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A FAMILY OF HYPERBOLAS ASSOCIATED TO A TRIANGLE

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Abstract. In this note, we explore an apparently new one parameter family of conics associated to a triangle. Given a triangle we study ellipses whose one axis is parallel to one of sides of the triangle. The centers of these ellipses move along three hyperbolas, one for each side of the triangle. These hyperbolas intersect in four common points, which we identify as centers of incircle and the three excircles of the triangle. Thus they belong to a pencil of conics. We trace centers of all conics in the family and establish a surprising fact that they move along the excircle of the triangle. Even though our research is motivated by a problem in elementary geometry, its solution involves some non-trivial algebra and appeal to effective computational methods of algebraic geometry. Our work is illustrated by an animation in Geogebra and accompanied by a Singular file.

1. Introduction

Let $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2)$ be non-collinear points in the real affine plane. Their coordinates satisfy thus the condition

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \neq 0$$

In [8] we proved the following result.

Theorem 1.1. *We consider ellipses whose one axis is parallel to a fixed side of the triangle (the line containing its two vertices). Then*

- *such ellipses form a one dimensional family;*
- *their centers move along a hyperbola which passes through the incenter S and the three excenters A', B', C' of the triangle.*

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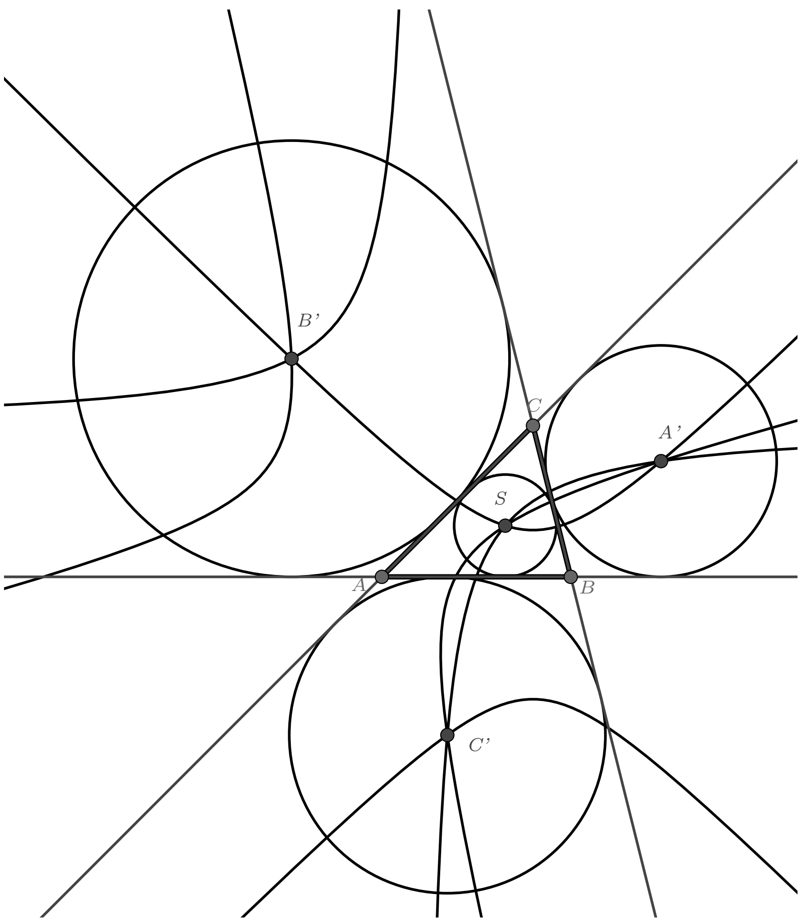
Remark 1.2. By a center of a conic, we understand its center of symmetry. If the conic is degenerate and consists of two intersecting lines then its center is the intersection point of both lines. If the lines are parallel, then we declare the corresponding point at the infinity as their center. We are aware of the fact that for two parallel lines there are infinitely many centers of symmetry but it is convenient and consistent with our approach to declare the point they share at infinity as their center.

Corollary 1.3. *It follows immediately from Theorem 1.1 that taking the three hyperbolas corresponding to each of the sides of the triangle, they all belong to a pencil of conics determined by points* S, A', B' *and* C'*. This is illustrated in Figure 1.*

Figure 1.



Remark 1.4. Note that another three hyperbolas associated to a triangle have been identified in 1957 by Court. However his construction is not related to ours.

There are well-known formulas we allow us to compute coordinates of points S, A', B' and C' explicitly:

$$
\begin{aligned}
S &= \left(\frac{a_1\widetilde{a} + b_1\widetilde{b} + c_1\widetilde{c}}{\widetilde{a} + \widetilde{b} + \widetilde{c}},\ \frac{a_2\widetilde{a} + b_2\widetilde{b} + c_2\widetilde{c}}{\widetilde{a} + \widetilde{b} + \widetilde{c}} \right),\\
A' &= \left(\frac{-a_1\widetilde{a} + b_1\widetilde{b} + c_1\widetilde{c}}{-\widetilde{a} + \widetilde{b} + \widetilde{c}},\ \frac{-a_2\widetilde{a} + b_2\widetilde{b} + c_2\widetilde{c}}{-\widetilde{a} + \widetilde{b} + \widetilde{c}} \right),\\
B' &= \left(\frac{a_1\widetilde{a} - b_1\widetilde{b} + c_1\widetilde{c}}{\widetilde{a} - \widetilde{b} + \widetilde{c}},\ \frac{a_2\widetilde{a} - b_2\widetilde{b} + c_2\widetilde{c}}{-\widetilde{a} - \widetilde{b} + \widetilde{c}} \right),\\
C' &= \left(\frac{a_1\widetilde{a} + b_1\widetilde{b} - c_1\widetilde{c}}{\widetilde{a} + \widetilde{b} - \widetilde{c}},\ \frac{a_2\widetilde{a} + b_2\widetilde{b} - c_2\widetilde{c}}{\widetilde{a} + \widetilde{b} - \widetilde{c}} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{a} &= \sqrt{(b_1 - c_1)^2 + (b_2 - c_2)^2},\\
\widetilde{b} &= \sqrt{(a_1 - c_1)^2 + (a_2 - c_2)^2},\\
\widetilde{c} &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.
\end{aligned}
$$

However, since the formulas involve taking roots, they are not so convenient for symbolic computations. We circumvent this difficulty in the next section.

2. Conics determined by four points.

Let $Z = \{P_1, \ldots, P_4\}$ be a set of four points in the plane such that no three of them are collinear. For a subset $V \subset \mathbb{R}^2$ we denote by $I(V)$ its saturated ideal, i.e., the ideal in the polynomial ring $\mathbb{R}[x, y]$ consisting of all polynomials vanishing at all points of V. Then we have

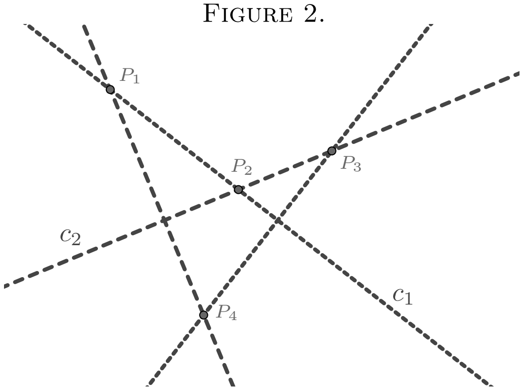
$$I(Z) = I(P_1) \cap \ldots \cap I(P_4).$$

Using the geometry of Z, it is easy to determine generators of $I(Z)$. To this end note that $I(P_i \cup P_j)$ for $i \neq j$ contains a unique (up to a multiplicative factor) element $\ell_{i,j}$ of degree 1 (namely the equation of the line through P_i and P_j). Then, with $c_1 = \ell_{1,2} \cdot \ell_{3,4}$ and $c_2 = \ell_{1,3} \cdot \ell_{2,4}$ we have

$$I(Z) = \langle c_1,\ c_2 \rangle.$$

Geometrically, the set Z is then the intersection of conics c_1 and c_2. This is illustrated in Figure 2.

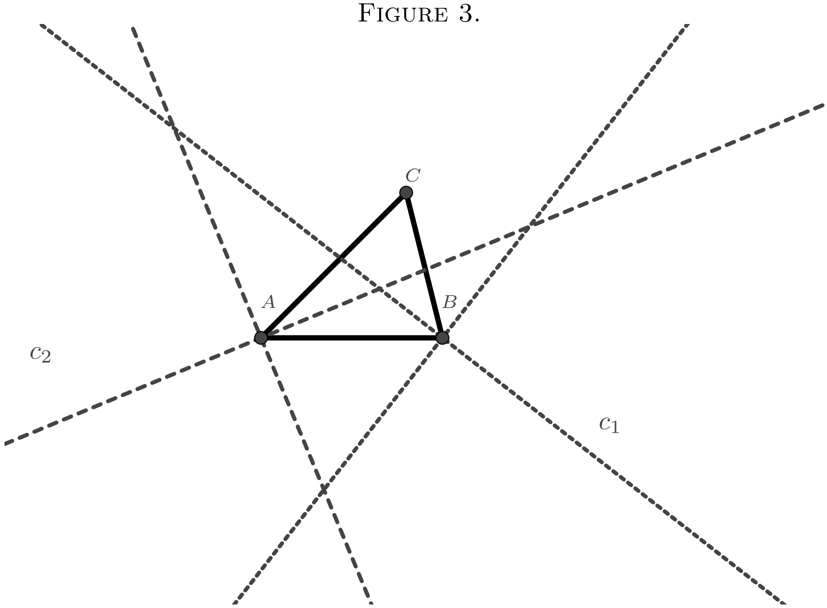
FIGURE 2.



Note that also $c_3 = \ell_{1,4} \cdot \ell_{2,3}$ is an element of $I(Z)$. It can be written down as a linear combination of c_1 and c_2. In the linear system of all conics determined by Z there are exactly three degenerate conics c_1, c_2 and c_3.

Turning back to our situation, we consider $Z = \{S, A', B', C'\}$. Then the lines joining pairs of points in Z have additional geometric meaning: They are either bisectors of angles of the triangle or bisectors of its exterior angles. This is depicted in Figure 3.

FIGURE 3.



Since we are interested in the union of the bisector of an angle of a triangle and the bisector of the exterior angle rather than one of these lines separately, by a slight abuse of the language, we introduce the folowing notion.

Definition 2.1. Let ℓ_1 and ℓ_2 be two distinct lines intersecting at a point P. The *bibisector* of ℓ_1 and ℓ_2 is the union of bisectors of angles formed by the two lines. We denote the bibisector by $\mathrm{bibi}(\ell_1, \ell_2)$, or if there is no ambiguity about the lines just by $\mathrm{bibi}(P)$.

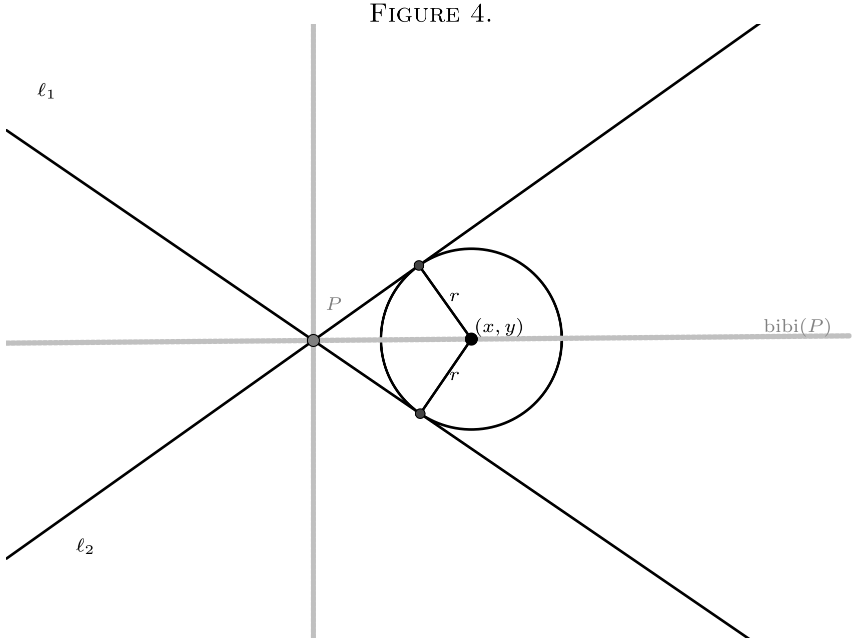
The next Lemma shows how surprisingly easy it is to derive the equation of $\mathrm{bibi}(\ell_1, \ell_2)$ out of equations of lines ℓ_1 and ℓ_2.

Lemma 2.2. *Let ℓ_1 be given by the equation $Ax + By + C = 0$ and let ℓ_2 be given by $\widetilde{A}x + \widetilde{B}y + \widetilde{C} = 0$. Then the bibisector of ℓ_1 and ℓ_2 is given by*

$$\frac{(Ax + By + C)^2}{A^2 + B^2} = \frac{(\widetilde{A}x + \widetilde{B}y + \widetilde{C})^2}{\widetilde{A}^2 + \widetilde{B}^2}. \tag{1}$$

Proof. A geometric property of the bibisector is that it consists of points equidistant to both lines. In other words, we are looking for the locus of points (x, y) subject to the condition for certain $r > 0$, the circle centered at (x, y) is tangent to lines ℓ_1 and ℓ_2, see Figure 4.

FIGURE 4.



A point (x, y) is equidistant to lines ℓ_1 and ℓ_2 if and only if its coordinates satisfy (1) and we are done. As expected, the equation is quadratic in x and y. □

Corollary 2.3. *In the set up of the triangle ABC, Lemma 2.2 we obtain*

$$\begin{aligned}\operatorname{bibi}(A):\ & \frac{\left((a_2-b_2)\,x+(b_1-a_1)\,y+a_1\,b_2-a_2\,b_1\right)^2}{(a_2-b_2)^2+(b_1-a_1)^2}\\ & -\frac{\left((a_2-c_2)\,x+(-a_1+c_1)\,y+a_1\,c_2-a_2\,c_1\right)^2}{(a_2-c_2)^2+(a_1-c_1)^2}\end{aligned}$$

$$\begin{aligned}\operatorname{bibi}(B):\ & \frac{\left((b2-a2)\,x+(a1-b1)\,y+a2\;b1-a1\;b2\right)^2}{(b2-a2)^2+(a1-b1)^2}\\ & -\frac{\left((b2-c2)\,x+(-b1+c1)\,y+b1\;c2-b2\;c1\right)^2}{(b2-c2)^2+(-b1+c1)^2}\end{aligned}$$

Corollary 2.4. *Since $Z=\{S,A',B',C'\}$ has two generators in degree 2, every element $f\in(I(Z))_2$ can be written as*

$$f=s\cdot\operatorname{bibi}(A)+t\cdot\operatorname{bibi}(B)$$

for some real numbers s,t.

3. Main result

We begin with a Lemma which provides coordinates of the center of a conic.

Lemma 3.1. *Let $g(x,y)=ax^2+by^2+c+2dxy+2ex+2fy$ be a polynomial of degree 2 in an affine real plane. We assume $ab-d^2\neq 0$, i.e., we assume that the set of zeroes of g is not a parabola. Then g describes either an ellipse, if $d^2-ab<0$ or a hyperbola if $d^2-ab>0$. In both cases the curves could be degenerate but in both cases they poses a center of symmetry. More precisely, the point*

$$S=\left(\tfrac{df-be}{ab-d^2},\ \tfrac{de-af}{ab-d^2}\right). \tag{2}$$

is the center of the conic $\{\ g=0\ \}$.

Proof. Since the proof is elementary but also rather technical and lengthy, we refer to [6, Section 6.3] for details. □

From now on, it is convenient to work with projective coordinates, rather than with affine. In particular, this approach allows us to express coordinates of the center of a conic by polynomials in the coefficients of the conic, rather than by rational functions of these coefficients. Indeed, we have in (2)

$$S=(df-be\ :\ de-af\ :\ ab-d^2). \tag{3}$$

We shall need also the following property of a circle viewed as a complex projective conic.

Lemma 3.2. *Let Γ be a circle. Then its complex projective completion contains points $J_1=(1:i:0)$ and $J_2=(1:-i:0)$.*

Proof. Let Γ be given by the equation

$$(x-a)^2 + (y-b)^2 = r^2,$$

where (a, b) are coordinates of its center and r is the radius. We homogenize the equation with a new variable z and obtain

$$(x - az)^2 + (y - bz)^2 = r^2 z^2. \tag{4}$$

Computing the points at infinity, we insert $z = 0$ and get

$$x^2 + y^2 = 0.$$

It is now clear that the points J_1 and J_2 satisfy this equation. □

Remark 3.3. It is easy to see that Lemma 3.2 has an inverse. By this we mean that any complex conic Γ passing through points J_1 and J_2 can be written down in the form of equation (4) for some *complex* numbers a, b and r.

Now we are in the position to state the main result of this note. Animation [9], prepared in Geogebra and available online, illustrates this result.

Theorem 3.4. *Let S, A', B', C' be the incenter and the excenters of a triangle ABC. Let $\mathcal{C}$ be the pencil of conics passing through these 4 points. Then the locus of centers of conics in $\mathcal{C}$ is the excircle of the triangle ABC.*

Proof. According to Corollary 2.4 any element $\mathcal{C}_{(s:t)}$ of $\mathcal{C}$ is defined by an equation of the form

$$f_{(s:t)} = s\,\mathrm{bibi}(A) + t\,\mathrm{bibi}(B),$$

where $(s : t) \in \mathbb{P}^1$, $\mathrm{bibi}(A)$ and $\mathrm{bibi}(B)$ are conics defined in Corollary 2.3.

Using Lemma 3.1 we obtain projective coordinates of the centers $S_{s:t}$ of $\mathcal{C}_{(s:t)}$ expressed as polynomials depending on parameters $(s : t)$. Since the particular formulas are rather obscure, we omit them in this presentation. A motivated reader will easily recover them using any symbolic algebra system. We used Singular.

Eliminating the parameters $(s : t)$ from the equations of the coordinates of $S_{(s:t)}$ and dehomogenizing (i.e. setting $z = 1$) we obtain the following quadratic equation in variables x and y.

$$\begin{aligned}
N(x,y) =&(a_1b_2 - a_1c_2 - a_2b_1 + a_2c_1 + b_1c_2 - b_2c_1)x^2\\
&+ (a_1b_2 - a_1c_2 - a_2b_1 + a_2c_1 + b_1c_2 - b_2c_1)y^2\\
&+ (-a_1^2b_2 + a_1^2c_2 - a_2^2b_2 + a_2^2c_2 + a_2b_1^2 + a_2b_2^2\\
&- a_2c_1^2 - a_2c_2^2 - b_1^2c_2 - b_2^2c_2 + b_2c_1^2 + b_2c_2^2)x\\
&+ (a_1^2b_1 - a_1^2c_1 - a_1b_1^2 - a_1b_2^2 + a_1c_1^2 + a_1c_2^2\\
&+ a_2^2b_1 - a_2^2c_1 + b_1^2c_1 - b_1c_1^2 - b_1c_2^2 + b_2^2c_1)y\\
&+ (-a_1^2b_1c_2 + a_1^2b_2c_1 + a_1b_1^2c_2 + a_1b_2^2c_2 - a_1b_2c_1^2 - a_1b_2c_2^2\\
&- a_2^2b_1c_2 + a_2^2b_2c_1 - a_2b_1^2c_1 + a_2b_1c_1^2 + a_2b_1c_2^2 - a_2b_2^2c_1) = 0
\end{aligned}$$

Thus N is the equation of a curve of degree 2 which contains all centers of conics in the pencil $\mathcal{C}$.

It remains to check that N defines the excircle of the triangle ABC. To this end we just check that coordinates of points A, B and C satisfy N. We omit easy calculations. Finally we check that also points J_1 and J_2 defined in Lemma 3.2 belong to the zero locus of N. Hence N is a circle passing through A, B and C. But there is just one such circle, namely the excircle of the triangle ABC and we are done. □

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