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A NON-CONTAINMENT EXAMPLE ON LINES AND A SMOOTH CURVE OF GENUS 10

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Abstract. The containment problem between symbolic and ordinary powers of homogeneous ideals has stimulated a lot of interesting research recently. In the most basic case of points in $\mathbb{P}^2$ and powers $I^{(3)}$ and I^2, there is a number of non-containment results based on arrangements of lines. In a joint paper with Lampa-Baczyńska we discovered the first example of non-containment based on an arrangement of axes and a singular irreducible curve of high degree.

In the present note we show a similar example based on lines and a smooth curve of degree 6.

1. Introduction

For algebraic geometers the most intriguing thing is the question about relations between algebraic structures and the geometry that is represented by these objects. One of the famous question related to this investigations is connected to the so-called containment problem which can be formulated as follows.

Containment Problem. Assume that $I \subseteq \mathbb{K}[x_0, \dots, x_N]$ is a homogeneous ideal in the ring of polynomials over a field $\mathbb{K}$. For which pairs of positive integers (m, r) there is a containment

$$I^{(m)} \subset I^r?$$

A break through in finding an answer to this problem appeared in the paper by Ein, Lazarsfeld and Smith [6], which gives almost optimal relations between numbers m and r. What they proved is that the inclusion $I^{(m)} \subset I^r$ always holds

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under the condition $m \geq Nr$. Since then the problem if the containment holds for m strictly smaller than Nr remained open. Right after [6], some conjectures about the relation between symbolic and ordinary powers of ideals appeared (see [1, Conjecture 8.4.2], or [12, Conjecture 4.1.1], or [2, Conjecture 1.1]). We can collect all these ideals in the following conjecture attributed to Harbourne.

Conjecture 1.1. *Let I be a proper homogeneous ideal such that bigheight of $I = e$. Then*

$$I^{(m)} \subseteq I^r$$

whenever $m \geq er - (e-1)$.

If I describes a smooth variety, then e is its codimension. In general, it can be thought of as the maximal codimension of an embedded component.

If we consider the ideal of points $I \subset \mathbb{K}[x, y, z]$ then the conjecture, for the first non-trivial case of $r = 2$, claims that $I^{(3)} \subseteq I^2$. As we know now, it is not true in general. The first counterexample to $I^{(3)} \subseteq I^2$ was given in [5] and it involves a special point-line configuration defined over the complex numbers. This configuration is known in the literature under many names: dual Hesse configuration, Fermat or Ceva arrangement. It is also denoted by $G(3, 3, 3)$ in the Shephard, Todd classification [18], so it belongs to the irreducible complex reflection group. In a short time a lot of other examples of ideals of points in $\mathbb{P}^2$ were found, for which Conjecture 1.1 fails. In the chronological order of appearing there are:

- A whole family of real counterexamples, coming from the so-called Böröczky configurations [3],
- A collection of counterexamples over the field of finite characteristic [13],
- Examples over the rational numbers [15].

After the appearance of the papers mentioned above, many mathematicians tried to find out a criterion to detect wherever a given configuration of points leads to a counterexample to Conjecture 1.1. The first guess about which property of the set of lines is important to be a counterexample, was the overall number of tripe points in relation to double points. Indeed, Fermat-type arrangements, which are defined over $\mathbb{C}$, have no double points, and for almost all members of this family the inclusion $I^{(3)} \subseteq I^2$ fails (see [13]). Similarly for Böröczky configurations for which the number of double points is the smallest possible and the number of triple points attains the highest possible value (see [9] and [10] for more information).

As the first counterexamples of configurations with relatively high number of double points were discovered (see [16]) another guess about what distinguish configurations of lines $\mathcal{A}$ to give a counterexample was a weak combinatorics, expressed in terms of the t-vector associated to $\mathcal{A}$. But that also occurred not to be essential, as shown in [8]. Nevertheless, all mentioned so far articles deal with the set of at least triple points configuration coming from intersection of lines and the element from $I^{(3)}$ not belonging to I^2 was always the product of equations of all lines. The last results in this subject [14] showed some surprising facts: the ideal of points I

which is the counterexample to Conjecture 1.1 can be generated by triple points, as well as, a big number of double points; the element from the set $I^{(3)} \setminus I^2$ can be generated by a polynomial of degree higher than 2, whose set of zeroes is an irrational curve.

In this paper we continue investigation in finding geometric and combinatorial properties for the set of points which detects wherever this set is generating an ideal giving the counterexample or not. The latest results shows a new direction of researches. Thus in this paper, instead of taking the points of given multiplicity from the configuration of lines, we consider the set of points in some special position. Namely, we start from one of the real reflection arrangement, known as $A(31, 3)$, and we take some orbits of points. What we get can be formulated as the following theorem

Main Theorem. *The ideal I defined as the intersection*

$$I = \bigcap_{P_i \in \mathcal{P}} I(P_i),$$

of 48 points from the Table 2 is an example of failure for the containment $I^{(3)} \subseteq I^2$. Moreover, there exists an element $F \in I^{(3)} \setminus I^2$ which consists of 13 lines and an irreducible curve of degree 6.

2. Preliminaries

We begin this section with recalling some basic facts and definitions concerning containment problem (see [19] for a detailed survey).

Take any homogeneous ideal $I \subset \mathbb{K}[x_0, \dots, x_N]$. The m-th symbolic powers of ideal I is defined as

Definition 2.1. (Symbolic power) For $m \geq 1$, the m-th symbolic power of I is the ideal

$$I^{(m)} = \mathbb{K}[x_0, \dots, x_N] \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^m)_{\mathfrak{p}} \right),$$

where the intersection is taken over all associated primes of I.

Definition 2.2. Let $I, J \subset \mathbb{K}[x_0, \dots, x_N]$ be ideals. The saturation of I with respect to J is

$$I : J^{\infty} = \bigcup_{j=0}^{\infty} I : (J^j).$$

From now on we assume that we work over complex projective plane $\mathbb{P}^2(\mathbb{C})$ and that the ideal I is the radical ideal of a finite number of points $P_1, \dots, P_s$. Under our assumption about the ground field, the symbolic power can be translated into the language of geometry due to Nagata-Zariski theorem ([7, Theorem 3.14]). By considering all homogeneous forms which vanish up to order m, this theorem tells us that we can think about symbolic power of I as the intersection

$$I^{(m)} = \bigcap_{i=1}^{s} I(P_i)^m,$$

where $I(P_i)$ denotes the ideal of point P_i.

Denote by $\mathcal{A}$ a finite set of lines $L_1, \ldots, L_r$ and by $t_i(\mathcal{A})$ the number of points from the set of all intersection points of L_j where exactly i lines from $\mathcal{A}$ intersect. Thus we obtain a t-vector, which we can associate to any configuration of lines, defined as follows $t(\mathcal{A}) = (t_2(\mathcal{A}), \ldots, t_r(\mathcal{A}))$.

The subject of our consideration is the configuration of points which comes from the line arrangement denoted by $A(31, 1)$ in the Grünbaum list [11]. This special configuration was described in details in [14], where the reader can find informations how to construct this configuration over rational numbers. Let us recall all information relevant for concerning this configuration here.

Put $e = \frac{\sqrt{3}}{2}$, then $A(31, 1)$ consists of 31 lines which equations are given by

$$x \pm aez = 0, \quad x \pm ey \pm bez = 0, \quad ex \pm y \pm cz = 0, \quad y \pm dz = 0, \quad z = 0,$$

where $a \in \{0, \frac{1}{2}, 1, 2\}$, $b \in \{0, 1, 2, 4\}$, $c \in \{0, 2\}$ and $d \in \{0, 1\}$. This configuration is indicated on Figure 2. Line arrangement $A(31, 3)$ is invariant under group

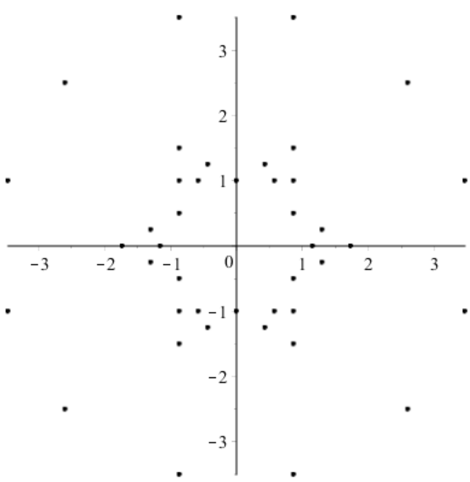

FIGURE 1. Points from set $\mathcal{P}$. Orbit of 6 points at infinity is not indicated.

action $H = Z_3 \times Z_2$, thus all 127 intersection points can be viewed as a groups of points which lies on orbits. It turns out that for these intersection points a different group can be applied, $G = Z_6 \times Z_2$, and from now on we work with this group action. We can distinguish 16 orbits among all 127 points under the action of G. One of length 1, nine of length 6 and six of length 12. We choose six of the orbits (4 of length 6 and 2 of length 12) and denote their union as $\mathcal{P}$. All 48 chosen points with coordinates are collected in Table 2 and visualized on Figure 1.

number of orbits	length of orbit	points
1	6	$(e:3:-3)$, $(e:-3:3)$, $(-2e:0:3)$, $(2e:0:3)$, $(e:-3:-3)$, $(e:3:3)$
2	6	$(e:-2:2), (e:1:2), (0:-1:1)$, $(0:1:1)$,$(e:1:-2)$, $(e:-1:-2)$
3	12	$(e:5:4), (-3e:1:4), (3e:-1:4)$, $(e:5:-4)$, $(3e:1:-4)$, $(e:-5:4)$, $(e:-5:-4)$, $(3e:1:4)$, $(e:-2:2)$, $(e:2:2), (e:2:-2), (e:-2:-2)$
4	6	$(e:0:1)$, $(e:-3:2)$, $(e:3:2)$, $(e:3:-2)$, $(e:-3:-2)$, $(e:0:-1)$
5	12	$(3e:5:-2)$, $(3e:-5:2)$, $(3e:-5:-2)$, $(3e:5:2)$, $(2e:-1:1)$, $(2e:1:1)$, $(e:-7:2)$, $(e:7:2)$, $(e:-7:-2)$, $(e:7:-2)$, $(-2e:-1:1)$, $(-2e:1:1)$
6	6	$(1:-e:0)$, $(3:e:0)$, $(1:e:0)$, $(3:-e:0)$, $(1:0:0)$, $(0:e:0)$

With all preceding notation we can formulate the following theorem

Main Theorem. *The ideal I defined as the intersection*

$$I = \bigcap_{P_i \in \mathcal{P}} I(P_i),$$

of 48 points from the Table 2 is an example of failure for the containment $I^{(3)} \subseteq I^2$. Moreover, there exists an element $F \in I^{(3)} \setminus I^2$ which consists of 13 lines and irreducible curve of degree 6.

Proof. Define the lines

$$L_{1,2,3,4}: \; x \pm ey \pm ez, \quad L_{5,6,7,8}: 3x \pm ey \pm 2ez,$$

$$L_{9,10}: \; y \pm z, \quad L_{11,12}: \; 2x \pm ez, \quad L_{13}: \; z,$$

and put

$$f := x^6 - \tfrac{9}{2}x^4y^2 + 8x^2y^4 + \tfrac{1}{6}y^6 - \tfrac{103}{12}x^4z^2 - \tfrac{103}{6}x^2y^2z^2 - \tfrac{103}{12}y^4z^2 + \tfrac{269}{12}x^2z^4 + \tfrac{269}{12}y^2z^4 - 17z^6.$$

We claim that

$$F = f \cdot \prod_{i=1}^{13} L_i \in I^{(3)} \setminus I^2.$$

It is a straightforward but tedious calculations to check that $F \in I^{(3)}$. For the reader convenience we prepared Singular [4] script which can be downloaded from web page (see [17]). Polynomial F, together with all visible in affine part $z = 1$ points, are indicated on Figure 3.

It is highly non trivial to check that $F \notin I^2$, due to fact that ideal I^2 is generated by 28 polynomials with very big coefficients. As before, for reader convenience, it can be done by running our script.

In order to prove irreducibility of polynomial f, observe that the Jacobian ideal is generated by

$$\begin{aligned} J = \text{jacob}(f) = \quad & \Big(x(6x^4 - 18x^2y^2 + 16y^4 - \tfrac{103}{3}x^2z^2 - \tfrac{103}{3}y^2z^2 + \tfrac{269}{6}z^4), \\ & y(-9x^4 + 32x^2y^2 + y^4 - \tfrac{103}{3}x^2z^2 - \tfrac{103}{3}y^2z^2 + \tfrac{269}{6}z^4), \\ & z(-\tfrac{103}{6}x^4 - \tfrac{103}{3}x^2y^2 - \tfrac{103}{6}y^4 + \tfrac{269}{3}x^2z^2 + \tfrac{269}{3}y^2z^2 - 102z^4)\Big) \\ & = (xf_1, yf_2, zf_3). \end{aligned}$$

What we can show [17] is that

$$\begin{aligned} x^8 = \quad & \textstyle\sum_{i=1}^{3} c_i f_i, \\ y^8 = \quad & \textstyle\sum_{j=1}^{3} c_j f_j, \\ z^8 = \quad & \textstyle\sum_{k=1}^{3} c_k f_k, \end{aligned}$$

for some $c_i, c_j, c_k \in \mathbb{K}[x, y, z]$, and therefore $V(J : (x, y, z)^\infty) = \emptyset$, which basically means that curve defined by f is smooth and thus irreducible.

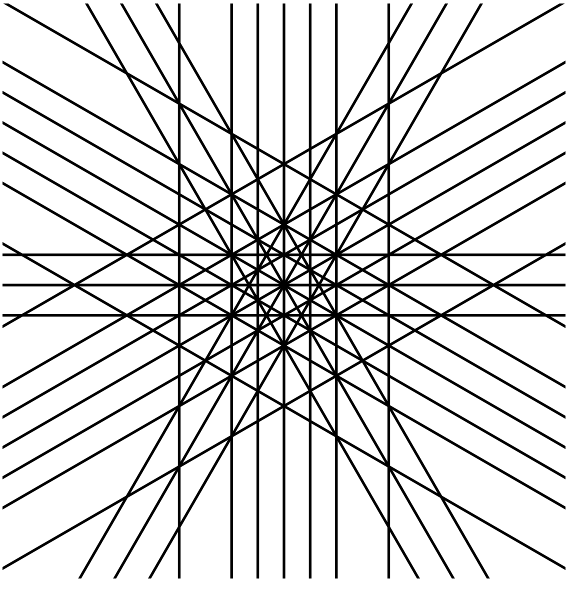

FIGURE 2. Configuration $A(31, 3)$. Line at infinity $z = 0$ is not shown.

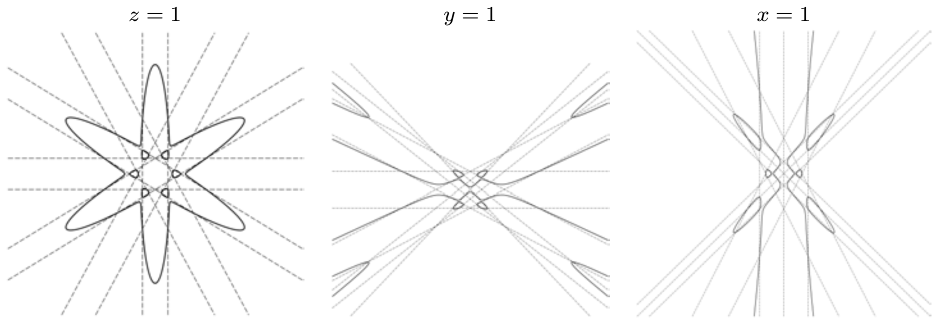

TABLE 1. The curve $f = 0$ with different affine projections.

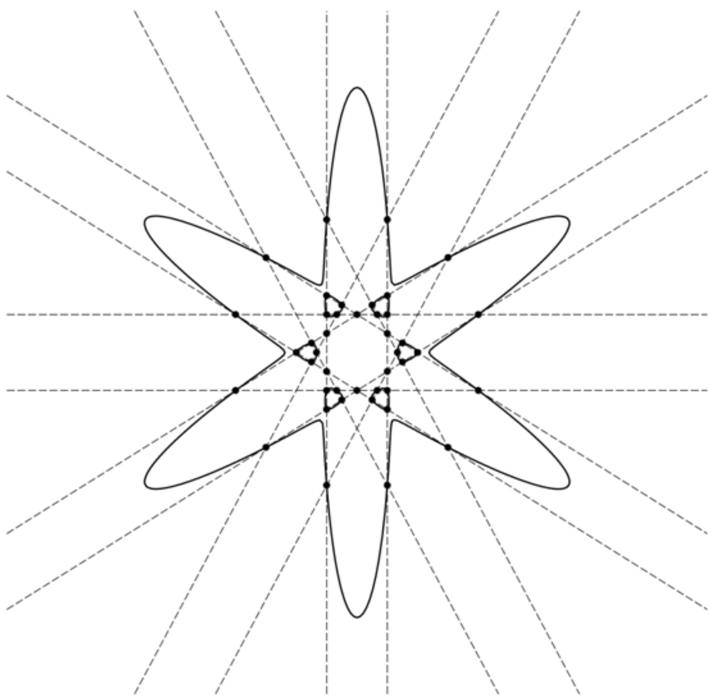

FIGURE 3. The affine part $z = 1$ of the curve $f = 0$ (solid curve), lines $L_i = 0$ (dashed lines) and points from the set $\mathcal{P}$.

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