

RINGS AND FIELDS OF CONSTANTS OF CYCLIC
FACTORIZABLE DERIVATIONS

JANUSZ ZIELIŃSKI

ABSTRACT. We present a survey of the research on rings of polynomial constants and fields of rational constants of cyclic factorizable derivations in polynomial rings over fields of characteristic zero.

1. MOTIVATIONS AND PRELIMINARIES

The first inspiration for the presented series of articles (some of them are joint works with Hegedűs and Ossowski) was the publication [20] of professor Nowicki and professor Moulin Ollagnier. The fundamental problem investigated in that series of articles concerns rings of polynomial constants ([26], [28], [33], [29], [8]) and fields of rational constants ([30], [31], [32]) in various classes of cyclic factorizable derivations. Moreover, we investigate Darboux polynomials of such derivations together with their cofactors ([33]) and applications of the results obtained for cyclic factorizable derivations to monomial derivations ([31]).

Let k be a field. If R is a commutative k -algebra, then k -linear mapping $d : R \rightarrow R$ is called a k -derivation (or simply a derivation) of R if $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$. The set $R^d = \ker d$ is called a ring (or an algebra) of constants of the derivation d . Then $k \subseteq R^d$ and a nontrivial constant of the derivation d is an element of the set $R^d \setminus k$. By $k[X]$ we denote $k[x_1, \dots, x_n]$, the polynomial ring in n variables. If $f_1, \dots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \rightarrow k[X]$ such that $d(x_1) = f_1, \dots, d(x_n) = f_n$.

2010 *Mathematics Subject Classification.* 13N15, 12H05, 34A34.

Key words and phrases. cyclic factorizable derivation, Lotka-Volterra derivation, ring of polynomial constants, field of rational constants.

More information on derivations one can find in the monographs [6] and [24]. They also present links of derivation theory with the Jacobian Conjecture. Because this still unsettled conjecture one may translate into the language of derivations. Such an equivalent formulation can be found for example in [23], Theorem 5.

There is no general effective procedure for determining the ring of constants of a derivation, although the subject has a long tradition. One of the possible approaches, a certain reduction of the problem, is the Lagutinskii's procedure. Namely, we can associate the factorizable derivation with a given derivation of the polynomial ring over a field of characteristic zero. A derivation $d : k[X] \rightarrow k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \dots, n$. That procedure of association is described for example in [25]. It turns out that in the generic case the problems of determining the fields of rational constants of the initial derivation and its associated derivation are equivalent. Which also gives us a good knowledge on polynomial constants. The challenge is that constants of factorizable derivations are still not sufficiently investigated. We know everything only in the case of number of variables ≤ 3 , mainly thanks to the papers by Moulin Ollagnier and Nowicki (e.g. [18], [19], [20]). For a greater number of variables there are examined only some special cases such as Lotka-Volterra derivations (e.g. [8], [32]), derivations that appear in the Lagutinskii's procedure applied to Jouanolou derivations ([16]) and factorizable derivations associated with cyclotomic derivations ([21]). For certain other classes of derivations there are indicated some constants without settling whether they are the complete set of generators of the ring of constants (e.g. Itoh [9], Cairó [4]).

The question of determining constants can be equivalently expressed in the language of differential equations. Namely, over an arbitrary field k of characteristic zero, if δ is a derivation of the ring $k[X]$ (respectively: of the field $k(X)$) such that $\delta(x_i) = f_i$ for $i = 1, \dots, n$, then the set $k[X]^\delta \setminus k$ (respectively: $k(X)^\delta \setminus k$) coincides with the set of all polynomial (respectively: rational) first integrals of a system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \dots, x_n(t)),$$

where $i = 1, \dots, n$ (for more details we refer the reader to [24], subsection 1.6).

The topic is also linked to invariant theory. Namely, for every connected algebraic group $G \subseteq \text{Gl}_n(k)$, where k is a field of characteristic zero, there exists a derivation d such that $k[X]^G = k[X]^d$ (more information can be found, among others, in [24], subsection 4.2).

From now on k is a field of characteristic zero. We will call a factorizable derivation $d : k[X] \rightarrow k[X]$ *cyclic* if $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \dots, n$ (in the cyclic sense, that is, we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). In particular, a derivation $d : k[X] \rightarrow k[X]$ is said to be

Lotka-Volterra derivation with parameters $C_1, \dots, C_n \in k$ (see e.g. [11], [18], [19]) if

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \dots, n$ (in the cyclic sense as above, that is, indices modulo n). The systems under consideration describe a wide range of phenomena and they appear in numerous domains of science such as population biology (inter-species interactions in the predator-prey model) [27], chemistry (oscillations of the concentration of substances in chemical reactions) [14], hydrodynamics (the convective instability in the Bénard problem) [3], plasma physics (the evolution of electrons and ions) [13], laser physics (the coupling of waves) [12], aerodynamics (the interaction of gases in a mixture) [15], economics [4], neural networks [22] and biochemistry [4]. Further motivations and applications are presented in [1], [2], [5] and many more.

The case of Lotka-Volterra derivations in three variables was settled in [18], [19], [20] by Moulin Ollagnier and Nowicki. For example, the existence of nontrivial polynomial constants is determined by the following theorem (here parameters C_i have opposite signs than in the notation above, which is of no account) from [18]:

Theorem 1. ([18], Theorem 1)

The Lotka-Volterra system

$$\begin{cases} d(x) = x(Cy + z) \\ d(y) = y(Az + x) \\ d(z) = z(Bx + y) \end{cases}$$

has a nontrivial polynomial constant if and only if one of the following cases holds:

- (i) $ABC + 1 = 0$,
- (ii) $-A - \frac{1}{B} = 1$, $-B - \frac{1}{C} = 1$ and $-C - \frac{1}{A} = 1$,
- (iii) $C = -k_2 - \frac{1}{A}$, $A = -k_3 - \frac{1}{B}$, $B = -k_1 - \frac{1}{C}$ where, up to a permutation, (k_1, k_2, k_3) is one of the following triples: $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$.

The rings of polynomial constants were determined in each of these cases in [20].

The article [25] contains a full description of monomial derivations (that is, derivations which values on variables are monic monomials) in two ([25], Proposition 5.4) and in three variables ([25], Theorem 8.6) with nontrivial rational constants (that is, in the field of rational functions). The results for three variables in the generic case are based on Lotka-Volterra derivations, thoroughly investigated before for three variables mainly in [17]. That complete characterization of all cases for monomial derivations in two and three variables has marked at the same time the limitations of the usefulness of the Lagutinskii's procedure, potentially general, and practically limited by the knowledge of constants of factorizable derivations. It was the motivation and a starting point to deal with determining of constants of cyclic factorizable derivations in $n \geq 4$ variables.

2. METHODS AND RESEARCH TECHNIQUES

In contrast to e.g. Jouanolou derivations ([16], [34]), where one has to show that a constant of a positive degree does not exist, here we also deal with some nontrivial constants. Therefore instead of obtaining a contradiction we have to prove that a constant is a polynomial in given generators. A direct investigation of constants of derivations of considered type came across serious problems. These constants did not subject to the induction on degree, and the calculations turned out to be virtually impossible to perform. Therefore, the idea proved valuable, was to analyze, instead of constants of a derivation d , polynomials φ that fulfill the condition $d(\varphi^A)^A = 0$, where f^A denotes the restriction of a polynomial f to the ring of polynomials in variables with indices in the set A , where $A \subseteq \{1, \dots, n\}$. Constants of a factorizable derivation d fulfill the condition $d(\varphi^A)^A = 0$, hence we obtained also some properties of these constants, which have been applied in the proofs of main theorems. However the properties of polynomials φ such that $d(\varphi^A)^A = 0$ have turned out to be possible to prove by combinatorial and inductive methods.

Moreover, our frequently used method of investigation of constants was the restriction of these constants to the polynomial ring in a smaller number of variables. And then, after the obtainment of their properties for various subsets of variables, we have tried to merge these data to receive some information about the shape of these initial constants.

Another important method was study of Darboux polynomials of a derivation d . This is a standard procedure in the case of determining rational constants, however much rarer in the case of determining polynomial constants, as here. Particularly important turned out to be characterizations of the coefficients of the cofactors of strict Darboux polynomials.

The next method was an investigation of the leading monomials according to fixed ordering. This approach originates from Gröbner bases theory, although we did not use these bases explicitly. Namely, we tried to establish as precisely as possible the shape of the leading monomials of elements from $k[X]^d$ according to the standard lexicographic ordering (after a convenient choice of the initial variable for this ordering). The aim was to delete the leading monomial using the generators, so as we could apply induction on the ordering.

Moreover, we often employ combinatorial methods, for instance to compare the coefficients of monomials of a given constant.

3. VOLTERRA DERIVATIONS

A Lotka-Volterra derivation with parameters $C_i = 1$ for all i is called a *Volterra* derivation (see e.g. [2]). The work [26] presents a description of the ring of constants of the Volterra derivation in four variables, which in this case has three algebraically independent generators:

Theorem 2. ([26], Theorem 3.1)

Let $R = k[x_1, \dots, x_4]$. Let $d : R \rightarrow R$ be the derivation of the form

$$d(x_i) = x_i(x_{i-1} - x_{i+1})$$

for $i = 1, \dots, 4$. Then

$$R^d = k[x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4].$$

In [26] there are also shown numerous facts for n variables. In particular, they concern the restrictions of constants to the polynomial rings in a smaller number of variables. Let $R_{(m)}$ denote the homogeneous component of $R = k[x_1, \dots, x_n]$ of degree m (since the derivation d is homogeneous, we need only search for homogeneous constants). For $\varphi \in R$ and for each subset $A \subseteq \{1, \dots, n\}$ denote by φ^A the sum of monomials of the polynomial φ that depend on variables with indices in A , that is, $\varphi^A = \varphi|_{x_j=0 \text{ for } j \notin A}$. As indicated in the previous section, to successfully perform complicated computations, the key idea was to investigate, instead of constants of the derivation d , polynomials φ such that $d(\varphi^A)^A = 0$ for various sets A (constants of the derivation d fulfill that condition, see [26], Corollary 2.8). This allowed to obtain some essential properties, ignoring at the same time in the calculations of a huge number of irrelevant data, greater than for the standard restriction. We quote below two examples of the results obtained in that way.

Proposition 3. ([26], Proposition 2.10)

Let $n \geq 3$. If $\varphi \in R_{(m)}$, $A = \{i, i + 1\} \subseteq \{1, \dots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A = c(x_i + x_{i+1})^m$ for some $c \in k$.

Proposition 4. ([26], Proposition 2.11)

Let $n \geq 4$. If $\varphi \in R_{(m)}$, $A = \{i, i + 1, i + 2\} \subseteq \{1, \dots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i + x_{i+1} + x_{i+2}, x_i x_{i+2}]$.

In [26] there are also given generalizations of various results also for Lotka-Volterra derivations with arbitrary parameters $C_i \in k$. And as it turned out once again (see: Jouanolou derivations, Hilbert's fourteenth problem), the cases of a small number of variables were more difficult (less independence in the cyclic sense, too high "density" of variables).

The work [28] gives a description of the ring of polynomial constants of the five-variable Volterra derivation.

Theorem 5. ([28], Theorem 4.1)

Let $R = k[x_1, \dots, x_5]$. Let $d : R \rightarrow R$ be the derivation of the form

$$d(x_i) = x_i(x_{i-1} - x_{i+1})$$

for $i = 1, \dots, 5$. Then

$$R^d = k\left[\sum_{j=1}^5 x_j, x_1x_3 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5, x_1x_2x_3x_4x_5\right].$$

Thus, starting from five variables there appear generators, which are linear forms of the shape: the sum of products of nonconsecutive variables (obviously the sum of all variables is also a trite case of this). It turns out that for n variables the ring of constants of the Volterra derivation is a polynomial ring, which generators are of a such form, plus the product of all variables for n odd (analogously to the case $n = 5$) and two products of variables of the same parity for n even (analogously to the case $n = 4$). It is conjectured in [28] that analogous results as for $n \leq 5$ remain valid also for an arbitrary number of variables, which was confirmed in [7].

In [28] the methods of proofs are based upon obtaining more extensive properties of polynomials φ fulfilling the condition $d(\varphi^A)^A = 0$ for suitable sets A . Amidst these facts was, among others:

Lemma 6. ([28], Lemma 3.1)

Let $n \geq 5$. If $\varphi \in R_{(m)}$, $A = \{i, i + 2, i + 3\} \subseteq \{1, \dots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i, x_{i+2} + x_{i+3}]$.

From which we can obtain the following, this time stronger properties of constants of the derivation d :

Lemma 7. ([28], Lemma 3.2)

Let $n \geq 5$. If $\varphi \in R_{(m)}^d$ and $A = \{i, i + 2, i + 3\} \subseteq \{1, \dots, n\}$, then $\varphi^A \in k[x_i + x_{i+2} + x_{i+3}, x_i(x_{i+2} + x_{i+3})]$.

Proposition 8. ([28], Proposition 3.5)

Let $n \geq 5$. If $\varphi \in R_{(m)}^d$ and $A = \{i, i + 1, i + 2, i + 3\} \subseteq \{1, \dots, n\}$, then $\varphi^A \in k[x_i + x_{i+1} + x_{i+2} + x_{i+3}, x_i x_{i+2} + x_i x_{i+3} + x_{i+1} x_{i+3}]$.

4. DARBOUX POLYNOMIALS AND LOTKA-VOLTERRA DERIVATIONS

Results of [33] are of two kinds. First, there are described the cofactors of strict Darboux polynomials of four-variable Lotka-Volterra derivations. Let $R = k[x_1, \dots, x_n]$. A polynomial $g \in R$ is called *strict* if it is homogeneous and not divisible by the variables x_1, \dots, x_n . For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we introduce the notation $X^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Clearly, every nonzero homogeneous polynomial $f \in R$ has a unique presentation $f = X^\alpha g$, where X^α is a monic monomial and g is a strict polynomial. Recall also that a nonzero polynomial f is said to be a *Darboux polynomial* of a derivation $\delta : R \rightarrow R$ if $\delta(f) = \Lambda f$ for some polynomial $\Lambda \in R$. We will call Λ a *cofactor* of f . In the following lemma we give the aforementioned description of cofactors:

Lemma 9. ([33], Lemma 3.2)

Let $n = 4$. Let $g \in R_{(m)}$ be a Darboux polynomial of a Lotka-Volterra derivation d with the cofactor $\lambda_1 x_1 + \dots + \lambda_4 x_4$. Let $i \in \{1, 2, 3, 4\}$. If g is not divisible by x_i , then $\lambda_{i+1} \in \mathbb{N}$. More precisely, if $g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_4) = x_{i+2}^{\beta_{i+2}} \overline{G}$ and $x_{i+2} \nmid \overline{G}$, then $\lambda_{i+1} = \beta_{i+2}$ and $\lambda_{i+3} = -C_{i+2} \lambda_{i+1}$.

Consequences of that lemma, which is a fairly technical nature, are more elegant:

Corollary 10. ([33], Corollary 3.3)

Let $n = 4$. If $g \in R_{(m)}$ is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in \mathbb{N} .

Lemma 11. ([33], Lemma 3.4)

Let $n = 4$. If $d(f) = 0$ and $f = X^\alpha g$, where g is a strict polynomial, then $d(X^\alpha) = 0$ and $d(g) = 0$.

In other words, for an arbitrary nonzero constant, in the factorization of the above type both the monomial factor and the strict factor are constants, too. Lemma 9 has turned out to be useful for investigation of polynomial constants ([33], [29]) and of rational constants ([30], [31], [32]). That lemma together with its potential generalizations seem crucial in a further study of rational constants.

The second result of [33] was a description of the ring of constants of four-variable Lotka-Volterra derivations in the generic case. It turns out that in such a case the ring of constants is trivial, that is, equal to k ([33], Theorem 5.1 and Corollary 5.2).

Among the methods used, besides the investigation of cofactors of strict Darboux polynomials, the second approach was to generalize results for Volterra derivations from [26] and [28] to cases of arbitrary parameters C_i , for example:

Proposition 12. ([33], Lemma 4.4)

Let $n \geq 3$. If $\varphi \in R_{(m)}$, $A = \{i, i + 1\} \subseteq \{1, \dots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i + C_i x_{i+1})^m$ for some $a \in k$.

The paper [29] gives a description of the rings of constants of four-variable Lotka-Volterra derivations depending on the parameters C_i , except the case when the product $C_1 C_2 C_3 C_4$ is a root of unity not equal to 1 (see [8]). Denote by \mathbb{N}_+ the set of positive integers, and by \mathbb{Q}_+ the set of positive rationals.

Consider the three sentences:

$$s_1 : C_1 C_2 C_3 C_4 = 1.$$

$$s_2 : C_1, C_3 \in \mathbb{Q}_+ \text{ and } C_1 C_3 = 1.$$

$$s_3 : C_2, C_4 \in \mathbb{Q}_+ \text{ and } C_2 C_4 = 1.$$

In case s_2 let $C_1 = \frac{p}{q}$, where $p, q \in \mathbb{N}_+$ and $\gcd(p, q) = 1$. In case s_3 let $C_2 = \frac{r}{t}$, where $r, t \in \mathbb{N}_+$ and $\gcd(r, t) = 1$. Denote by $\neg s_i$ the negation of the sentence s_i . We assume that $C_1 C_2 C_3 C_4$ is not nontrivial root of unity.

Theorem 13. ([29], Theorem 5.1)

Let $R = k[x_1, x_2, x_3, x_4]$ and $d : R \rightarrow R$ be a derivation of the form

$$d = \sum_{i=1}^4 x_i (x_{i-1} - C_i x_{i+1}) \frac{\partial}{\partial x_i},$$

where $C_1, C_2, C_3, C_4 \in k$. Then the ring of constants of d is always finitely generated over k with at most three generators. In each case it is a polynomial ring, more precisely:

- (1) if $s_1 \wedge \neg s_2 \wedge \neg s_3$, then $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$,
- (2) if $\neg s_1 \wedge \neg s_2 \wedge \neg s_3$, then $R^d = k$,
- (3) if $\neg s_1 \wedge \neg s_2 \wedge s_3$, then $R^d = k[x_2^t x_4^r]$,
- (4) if $\neg s_1 \wedge s_2 \wedge \neg s_3$, then $R^d = k[x_1^q x_3^p]$,
- (5) if $s_1 \wedge \neg s_2 \wedge s_3$, then $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4, x_2^t x_4^r]$,
- (6) if $s_1 \wedge s_2 \wedge \neg s_3$, then $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4, x_1^q x_3^p]$,
- (7) if $s_2 \wedge s_3$, then $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4, x_1^q x_3^p, x_2^t x_4^r]$.

Techniques of proofs used in [29] are extension of the methods from the former articles. For example, a useful tool is:

Lemma 14. ([29], Lemma 3.1)

Let $n \geq 4$, $\varphi \in R_{(m)}$, $i \in \{1, \dots, n\}$ and $A = \{i, i + 1, i + 2\}$. Let $C_i \in \mathbb{Q}_+$ and $C_i = \frac{p}{q}$, where $p, q \in \mathbb{N}_+$ and $\gcd(p, q) = 1$. If $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i + C_i x_{i+1} + C_i C_{i+1} x_{i+2}, x_i^q x_{i+2}^p]$.

While the complementary to the Lemma 14 case of $C_i \notin \mathbb{Q}_+$ had already been resolved in [33] (Lemma 4.5).

5. FIELDS OF RATIONAL CONSTANTS

The article [30] explores the fields of rational constants, that is, constants belonging to the field of rational functions. Recall that for any derivation $\delta : k[X] \rightarrow k[X]$ of the polynomial ring in n variables there exists exactly one derivation $\bar{\delta} : k(X) \rightarrow k(X)$ of the field of rational functions in n variables such that $\bar{\delta}|_{k[X]} = \delta$. By a *rational constant* of the derivation $\delta : k[X] \rightarrow k[X]$ we mean the constant of its corresponding derivation $\bar{\delta} : k(X) \rightarrow k(X)$. The rational constants of δ form a field. For simplicity, we write δ instead of $\bar{\delta}$. In [30] it is shown the following theorem:

Theorem 15. ([30], Theorem 2)

If d is the four-variable Volterra derivation, then

$$k(X)^d = k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4).$$

Thus, in this case $k(X)^d$ is the field of fractions of the ring $k[X]^d$ (which is not true in general, see e.g. [31], Example 1).

Theorem 15 was proved using an analysis of the cofactors of strict Darboux polynomials. In particular, it was shown ([30], Lemma 2) that every strict Darboux polynomial of the four-variable Volterra derivation is also a constant of this derivation.

The field of rational constants of a four-variable Lotka-Volterra derivation in the generic case was determined in [31]. Namely, it was demonstrated that in such a

generic case nontrivial rational constants do not exist ([31], Theorem 2). It was also determined the field of rational constants in some another case ([31], Theorem 2 again). In both cases these fields of constants are the fields of fractions of the rings of constants.

An important aspect of the results obtained are their applications. Therefore, in [31] there were investigated, similarly to [25], monomial derivations, but this time in four variables. Recall that a derivation $\delta : k(X) \rightarrow k(X)$ is monomial if

$$\delta(x_i) = x_1^{\beta_{i1}} \dots x_n^{\beta_{in}}$$

for $i = 1, \dots, n$, where each β_{ij} is an integer. It is shown how one can determine its rational constants using two tools, which are a description of strict Darboux polynomials ([33], Lemma 3.2) and so far proven results on constants of Lotka-Volterra derivations. This was demonstrated on the following example of a class of monomial derivations depending on four natural parameters s_i .

Theorem 16. ([31], Theorem 5)

Let $s_1, \dots, s_4 \in \mathbb{N}_+$, where $(s_1, s_3) \neq (1, 1)$ and $(s_2, s_4) \neq (1, 1)$. Let $D : k(X) \rightarrow k(X)$ be a derivation of the form

$$D(x_i) = x_{i-1}^{s_{i-1}+1} x_i^{s_i+1} x_{i+2}^{s_{i+2}}$$

for $i = 1, \dots, 4$ (in the cyclic sense). Then $k(X)^D = k$.

As we remember, a factorizable derivation $d : k[X] \rightarrow k[X]$ is called cyclic if $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \dots, n$ (and we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). Suppose that $A_i \neq 0$ for all i . Consider an automorphism $\sigma : k[X] \rightarrow k[X]$ defined by $\sigma(x_i) = A_{i+1}^{-1} x_i$ for $i = 1, \dots, n$. Then $\Delta = \sigma d \sigma^{-1}$ is also a derivation of the ring $k[X]$. Moreover, f is a nontrivial polynomial (respectively: rational) constant of a derivation d if and only if $\sigma(f)$ is a nontrivial polynomial (respectively: rational) constant of a derivation Δ . Clearly $\sigma^{-1}(x_i) = A_{i+1} x_i$ and $\Delta(x_i) = x_i(x_{i-1} - C_i x_{i+1})$ for $C_i = -B_i A_{i+2}^{-1}$ (we allow $C_i = 0$) and $i = 1, \dots, n$. We can proceed similarly if $A_i = 0$ for some i but $B_i \neq 0$ for all i .

A characterization of all four-variable Lotka-Volterra derivations with a nontrivial constant in the field of rational functions is given in [32]:

Proposition 17. ([32], Corollary 2)

If d is a four-variable Lotka-Volterra derivation, then $k(X)^d$ contains a nontrivial rational constant if and only if at least one of the following four conditions is fulfilled:

- (1) $C_1 C_2 C_3 C_4 = 1$,
- (2) $C_1, C_3 \in \mathbb{Q}$ and $C_1 C_3 = 1$,
- (3) $C_2, C_4 \in \mathbb{Q}$ and $C_2 C_4 = 1$,
- (4) $C_1 C_2 C_3 C_4 = -1$ and $C_i = 1$ for two consecutive indices i .

Note that the existence of a nontrivial polynomial constant is equivalent to similar four conditions, wherein in conditions (2) and (3) the set \mathbb{Q} is replaced by \mathbb{Q}_+ (a consequence of Theorem 1.2 from [8]).

In many of the cases we can describe the full fields of constants. Namely, consider the sentences:

$$\tilde{s}_2 : C_1, C_3 \in \mathbb{Q} \text{ and } C_1 C_3 = 1.$$

$$\tilde{s}_3 : C_2, C_4 \in \mathbb{Q} \text{ and } C_2 C_4 = 1.$$

Sentences s_1, s_2, s_3 and numbers p, q, r, t are as in Theorem 13. We define the sentence:

$$s_4 : C_1 C_2 C_3 C_4 = -1 \text{ and } C_i = 1 \text{ for two consecutive indices } i.$$

If the sentence s_4 is true we define the polynomial f_4 , namely for $C_1 = C_2 = 1$ let

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2 x_4^2 + 2x_1 x_2 - 2x_1 x_3 - 2C_3 x_1 x_4 + 2x_2 x_3 - 2C_3 x_2 x_4 + 2C_3 x_3 x_4,$$

for the other possibilities one has to rotate the indices appropriately.

Theorem 18. ([32], Theorem 4.1)

Let $d : k(X) \rightarrow k(X)$ be a four-variable Lotka-Volterra derivation with parameters $C_1, C_2, C_3, C_4 \in k$. Then:

- (1) if $\neg s_1 \wedge \neg \tilde{s}_2 \wedge \neg \tilde{s}_3 \wedge \neg s_4$, then $k(X)^d = k$,
- (2) if $s_1 \wedge \neg \tilde{s}_2 \wedge \neg \tilde{s}_3$, then $k(X)^d = k(x_1 + C_1 x_2 + C_1 C_2 x_3 + C_1 C_2 C_3 x_4)$,
- (3) if $\neg s_1 \wedge \neg \tilde{s}_2 \wedge \neg \tilde{s}_3 \wedge s_4$, then $k(X)^d = k(f_4)$,
- (4) if $\neg s_1 \wedge \neg \tilde{s}_2 \wedge s_3 \wedge \neg s_4$, then $k(X)^d = k(x_2^t x_4^r)$,
- (5) if $\neg s_1 \wedge s_2 \wedge \neg \tilde{s}_3 \wedge \neg s_4$, then $k(X)^d = k(x_1^q x_3^p)$,
- (6) if $\neg s_1 \wedge \neg \tilde{s}_2 \wedge s_3 \wedge s_4$, then $k(X)^d = k(f_4, x_2^t x_4^r)$,
- (7) if $\neg s_1 \wedge s_2 \wedge \neg \tilde{s}_3 \wedge s_4$, then $k(X)^d = k(f_4, x_1^q x_3^p)$,
- (8) if $s_1 \wedge \neg \tilde{s}_2 \wedge s_3$, then $k(X)^d = k(x_1 + C_1 x_2 + C_1 C_2 x_3 + C_1 C_2 C_3 x_4, x_2^t x_4^r)$,
- (9) if $s_1 \wedge s_2 \wedge \neg \tilde{s}_3$, then $k(X)^d = k(x_1 + C_1 x_2 + C_1 C_2 x_3 + C_1 C_2 C_3 x_4, x_1^q x_3^p)$,
- (10) if $s_2 \wedge s_3$, then $k(X)^d = k(x_1 + C_1 x_2 + C_1 C_2 x_3 + C_1 C_2 C_3 x_4, x_1^q x_3^p, x_2^t x_4^r)$.

The proof of the above theorem also uses investigations of the cofactors of strict Darboux polynomials.

6. RINGS OF CONSTANTS IN n VARIABLES

The paper [8] resolves in a complete way the problem of describing the rings of constants of Lotka-Volterra derivations for an arbitrary number of variables. The case $C_i = 1$ for all i was determined in [7]. All other cases are included in the following theorem:

Theorem 19. ([8], Theorem 1.1 and Theorem 1.2)

The ring of constants of Lotka-Volterra derivation in $n \geq 4$ variables is finitely generated over k with at most 3 generators, if there exists i such that $C_i \neq 1$. In every case it is a polynomial ring.

In [8] all of these rings of constants are determined in an effective way depending on n . It is presented in the following Theorems 20 and 25.

Let $f = \sum_{i=1}^n (\prod_{j=1}^{i-1} C_j) x_i = x_1 + C_1 x_2 + C_1 C_2 x_3 + \dots + C_1 C_2 \dots C_{n-1} x_n$. Moreover, consider nonempty subsets $\mathcal{A} \subseteq \mathbb{Z}_n$ of integers mod n closed under $i \mapsto i + 2$. If n is odd then $\mathcal{A} = \mathbb{Z}_n$, if n is even we have two additional subsets $\mathcal{E} = \{2i \mid i \leq n/2\}$ and $\mathcal{O} = \{2i - 1 \mid i \leq n/2\}$. For a given \mathcal{A} we define a polynomial $g_{\mathcal{A}}$ if there exist $\theta_i \in \mathbb{N}_+$ for $i \in \mathcal{A}$, such that $\theta_{i+2} = C_i \theta_i$. We can choose the set of θ_i coprime, then that numbers are uniquely determined. Then let $g_{\mathcal{A}} = \prod_{i \in \mathcal{A}} x_i^{\theta_i}$.

Theorem 20. ([8], Theorem 1.1)

Let $n > 4$ and let there exist i such that $C_i \neq 1$. Then the number of generators of the ring of constants of the Lotka-Volterra derivation with parameters C_1, \dots, C_n is equal to:

- 0 if $\prod C_i \neq 1$ and no $g_{\mathcal{A}}$ is defined;
- 3 if n is even and both $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ are defined;
- 2 if n is odd and $g_{\mathbb{Z}_n}$ is defined, or n is even and $\prod C_i = 1$ but only one of $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ is defined;
- 1 in all other cases.

The generators are always those polynomials $g_{\mathcal{A}}$ that are defined together with f if $\prod C_i = 1$.

To prove the theorem above, we have to show that the aforementioned generators are constants, which is a quick calculation, that these generators are algebraically independent, which can be shown by standard methods using the Jacobian, and that there are no constants not belonging to the polynomial ring with generators given above, which is practically entire difficulty of the proof. In order to establish this last condition we tried to describe as precisely as possible the shape of the leading monomials of polynomial constants according to the standard lexicographic ordering on monomials of a fixed degree. To then be able to eliminate such a leading monomial using generators and to be able to apply the induction (on the aforementioned ordering). This is achieved by several auxiliary facts. We quote below a selection of them.

Assume $C_n \neq 1$. Consider the standard lexicographic ordering. Suppose that h is a counterexample to Theorem 20 with the smallest leading monomial according to the ordering under consideration. Let $m_1 = \prod_{i=1}^n x_i^{\alpha_i}$ be that leading monomial. Let $M(h)$ denote the set of monomials occurring in h with a nonzero coefficient.

Proposition 21. ([8], Proposition 2.5)

Suppose $m = \prod_{i=1}^n x_i^{\gamma_i}$ is a monomial and r is a positive integer with the following properties:

- (1) $\gamma_n = \alpha_n$,
- (2) $\gamma_{2i-1} = \alpha_{2i-1}$ for $1 \leq i \leq r$,
- (3) $\gamma_{2i} = C_{2i-2}\gamma_{2i-2}$ for $1 \leq i \leq r-1$,
- (4) $\gamma_{2r} \neq C_{2r-2}\gamma_{2r-2}$.

Then $m \notin M(h)$.

Note that the above proposition implies that the even-indexed exponents of m are uniquely determined by α_n . The odd-indexed exponents are determined only up to a certain extent, as described in the following proposition.

Proposition 22. ([8], Proposition 2.7)

Suppose $m = \prod_{i=1}^n x_i^{\gamma_i} \in M(h)$ is a monomial and $r < n/2$ is a positive integer with the following properties:

- (1) $\gamma_n = \alpha_n$ (or $C_n = 0$),
- (2) $\gamma_{2i-1} = \alpha_{2i-1}$ for $1 \leq i \leq r$,
- (3) $\gamma_{2i} = \alpha_{2i}$ for $1 \leq i \leq r$.

Then there exists a nonnegative integer β_{2r-1} such that $C_{2r-1}(\gamma_{2r-1} - \beta_{2r-1}) = \gamma_{2r+1}$ and $m' = m(x_{2r}/x_{2r-1})^{\beta_{2r-1}} \in M(h)$. In particular, there exist nonnegative integers β'_{2i-1} such that $C_{2i-1}(\alpha_{2i-1} - \beta'_{2i-1}) = \alpha_{2i+1}$ for $1 \leq i < n/2$.

The next result enables further reductions and some kind of substitutions of monomials in a constant h and, on the other hand, forces certain conditions on the parameters C_i .

Proposition 23. ([8], Corollary 2.9)

Suppose $C_n \neq 0$ and $m = \prod x_i^{\gamma_i} \in M(h)$ is such that $\gamma_n = \alpha_n$, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2 = C_n\alpha_n$. Then $l = \gamma_1 - C_{n-1}\gamma_{n-1}$ is a nonnegative integer and $m' = m(x_n/x_1)^l \in M(h)$. In particular, $\alpha_1 - C_{n-1}\alpha_{n-1}$ is a nonnegative integer.

It also turns out that some types of constants may occur only under certain conditions on the product of parameters C_i .

Proposition 24. ([8], Proposition 2.10)

Let $h, g \in k[X]^d$, where g is a monomial. If the leading monomial of h is $m_1 = x_1^s g$ with $s > 0$, then $C_1 C_2 \cdots C_n$ is s -th root of unity, in particular, all $C_i \neq 0$. If further $n > 4$, then $C_1 C_2 \cdots C_n = 1$.

We now proceed to the case $n = 4$. In this case it may occur some generator of the ring of constants, which does not occur for $n > 4$. Namely, if $C_1 C_2 C_3 C_4 = -1$ and for two consecutive indices i we have $C_i = 1$ (see already [26], Proposition 4.4, condition (4)). Without loss of generality, if $C_1 = C_2 = 1$ and $C_4 = -1/C_3$, then that generator is equal to

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2 x_4^2 + 2x_1 x_2 - 2x_1 x_3 - 2C_3 x_1 x_4 + 2x_2 x_3 - 2C_3 x_2 x_4 + 2C_3 x_3 x_4.$$

The following theorem describes the ring of constants for 4 variables.

Theorem 25. ([8], Theorem 1.2)

Assume $n = 4$ and let there exist i such that $C_i \neq 1$. Then the number of generators of the ring of constants of the Lotka-Volterra derivation with parameters C_1, \dots, C_n is equal to:

- 0 if $\prod C_i \neq 1$ and none of $g_{\mathcal{O}}, g_{\mathcal{E}}, f_4$ is defined;
- 3 if both $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ are defined;
- 2 if $\prod C_i = 1$ but only one of $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ is defined or one of parameters C_i is equal to -1 and the other three are equal to 1;
- 1 in all other cases.

The generators are always those polynomials $g_{\mathcal{A}}$ that are defined together with f_4 if $\prod C_i = -1$ and two consecutive parameters are equal to 1 or together with f if $\prod C_i = 1$.

The case of 4 variables has turned out to be the most difficult to prove. Already in the previously cited facts occurred some distinctions for this case (see e.g. Proposition 24). Also needed were results specific to the four variables, for example:

Proposition 26. ([8], Proposition 2.11)

Let $n = 4$ and $h, g \in k[X]^d$, where g is a monomial. Assume either every C_i is positive rational, or C_4 is not. If the leading monomial of h is $m_1 = x_1^s g$ with $s > 0$, then $C_1 C_2 C_3 C_4 = \pm 1$. If $C_1 C_2 C_3 C_4 = -1$, then $C_2 = 1$ and at least one of $C_1 = 1$ or $C_3 = 1$ also holds.

REFERENCES

- [1] M. A. Almeida, M. E. Magalhães, I. C. Moreira, *Lie symmetries and invariants of the Lotka-Volterra system*, J. Math. Phys. 36 (1995), 1854–1867.
- [2] O. I. Bogoyavlenskii, *Algebraic constructions of integrable dynamical systems - extension of the Volterra system*, Russian Math. Surveys 46 (1991), 1–64.
- [3] F. H. Busse, *Transitions to Turbulence Via the Statistical Limit Cycle Route*, Synergetics, Springer-Verlag, Berlin 1978.
- [4] L. Cairó, *Darboux First Integral Conditions and Integrability of the 3D Lotka-Volterra System*, J. Nonlinear Math. Phys. 7 (2000), 511–531.
- [5] L. Cairó, J. Llibre, *Darboux integrability for 3D Lotka-Volterra systems*, J. Phys. A 33 (2000), 2395–2406.
- [6] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, 190. Birkhäuser Verlag, Basel 2000.
- [7] P. Hegedús, *The constants of the Volterra derivation*, Cent. Eur. J. Math. 10 (2012), 969–973.
- [8] P. Hegedús, J. Zieliński, *The constants of Lotka-Volterra derivations*, Eur. J. Math. 2 (2016), 544–564.
- [9] Y. Itoh, *Integrals of a Lotka-Volterra System of Odd Number of Variables*, Progr. Theoret. Phys. 78 (1987), 507–510.
- [10] S. Kuroda, *Fields defined by locally nilpotent derivations and monomials*, J. Algebra 293 (2005), 395–406.
- [11] S. Labrunie, *On the polynomial first integrals of the (a, b, c) Lotka-Volterra system*, J. Math. Phys. 37 (1996), 5539–5550.

- [12] W. E. Lamb, *Theory of an Optical Maser*, Phys Rev. A 134 (1964), 1429.
- [13] G. Laval, R. Pellat, *Plasma Physics*, Proceedings of Summer School of Theoretical Physics, Gordon and Breach, New York 1975.
- [14] A. J. Lotka, *Analytical Note on Certain Rhythmic Relations in Organic Systems*, Proc. Natl. Acad. Sci. U.S. 6 (1920), 410–415.
- [15] R. Lupini, G. Spiga, *Chaotic Dynamics of Spatially Homogeneous Gas Mixtures*, Phys. Fluids. 31 (1988), 2048–2051.
- [16] A. J. Maciejewski, J. Moulin Ollagnier, A. Nowicki, J.-M. Strelcyn, *Around Jouanolou non-integrability theorem*, Indag. Math. (N.S.) 11 (2000), 239–254.
- [17] J. Moulin Ollagnier, *Liouvillian first integrals of homogeneous polynomial 3-dimensional vector fields*, Colloq. Math. 70 (1996), 195–217.
- [18] J. Moulin Ollagnier, *Polynomial first integrals of the Lotka-Volterra system*, Bull. Sci. Math. 121 (1997), 463–476.
- [19] J. Moulin Ollagnier, *Rational integration of the Lotka-Volterra system*, Bull. Sci. Math. 123 (1999), 437–466.
- [20] J. Moulin Ollagnier, A. Nowicki, *Polynomial algebra of constants of the Lotka-Volterra system*, Colloq. Math. 81 (1999), 263–270.
- [21] J. Moulin Ollagnier, A. Nowicki, *Constants of cyclotomic derivations*, J. Algebra 394 (2013), 92–119.
- [22] V. W. Noonburg, *A Neural Network Modeled by an Adaptive Lotka-Volterra System*, SIAM J. Appl. Math. 49 (1989), 1779–1792.
- [23] A. Nowicki, *Commutative bases of derivations in polynomial and power series rings*, J. Pure Appl. Algebra 40 (1986), 275–279.
- [24] A. Nowicki, *Polynomial derivations and their rings of constants*, Nicolaus Copernicus University, Toruń 1994.
- [25] A. Nowicki, J. Zieliński, *Rational constants of monomial derivations*, J. of Algebra 302 (2006), 387–418.
- [26] P. Ossowski, J. Zieliński, *Polynomial algebra of constants of the four variable Lotka-Volterra system*, Colloq. Math. 120 (2010), 299–309.
- [27] V. Volterra, *Leçons sur la Théorie Mathématique de la Lutte pour la vie*, Gauthier Villars, Paris 1931.
- [28] J. Zieliński, *The five-variable Volterra system*, Cent. Eur. J. Math. 9 (2011), 888–896.
- [29] J. Zieliński, *Rings of constants of four-variable Lotka-Volterra systems*, Cent. Eur. J. Math. 11 (2013), 1923–1931.
- [30] J. Zieliński, *The field of rational constants of the Volterra derivation*, Proc. Est. Acad. Sci. 63 (2014), 133–135.
- [31] J. Zieliński, *Rational constants of generic LV derivations and of monomial derivations*, Bull. Pol. Acad. Sci. Math. 61 (2013), 201–208.
- [32] J. Zieliński, *Fields of rational constants of cyclic factorizable derivations*, Electron. J. Differential Equations 2015 (2015), 1–7.
- [33] J. Zieliński, P. Ossowski, *Rings of constants of generic 4D Lotka-Volterra systems*, Czechoslovak Math. J. 63 (2013), 529–538.
- [34] H. Żołądek, *Multi-dimensional Jouanolou system*, J. Reine Angew. Math. 556 (2003), 47–78.

NICOLAUS COPERNICUS UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND

E-mail address: `ubukrool@mat.uni.torun.pl`