h(z)}{\partial z_1}, \dots, z_n \cdot \frac{\partial h(z)}{\partial z_n}\right)}$ *is the* toric gradient of $h$. *More generally, if $w = (w_1, \dots, w_k)$, where $1 \leqslant k \leqslant n$, is a subsequence of the sequence of variables $z = (z_1, \dots, z_n)$, then we put*

$$\boxed{\nabla^w_{\text{tor}} h(z) := \left(w_1 \cdot \frac{\partial h(z)}{\partial w_1}, \dots, w_k \cdot \frac{\partial h(z)}{\partial w_k}\right).}$$

We note the following straightforward

**Lemma 8.** *If $h \in \mathbb{K}[[z]]$ is a non-invertible Kouchnirenko non-degenerate formal power series, then for any formal parametrization $\varphi \in \mathbb{K}[[t]]^n$, with $\varphi(0) = 0$, and such that $l := \operatorname{vord}\varphi$ satisfies $l \in \mathbb{N}^n$, we have*

$$\operatorname{ord}((\nabla_{\mathrm{tor}} h) \circ \varphi) = \operatorname{ord}((\nabla_{\mathrm{tor}} h_{\Delta(l)}) \circ \varphi) = \operatorname{ord}_l(h). \quad \square$$

Clearly, the above-introduced notations, as well as the lemma, are valid in the complex analytic setting.

We have

**Proposition 9.** *Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible Kouchnirenko non-degenerate formal power series. Then*

$$\overline{\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]} = \overline{\{z^\alpha : \alpha \in \operatorname{vert}(\Gamma(h))\}\mathbb{K}[[z]]} = \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[[z]].$$

Here and below, "vert" denotes the set of all vertices of a given polyhedron.

*Proof.* The second equality follows from standard properties of integral closure of monomial ideals and does not require Kouchnirenko non-degeneracy. Namely, by [7, Proposition 1.4.6] we get $\overline{\{z^\alpha : \alpha \in \operatorname{vert}(\Gamma(h))\}\mathbb{K}[z]} = \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[z]$ in the polynomial ring $\mathbb{K}[z]$. But it is immediate to see that both these ideals are unchanged under passage to the ring of formal power series $\mathbb{K}[[z]]$.

We will prove the first equality. Since, obviously, we have $\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]] \subset \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[[z]]$, we only need to check that any given $z^\alpha$, where $\alpha \in \operatorname{vert}(\Gamma(h))$, is integral over $\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]$. Take $\varphi \in \mathbb{K}[[t]]^n$ such that $\varphi(0) = 0$ and $\varphi$ has non-zero components, so that $l := \operatorname{vord}\varphi$ satisfies $l \in \mathbb{N}^n$. From Lemma 8 we infer that

$$\operatorname{ord}((\nabla_{\mathrm{tor}} h) \circ \varphi) = \operatorname{ord}_l(h) \leqslant \operatorname{ord}_l(z^\alpha) = \operatorname{ord}(\varphi^*(z^\alpha)).$$

Exploiting the parametric valuative criterion (Theorem 6), we conclude that $z^\alpha \in \overline{\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]}$. $\square$

**Corollary 10.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic, Kouchnirenko non-degenerate function. Then*

$$\overline{\nabla_{\mathrm{tor}}(f)\mathbb{C}\{z\}} = \overline{\{z^\alpha : \alpha \in \operatorname{vert}(\Gamma(f))\}\mathbb{C}\{z\}} = \{z^\alpha : \alpha \in \Gamma_+(f) \cap \mathbb{N}_0^n\}\mathbb{C}\{z\}.$$

*Proof.* The first equality follows from Proposition 9 and [7, Proposition 1.6.2] (cf. the proof of Corollary 7). The second equality holds because both involved sets are unchanged when $\mathbb{C}\{z\}$ gets replaced by $\mathbb{C}[[z]]$. $\square$

**Comment.** Both Proposition 9 and Corollary 10 can be improved by stating that Kouchnirenko non-degeneracy is actually equivalent to the equalities of the various integral closures (for a proof of this result in the analytic case see [18, Theorem 1.7] or [16, Theorem 3.4]; a generalization to mappings can be found in

[14, Corollary 4.5]). The proof is easy: if $(\varrho_1, \ldots, \varrho_n) \in (\mathbb{K}^*)^n$ is a solution to some system $\{\nabla h_\Delta = 0\}$, then choosing: a vector $l \in \mathbb{N}^n$ such that $\Delta = \Delta(l)$, $\alpha \in \mathrm{vert}(\Gamma(h)) \cap \Delta$ and $\varphi(t) := ((\varrho_1 + \pi \cdot t) \cdot t^{N \cdot l_1}, \ldots, (\varrho_n + \pi \cdot t) \cdot t^{N \cdot l_n})$ with generic $\pi \in \mathbb{K}$ and $N \gg 1$, we get

$$\mathrm{ord}((\nabla_{\mathrm{tor}} h) \circ \varphi) = \mathrm{ord}((\nabla_{\mathrm{tor}} h_\Delta) \circ \varphi) > \mathrm{ord}_{N \cdot l}\, h_\Delta = \mathrm{ord}_{N \cdot l}(z^\alpha) = \mathrm{ord}(\varphi^*(z^\alpha)),$$

so that, according to Theorem 6, the monomial $z^\alpha$ is not integral over the ideal generated by $\nabla_{\mathrm{tor}} h$.

As a simple application of the information delivered above, let us note:

**Corollary 11.** *Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible Kouchnirenko non-degenerate formal power series. Assume that $h$ is convenient. Set $m_i := \min\{p : \Gamma_+(z_i^p) \subset \Gamma_+(h)\}$ $(i = 1, \ldots, n)$ and $m := \max_{1 \leqslant i \leqslant n}\{m_i\}$. Then*

$$\text{ł}(h) \leqslant m - 1.$$

*The same estimation holds if $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a holomorphic function.*

*Proof.* Let $\mathfrak{n}$ denote the maximal ideal in $\mathbb{K}[[z]]$. Since we have the containment of ideals $\mathfrak{n}(\nabla h)\mathbb{K}[[z]] \supset (\nabla_{\mathrm{tor}}(h))\mathbb{K}[[z]]$, we infer that

$$\text{Ł}_{\mathfrak{n}}(\mathfrak{n}(\nabla h)\mathbb{K}[[z]]) \leqslant \text{Ł}_{\mathfrak{n}}(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]).$$

From Proposition 9 we know that $\overline{\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]} = \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[[z]]$. By assumption, this last set contains some powers of all the variables so that $m$ is indeed well-defined. From Proposition 4 and Theorem 6 it easily follows that $\text{Ł}_{\mathcal{J}}(\mathcal{I}) = \text{Ł}_{\overline{\mathcal{J}}}(\overline{\mathcal{I}})$ for any ideals $\mathcal{I}$, $\mathcal{J}$. Using this, we get $\text{Ł}_{\mathfrak{n}}(\nabla_{\mathrm{tor}}(h)\,\mathbb{K}[[z]]) \leqslant \text{Ł}_{\mathfrak{n}}(\{z_i^{m_i}\}_{1 \leqslant i \leqslant n}\mathbb{K}[[z]])$. But this last number is immediately seen to be equal to $m$. Thus,

$$\text{Ł}_{\mathfrak{n}}(\mathfrak{n}(\nabla h)\mathbb{K}[[z]]) \leqslant m.$$

Now, exploiting Proposition 4, we see that $\text{Ł}_{\mathfrak{n}}(\mathfrak{n}(\nabla h)\mathbb{K}[[z]]) = 1 + \text{Ł}_{\mathfrak{n}}((\nabla h)\mathbb{K}[[z]]) = 1 + \text{ł}(h)$. Combining this with the relation above, we finish the proof in the formal setting. The holomorphic case is treated the same way. □

**Comment.** Choosing an appropriate parametrization of the form $\varphi(t) = (0, \ldots, 0, t, 0, \ldots, 0)$ we immediately see that under the above assumptions it actually holds $\text{Ł}_{\mathfrak{n}}(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]) = m$. A proof of this fact for complex analytic mappings can be found in [1, Corollary 3.6] and [14, Theorem 2.7].

## 4. Constant-Newton-diagram deformations of non-degenerate singularities

In order to prove the main result, we need to have information about special deformations of Kouchnirenko non-degenerate holomorphic functions. Similarly as above, this result turns out to hold more generally – for formal power series with coefficients in an algebraically closed field.

**Definition 12.** *Let $\mathbb{K}$ be a field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible formal power series. Let $s$ be a new variable over $\mathbb{K}[[z]]$. We say that $h_\times \in \mathbb{K}[[s,z]]$ is a* deformation of $h$ *if $h_\times(s, 0) = 0$ and $h_\times(0, z) = h$.*

**Definition 13.** *We say that a deformation $h_\times \in \mathbb{K}[[s,z]]$ of $h \in \mathbb{K}[[z]]$, where $z = (z_1, \dots, z_n)$, satisfies* condition (c) *if*

$$\frac{\partial h_\times}{\partial s} \in \overline{(z_1, \dots, z_n) \cdot \nabla_z(h_\times)\mathbb{K}[[s,z]]}.$$

**Comments.**

I. Naturally, above it is meant that $\nabla_z(h_\times) := \left(\frac{\partial h_\times}{\partial z_1}, \dots, \frac{\partial h_\times}{\partial z_n}\right)$. Note that condition (c), as stated, is weaker than the condition

$$\frac{\partial h_\times}{\partial s} \in \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]]}, \tag{2}$$

which we shall actually work with below (cf. Example 16).

II. Teissier in [17] requires that the deformation is not smooth i.e. it must hold $\nabla_z(h_\times)(s, 0) = 0$. We decided to remove this restriction here.

III. If $h_\times \in \mathbb{C}\{s,z\}$ then relation (2) is equivalent to

$$\frac{\partial h_\times}{\partial s} \in \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{C}[[s,z]]} \cap \mathbb{C}\{s,z\} \underset{[7,\text{ Prop. } 1.6.2]}{=} \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{C}\{s,z\}} \tag{3}$$

and, similarly, the relation from Definition 13 is equivalent to

$$\frac{\partial h_\times}{\partial s} \in \overline{(z_1, \dots, z_n) \cdot \nabla_z(h_\times)\mathbb{C}\{s,z\}}.$$

This is the original Teissier condition (c) given in [17, § 2] in the complex analytic setting. We also remark that, according to Teissier, elements of the family $h_\times(\sigma, z)$, for $\sigma \ll 1$, are called *(c)-cosecant*.

The result below shows that simple enough deformations of Kouchnirenko non-degenerate formal power series do satisfy condition (c).

**Proposition 14.** *Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible Kouchnirenko non-degenerate formal power series. Define a deformation $h_\times \in \mathbb{K}[s][[z]]$ of $h$ by the formula $h_\times := h + s \cdot z^\alpha$, where $\alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n$. Then $h_\times$ satisfies Teissier condition (c), and even condition* (2).

*Proof.* We must check that $z^\alpha = \frac{\partial h_\times}{\partial s} \in \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]]}$. Take an arbitrary $\varphi = (\varphi_0, \hat{\varphi}) \in \mathbb{K}[[t]]^{n+1}$ such that $\varphi(0) = 0$. By Proposition 9 we have $z^\alpha \in \overline{\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]}$, hence using Theorem 6 we get

$$\mathrm{ord}(\hat{\varphi}^* z^\alpha) \geqslant \mathrm{ord}(\hat{\varphi}^*(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]])), \tag{4}$$

where we substitute $z = \hat{\varphi}$. This implies

$$\mathrm{ord}(\varphi^*(s \cdot z^\alpha)) > \mathrm{ord}(\varphi^*(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[s,z]])),$$

where we substitute $(s,z)=(\varphi_0,\hat{\varphi})$. Consequently, upon noticing that $\nabla^z_{\mathrm{tor}}(h_\times)=\nabla_{\mathrm{tor}}(h)+sz^\alpha\alpha$,

$$\operatorname{ord}(\varphi^*(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[s,z]]))=\operatorname{ord}(\varphi^*(\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]])). \tag{5}$$

Now, (4) and (5) give

$$\operatorname{ord}(\varphi^*(z^\alpha))\geqslant\operatorname{ord}(\varphi^*(\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]])), \tag{6}$$

where we substitute $(s,z)=(\varphi_0,\hat{\varphi})$. As $\varphi$ was chosen arbitrarily, Theorem 6 ensures that $z^\alpha\in\overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]]}$. This proves the result. □

**Remark.** Essentially the same proof as the one given above shows that every deformation $h_\times$ of $h$ whose Newton diagram built over the field $\mathbb{K}((s))$ of Laurent series is equal to that of $h$, that is $\Gamma_+^{\mathbb{K}((s))}(h_\times)=\Gamma_+^{\mathbb{K}}(h)$, also satisfies condition (c).

**Corollary 15.** *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic, Kouchnirenko non-degenerate function. Define a deformation $f_s:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ of $f$ by the formula $f_s:=f+s\cdot z^\alpha$, where $\alpha\in\Gamma_+(f)\cap\mathbb{N}_0^n$. Then $f_s$ satisfies Teissier condition (c), and even condition* (3)*.*

*Proof.* Follows from the above proposition and Comment III. on page 35. □

Let us consider the following

**Example 16.** It is not enough to assume the (weaker) non-degeneracy considered by Mondal (see [11]) in order for Proposition 14 (or Corollary 15) to hold with condition (2) in their assertions. Take the Kouchnirenko's example [8, Remarque 1.21], where $f:=(x+y)^2+xz+z^2\in\mathbb{C}\{x,y,z\}$. Then $f$ is Kouchnirenko degenerate with respect to the vector $l:=(1,1,2)$ supporting the segment $\Delta=\Delta(l)=\operatorname{conv}(\{(1,0,0),(0,1,0)\})\subset\mathbb{R}^3$. Indeed, the system $\{\nabla\operatorname{in}_\Delta f=0\}=\{\nabla(x+y)^2=0\}$ possesses solutions in $(\mathbb{C}^*)^3$. At the same time, $f$ is *Milnor non-degenerate* (see [11, Definition 5.1]) and, consequently, its Milnor number can be read off the Newton diagram of $f$ by the usual Kouchnirenko formula: $\mu(f)=1$.

Consider the deformation $f_s:=f+s\cdot xy\in\mathbb{C}\{s,x,y,z\}$ and let $\varphi(t):=(0,t,-t,0)\in\mathbb{C}\{t\}^4$. Since $f_\sigma$, for $0\neq\sigma\ll1$, are Kouchnirenko non-degenerate, we get $\mu(f_\sigma)=\mu(f)$, which by Lê-Saito-Teissier criterion of $\mu$-constancy gives that $\frac{\partial f_s(z)}{\partial s}\in\overline{\nabla_{(x,y,z)}f_s(z)}\mathbb{C}\{s,x,y,z\}$ (see [6]). Nevertheless, this last relation cannot be improved to get condition (2) (or (3)), as the following calculation reveals:

$$\nabla^{(x,y,z)}_{\mathrm{tor}}(f_s)=(2x\cdot(x+y)+x\cdot z+s\cdot x\cdot y,2y\cdot(x+y)+s\cdot x\cdot y,z\cdot x+2z^2),$$

so, substituting $(s,x,y,z)=\varphi(t)$, we get

$$\operatorname{ord}\varphi^*\left(\frac{\partial f_s}{\partial s}\right)=\operatorname{ord}\varphi^*(xy)=2<\infty=\operatorname{ord}\varphi^*(\nabla^{(x,y,z)}_{\mathrm{tor}}(f_s)\mathbb{C}\{s,x,y,z\}).$$

By Corollary 7, $\frac{\partial f_s}{\partial s}$ is not integral over the ideal $\nabla^{(x,y,z)}_{\mathrm{tor}}(f_s)\mathbb{C}\{s,x,y,z\}$, hence condition (3) does not hold for $f_s$.

Still, $\operatorname{ord}\varphi^*((x,y,z)\cdot\nabla_{(x,y,z)}(f_s)\mathbb{C}\{s,x,y,z\}) = 2$ and one can check that actually $f_s$ does satisfy condition (c), because, as ideals, $\nabla_{(x,y,z)}(f_s)\mathbb{C}\{s,x,y,z\} = (x,y,z)\mathbb{C}\{s,x,y,z\}$ and $\frac{\partial f_s}{\partial s} = xy \in (x,y,z)^2\mathbb{C}\{s,x,y,z\}$.

Finally, let us note that $\text{ł}(f_\sigma) = \text{ł}(f) = 1$ for small $\sigma$.

These observations inspire several problems (see Section 6).

## 5. The proof of the main result

In this section, we go back to the complex analytic world. We need the following result:

**Theorem 17** (Teissier). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated singular point at 0 and $f_s : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ – its deformation such that $\nabla_z f_s(0,s) = 0$. Assume that $f_s$ satisfies condition (c). Then*

$$\text{ł}(f_\sigma) = \text{ł}(f),$$

*for $0 \neq \sigma \ll 1$.*

*Proof.* This is a consequence of two results of B. Teissier. The first one, [17, Théorème 6], asserts that, under the above assumptions, the set of so-called *polar quotients* (see [17]) attached to an isolated singularity is invariant in deformations satisfying condition (c). The second one, [17, §1.7., Corollaire 2], explains that the biggest polar quotient is exactly the Łojasiewicz exponent ł. Hence, the assertion of the theorem follows. □

For convenience, let us state Theorem 1 once again.

**Main Theorem.** *For any two Kouchnirenko non-degenerate functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having isolated singularities at 0 and the same Newton diagrams their Łojasiewicz exponents are equal: $\text{ł}(f) = \text{ł}(g)$.*

*Proof.* Set $f = \Sigma_{i\in\mathbb{N}_0^n} a_i z^i$ and $g = \Sigma_{i\in\mathbb{N}_0^n} b_i z^i$ in a neighborhood of 0.

Firstly, note that we may assume that both $f$ and $g$ are polynomials (of degrees $\leqslant \mu(f) + 1 = \mu(g) + 1$). This is a consequence of their finite determinacy for the right (biholomorphic) equivalence (see [5, Theorem 9.1.4]).

Secondly, we may make $f$ and $g$ be supported only on $\Gamma = \Gamma(f) = \Gamma(g)$. Indeed, choose e.g. $j \in (\operatorname{Supp} f)\backslash\Gamma$ and consider the deformation $f_s(z) := f(z) - s\cdot a_j z^j$ of $f$. Then $f_\sigma$ ($\sigma \in \mathbb{C}$) are all Kouchnirenko non-degenerate functions with the same Newton diagrams as $f$ has, and, actually, all of them differ by only one term, above $\Gamma$. In particular, $\mu(f_\sigma) = \mu(f) < \infty$, so all $f_\sigma$ have isolated singularities at 0. From Corollary 15 we know that locally, at each $\sigma \in [0,1]$, the deformation $f_{s+\sigma} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of $f_\sigma$ satisfies condition (c) and hence, by Theorem 17, has constant Łojasiewicz exponent. Consequently, $\text{ł}(f_1) = \text{ł}(f_0) = \text{ł}(f)$ and the function $f_1$ has got one monomial less above $\Gamma$ than $f$ has. Continuing in this way, after finitely many steps we will change $f$ into a function without terms above $\Gamma$. Similar procedure does the same to $g$.

Lastly, we essentially repeat the above argument for a monomial $z^j$ with $j \in \Gamma \cap \mathbb{N}_0^n$ and $a_j \neq b_j$ but this time this requires some care. Namely, since the set

$$H := \left\{ (\xi_\alpha)_{\alpha \in \Gamma \cap \mathbb{N}_0^n} : \text{ the function } \sum \xi_\alpha z^\alpha \text{ is Kouchnirenko non-degenerate} \right\}$$

is Zariski open in $\mathbb{C}^{\#(\Gamma \cap \mathbb{N}_0^n)}$ (see e.g. [8, Théorème 6.1] or [13, Appendix]), we may choose a real (piecewise linear) simple curve $\delta$ lying entirely in $H$ and joining the coefficients of $f$ with these of $g$. Next, covering the curve by a finite number of small enough closed cubes contained in $H$, we may sequentially modify the curve inside each of these cubes to make it only be built of segments parallel to the coordinate axes in $\mathbb{R}^{2 \cdot \#(\Gamma \cap \mathbb{N}_0^n)}$ (see Figure 1). Then, locally, along each of these segments, one can apply the above reasoning to find that the Łojasiewicz exponent is constant there. Consequently, $\text{ł}(f) = \text{ł}(g)$. □

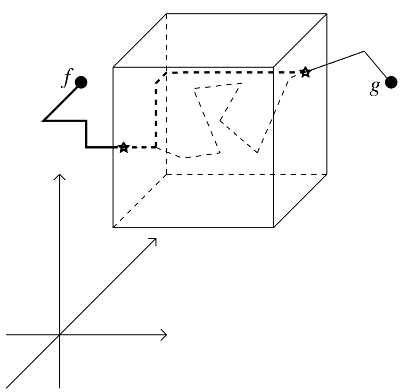


FIGURE 1. Modification of the curve $\delta$ ($i$-th step).

**Comment.** In the statement of Theorem 1 we can simply assume that the functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are Kouchnirenko non-degenerate and have the same Newton diagrams. Indeed, if $\operatorname{ord} f = \operatorname{ord} g = 1$, that is both $f$ and $g$ are smooth at 0, we clearly have $\text{ł}(f) = \text{ł}(g) = 0$. On the other hand, if e.g. $f$ possesses a non-isolated singularity at 0, then $\Gamma(f) = \Gamma(g)$ cannot be the Newton diagram of any isolated singularity (see [2]), hence $g$ is also non-isolated and $\text{ł}(f) = \text{ł}(g) = \infty$.

## 6. PROBLEMS

Here we ask several questions that may be worthwhile addressing.

I. *Does Theorem 17 hold for formal power series over an algebraically closed field?*

II. *And what about Theorem 17 with condition (c) replaced by condition* (2)*?*
III. *And what about the Main Theorem?*
IV. *And what about the Łojasiewicz exponent for Milnor non-degenerate singularities?* (consult [11]).
V. For a function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, one can consider the Łojasiewicz exponent $Ł_f(\nabla f)$ of $f$ relative to its gradient. This number is always finite, even in the case of non-isolated singularities. For isolated singularities, we have the identity $Ł_f(\nabla f) = \frac{ł(f)}{1+ł(f)}$ ([17, § 1.7, Corollaire 2]) so this number, too, depends only on the Newton diagram for Kouchnirenko non-degenerate isolated singularities. *Does the same hold in the non-isolated case? And over algebraically closed fields?* Note that, by [12], this is indeed the case in the holomorphic setting in dimension 2.

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