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AN INVITATION TO THE POSITIVITY AND GEOMETRY OF ALGEBRAIC CYCLES

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1. Problems

The purpose of this work is an introduction and overview of geometric and numeric properties of algebraic cycles in smooth projective varieties. We recall or propose several problems, which we consider worth to study. We are mainly interested in, but do not restrict our story to, codimension 2 cycles in projective spaces. These are points in $\mathbb{P}^2$, curves in $\mathbb{P}^3$, surfaces in $\mathbb{P}^4$ and so on.

We focus on the *positivity aspects* of such cycles on the one hand, and on attached *asymptotic invariants* on the other hand.

Whereas positivity for divisors (i.e. codimension 1 cycles) is either well understood or there is at least a clear conjectural picture, the study of positivity of higher codimension cycles has been taken on seriously in this century and the theory is much less developed.

Historically, there has been a lot of interest in the geometry of space curves. A lot of research focused on the study of Hilbert schemes $H(d, g)$ of smooth, irreducible curves of degree d and genus g in $\mathbb{P}^N$. The Hilbert scheme perspective is naturally associated to degeneration techniques. Whereas these techniques are inevitable our objective is to inform about another approach motivated by recent results of Fulger, Lehmann and others.

One of the most fundamental questions asked, in the context of positivity, about divisors is whether a divisor is effective and if it is so, the next question is: what is the dimension of the linear system it lives in. Similar approach can be taken on studying cycles of higher codimension. We recall here the relevant invariant, which

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is the *mobility count* as introduced in early works of Daniel Perrin [25] and which is a natural generalisation of the dimension of a linear system of divisors.

Let X be a smooth variety of dimension n and let $\alpha \in N_k(X)$ be an effective integral k-cycle (k is here the *dimension* of the support of α, thus $k = n-1$ if α is a divisor). The *mobility count* $\mathrm{mc}(\alpha)$ of α is the maximal number of general points in X that can be imposed on the class of α (i.e. there exists an effective cycle, numerically equivalent to α, passing through all these points). Note that for a divisor, the mobility count is essentially (up to some issues between the numerical and linear equivalence) the dimension of the linear system generated by this divisor.

It is natural to expect that if X has a simple, or well-known structure, the mobility count should be easily performed. Surprisingly, this is not the case! Already in the case of curves in $\mathbb{P}^3$ the picture is far from being complete. Indeed, given a positive number s, Perrin asked what is the minimal degree $d(s)$ of a curve $C \subset \mathbb{P}^3$, with $\mathrm{mc}(C) \geqslant s$. Interestingly, it is not known in general. Note that for divisors the same question is an elementary exercise. This motivates the first problem we propose to study.

Problem 1.1. *Find new, lower and upper, bounds on the numbers $d(s)$. Additionally, introduce and study numbers $d(s,m)$ corresponding to cycles which pass through s general points and have there multiplicity at least m (thus $d(s) = d(s,1)$ for all $s \geqslant 1$).*

We expect that this is a hard problem. Introducing the multiplicities is hard already in the setting of divisors. On the other hand, even partial results in this direction would mark a landscape, which seems unexplored to large extend so far.

The volume of a divisor can be thought of as an asymptotic version of the dimension of the linear system. An important advantage of the volume when compared with the dimension of the linear series is that the volume depends only on the numerical equivalence class of the underlying divisor, whereas the dimension depends in general on the linear equivalence class. Motivated by the concept of the volume of divisors Lehman [20] and Xiao [30] introduced the notion of the *mobility*. For cycles of dimension k the mobility is defined as follows

$$\mathrm{mob}(\alpha) = \limsup_{m\to\infty} \frac{\mathrm{mc}(m\alpha)}{m^{n/(n-k)}/n!}.$$

Thus if the codimension of α is 1, then $\mathrm{mob}(\alpha) = \mathrm{vol}(\alpha)$, i.e., we recover the volume of the divisor. For cycles of higher codimension mobility parallels essential properties of the volume. In particular, Lehman [20, Theorem 1.2] showed that the mobility is a continuous $\frac{n}{n-k}$-homogeneous function on the space of k-cycles $N_k(X)$.

Whereas, at least for big and nef divisors, the count of the volume reduces to the simple computation of the self-intersection number of the divisor, the picture is much more mysterious for cycles of higher codimension. Suffices it to say that already the number $\mathrm{mob}(\ell)$ for a line $\ell \subseteq \mathbb{P}^3$ is not known! The best estimate up to date $1 \leqslant \mathrm{mob}(\ell) \leqslant 3.54$ is due to Lehmann [20]. Establishing the exact value of $\mathrm{mob}(\ell)$ is related to the enumerative problem mentioned above: what is the minimal degree of a curve in $\mathbb{P}^3$ passing through s general points? It is expected that the answer is governed by the values of $(6s)^{\frac{2}{3}}$ but this is not known. Accordingly, it is conjectured that the actual value is $\mathrm{mob}(\ell) = 1$. All this motivates he second problem we put forward.

Problem 1.2. *Improve the upper bound on the number* $\mathrm{mob}(\ell)$. *Or even show* $\mathrm{mob}(\ell) = 1$.

In fact, it is expected that complete intersection curves are subject to the following conjectural statement.

Conjecture 1.3 (Lehmann)**.** *Let X be a projective variety of dimension n and let L be an ample line bundle on X. Then for all k in the range $0 < k < n$ there is*

$$\mathrm{mob}(L^{n-k}) = \mathrm{vol}(L).$$

It is natural to wonder what happens for cycles which are not complete intersections. Thus we will be also interested in the number $\mathrm{mob}(T)$ for T, a twisted cubic. This is the first interesting example of a non-complete intersection curve in $\mathbb{P}^3$. As in the case of Problem 1.1 for points of higher multiplicity, we were not able to trace down any results in this direction in the literature. Note, that almost certainly it is not $\mathrm{mob}(T) = 3\,\mathrm{mob}(\ell)$.

Both problems put forward so far concern postulation on codimension 2 cycles. However, such cycles can be used themselves to formulate postulation problems on divisors. The best known examples of such problems concern points in $\mathbb{P}^2$. We mention here two most prominent conjectures in the field. Nagata's Conjecture, going back to 1959, is analogous to Problem 1.1. It predicts that the minimal degree $n(s, m)$ of a plane curve passing through $s \geqslant 10$ general points with multiplicity at least m is subject to the following inequality

$$n(s, m) > m\sqrt{s}.$$

The Segre-Harbourne-Gimigliano-Hirschowitz Conjecture (SHGH for short) is somewhat more geometrical in nature. Since this is not our objective to study this conjecture, we provide a somewhat simplified formulation, basically due to Segre. Given s general points in $\mathbb{P}^2$, the linear system of curves passing through these points with fixed multiplicities is either non-special (i.e. its dimension is provided by a simple calculation of Hilbert polynomials) or the system contains a non-reduced base curve.

Passing to the postulation of higher dimensional cycles, it is natural to replace, in the first step, points by flats, i.e., linear subspaces in projective spaces. The landscape here is much less explored. It has been proved by Hartshorne and Hirschowitz in [15] that a general collection of lines in $\mathbb{P}^N$ imposes independent conditions on forms of any degree d. In other words, their result shows that finding the minimal degree of a hypersurfaces containing s general lines in $\mathbb{P}^N$ boils down to an easy calculation of Hilbert functions. Allowing only one fat line introduces a lot of complications, see our work with Bauer, Di Rocco, Schmitz and Szemberg [3] and the work of Aladpoosh [1]. To the best of our knowledge the postulation problem for planes in $\mathbb{P}^4$ is not solved. An obvious, new complication, arising here, is that planes in $\mathbb{P}^4$ are not disjoint. This gets even more involved for unions of codimension 2 flats in higher dimensional projective spaces as not only the flats intersect each other but their intersections interact with other intersections as well. This makes our next problem pretty challenging.

Problem 1.4. *Study the postulation problem for unions of general flats of codimension* 2 *in projective spaces.*

We expect that there is no simple analogy of the Hartshorne-Hirschowitz result and that some special linear systems can be identified in this way. Such systems might in turn prove quite useful, for example, in the area of birational geometry in the spirit of [7].

As already indicated in Problem 1.1 there is an additional difficulty when we consider postulation of multiple (fat) points. This is especially transparent in the case of Nagata's Conjecture. For reduced points, its prediction is trivial and can be proved by linear algebra. Introducing singularities changes the problem dramatically.

Bocci and Chiantini initiated in [5] a new line of investigation. They considered ideals of sets Z of *arbitrary* points in $\mathbb{P}^2$ and asked how the assumption that the difference between the minimal degree of a curve passing through Z and that passing doubly through Z is minimal (i.e. equal 1) influences the geometry of Z. They showed that the constrain is serious and the points in Z either form what is now known as a star configuration (see [10]) or they are all collinear. This result has been generalised for flats of codimension 2 by Janssen [18] and Haghighi, Zaman Fashami and Szemberg [8] under the additional assumption that the union of studied flats is an arithmetically Cohen-Macaulay variety. Already in $\mathbb{P}^3$ it is natural to study the same problem for connected curves. This is exactly the next and the last problem we want to spell out.

Problem 1.5. *Let $C \subseteq \mathbb{P}^3$ be a smooth (or just connected) curve such that the minimal degree of a generator of its ideal $I(C)$ is α and the minimal degree of the second symbolic power $I(C)^{(2)}$ is $\alpha + 1$. Show that then C is contained in a hyperplane.*

In a sense, this is the most challenging problem, because there is no analogy in the literature to build on. Note that proving the reverse implication in Problem 1.5 is elementary.

2. Significance

On smooth varieties (for example in projective spaces) codimension 1 subvarieties (divisors) are in one-to-one correspondence with sections of line bundles.

Higher codimension cycles can be obtained by intersecting divisors. It is natural to wonder if one obtains all cycles in this way. A classical result along these lines is due to Noether and Lefschetz.

Theorem 2.1 (Noether-Lefschetz). *Let S be a general surface of degree $d \geqslant 4$ in $\mathbb{P}^3$. Then any curve C contained in S is a complete intersection, i.e., it is cut out by another surface $S' \subset \mathbb{P}^3$.*

This result, proved in the final form only in 1985 by Griffiths and Harris [11], stimulated a lot of research on higher codimension cycles. In particular, Griffiths and Harris raised the following interesting question.

Conjecture 2.2 (Degree Conjecture). *Let $X \subset \mathbb{P}^4$ be a general threefold of degree $d \geqslant 6$ and let C be a curve contained in X. Is then the degree of C a multiple of d?*

This conjecture would easily follow if the following statement, analogous to Theorem 2.1, would hold: Any curve as in the Degree Conjecture is the intersection of X with some surface $S \subset \mathbb{P}^4$. Voisin showed in 1988 that this statement is false. Of course her result shows that an even more naive hope that C might be an intersection of three threefolds in $\mathbb{P}^4$ fails, but it was already known at the point when the conjecture was stated. Interestingly, the Degree Conjecture is still open. Some strong evidence supporting the Conjecture has been provided recently by Kollár. This developments and names appearing here prove that *studying codimension 2 cycles in algebraic varieties is an important and hard problem in contemporary algebraic geometry.*

We have just seen that one cannot expect in general that cycles of high codimension come up as intersections of cycles of lower codimension. There is another possibility to extend the relation between effective divisors and sections of line bundles. To this end one can study sections in higher rank locally free sheaves, i.e., vector bundles. For codimension 2 cycles it is natural to look at vector bundles of rank 2. Whereas this relation again fails in general, even for cycles in projective spaces, there is a very challenging conjecture due to Hartshorne [19, Conjecture 3.2.8].

Conjecture 2.3 (Hartshorne). *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible subvariety of codimension 2. For $N \geqslant 6$ it follows that X is a complete intersection.*

This conjecture has stimulated a lot of research on codimension 2 subvarieties. The proof with available methods seems out of reach. However one can hope that assuming the Conjecture (possibly in a slightly stronger form) one can obtain progress in Problem 1.4 precisely in the cases where iterated intersections between involved flats become messy. It is also possible that progress on Problem 1.4 can be obtained without assuming Hartshorne's conjecture, but studying instead, from the perspective of our problem, its consequences. By this we mean the general yoga that small codimension subvarieties in projective spaces behave cohomologically like complete intersections and it is the cohomology we are interested in.

In the formulation provided above, Conjecture 2.3 is equivalent to the following statement, see [14].

Conjecture 2.4 (Hartshorne's splitting conjecture)**.** *Any rank* 2 *vector bundle on* $\mathbb{P}^N$ *with* $N \geqslant 6$ *splits as a direct sum of line bundles.*

It is well-known that this conjecture fails in lower dimensional projective spaces, perhaps the most prominent example being that of Horrocks-Mumford bundle [17]. In any case, it is clear that vector bundle techniques come naturally into the picture, when higher codimension subvarieties are considered. It is also worth to mention the following two facts. A well-known splitting criterion due to Horrocks asserts that a vector bundle E on the projective space $\mathbb{P}^N$ splits if and only if $H^i(\mathbb{P}^N, E \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0$ for all d and all $0 < i < N$. An improvement, due to Evans and Griffiths, asserts that it suffices to check the splitting for $i < \min\{N, \mathrm{rk}(E)\}$. Hence for the rank 2 case (i.e. codimension 2 cycles) it suffices to check the vanishing $H^1(\mathbb{P}^N, E) = 0$ *for all* rank 2 vector bundles E.

Among codimension 2 subvarieties in projective spaces, curves in $\mathbb{P}^3$ play a special role. This is due to the fact that, by an elementary classical theorem, any smooth curve can be embedded into the projective space of dimension 3 (very much as any smooth variety of dimension n can be embedded into $\mathbb{P}^{2n+1}$). Thus one cannot expect that curves in $\mathbb{P}^3$ enjoy any special properties, contrary e.g. to surfaces in $\mathbb{P}^4$ as only special surfaces can be embedded into $\mathbb{P}^4$. On the other hand there are certain constraints relating the genus and the degree of space curves, worked out by Gruson and Peskine [13]. Further refinements in term of the ideal $I(C)$ defining C have been obtained by Gruson, Lazarsfeld and Peskine in [12]. The ideals with the simplest structure are those generated by a regular sequence. For codimension two subvarieties there is a striking result due to Gaeta [9] to the effect that any arithmetically Cohen-Macaulay (ACM) subvariety X is minimally linked to a complete intersection subvariety, see [24] for the modern treatment of liason theory and much more. The study of non-ACM subschemes of projective spaces is an area of active research to which we hope to attract even more attention, see e.g. [23] for a nice introduction to this circle of ideas. In particular, Problem 1.5 is stated without assuming that the considered curve is an ACM-subscheme. Note that this assumption was inevitable in the approach taken on by Janssen [18] and in the generalisations proved in [8]. The reason is the application of the Hilbert-Burch theorem, which gives a useful description of the defining ideal of an arithmetically Cohen-Macaulay subvariety of codimension 2 in a projective space (or more generally: in a smooth projective variety). Of course, dealing with a connected curve C provides much stronger tools, including the normal sheaf and the infinitesimal neighbourhoods of C. In this context we quote the following illuminating result due to Ran [26].

Theorem 2.5 (Ran)**.** *Let* X *be a degree* d*, locally complete intersection, codimension* 2 *subvariety in* $\mathbb{P}^{n+2}$*. Let* N_X *be the normal sheaf of* X *and let* $\bigwedge^2 N_X = \mathcal{O}_X(a)$*. If*

either $a \geqslant d/n + n$*;*
or $d \leqslant n$*,*

then X *is a complete intersection.*

3. Possible path of research

As already stated in Section 1, even the conjectural picture of the positivity of codimension 2 subvarieties is far from being well understood. The most striking manifestation of how little is known is the problem to determine the mobility of the class of a line in $\mathbb{P}^3$. Therefor it seems reasonable to focus on the phenomenological part of the research, namely to explore connected codimension 2-cycles given by the intersections of hyperplane arrangements in projective spaces, e.g., defined as skeletons of Fermat arrangements [27] and thus collecting data before approaching the more general research problems. Such a strategy turned out to be quite successful in different areas of studies, for instance in questions concerning the existence of unexpected subvarieties and in the problems revolving around the containment conjectures. Skeletons of Fermat arrangements are an interesting testing ground for various hypotheses. In a sense, they resemble the so called star configurations, see [10], which are close to general arrangements on the one hand and special enough to exhibit interesting patterns on the other hand. Such patterns are particularly reach in algebraic objects connected to both classes varieties (i.e. skeletons of Fermat and star configurations), starting with their defining equations and various powers of them.

The proposed path of research consists of the following four questions:

(1) Find minimal degree $d(s)$ of a connected curve $C \subset \mathbb{P}^3$ passing though s general points.
(2) Improve an upper bound on the mobility of the class of a line in $\mathbb{P}^3$.
(3) Study postulation problems for unions of (general) flats of codimension 2 in $\mathbb{P}^N$.
(4) Explore the fattening effect for connected curves in $\mathbb{P}^3$.

These problems are clearly divided in two groups: one concerning curves and the other higher dimensional subvarieties. In both cases we propose to test working hypothesis with symbolic algebra programs. This approach is well established in this area, see for example the Crelle work of Holme and Schneider [16].

4. Related methods

The Castelnuovo-Mumford regularity (CM-regularity for short) is a fundamental invariant in commutative algebra and algebraic geometry. It has (informally) appeared in works of Guido Castelnuovo, long before it has been formally defined. Castelnuovo studied linear series on curves in $\mathbb{P}^3$ cut out by surfaces of fixed degree. In modern terms, he was interested in determining the dimension of the vector space of global sections $H^0(C, \mathcal{O}_C(d))$ for a curve C in $\mathbb{P}^3$. For d large enough, this dimension is, by the Riemann-Roch theorem, equal to $cd - g(C)$, where c is the degree of C and $g(C)$ is the genus. The CM-regularity of $I(C)$ makes sense of the phrase "large enough", turning it to an effective statement. Along these lines, there is the following interesting problem due to Eisenbud and Goto.

Conjecture 4.1 (Regularity Conjecture)**.** *For a non-reduced and connected in codimension* 1 *subscheme* $X \subset \mathbb{P}^N$*, there is*

$$\operatorname{reg}(I(X)) \leqslant \deg(X) - \operatorname{codim}(X) + 1.$$

This conjecture has been recently shown to fail spectacularly by McCullough and Peeva [22]. We suggest to explore methods from [22] in order to deal with problems listed in the previous section. In particular, it seems feasible to couple these new methods with those of Bertram, Ein and Lazarsfeld from [4]. One of results of that paper seems of direct interest.

Theorem 4.2 (Bertram-Ein-Lazarsfeld)**.** *Let* X *be a subvariety in projective space* $\mathbb{P}^N$ *defined scheme-theoretically by equations of degree* $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_m$ *(i.e. the equations in the minimal set of generators of the ideal* $I(X)$ *have these degrees). Then* X *is*

$$(d_1 + \ldots + d_e - e + 1) - \textit{regular},$$

where $e = \operatorname{codim}(X)$*.*

This result is of course in general far from the Regularity Conjecture but, on the other hand, it can be very useful towards solving Problem 1.5. Even if the Problem cannot be solved in the full generality, it could be more tractable, in particular with Theorem 4.2, for curves defined by quadratic equations.

In order to attack Problem 1.1, one of possible approaches is to study properties of the scheme $D_{m;d,g}$ which parametrizes couples of the form (M, C), where M is a finite scheme of length m and C is a curve of degree d and genus g such that $M \subset C$. If we denote by $m(d, g)$ the maximal number of general points in $\mathbb{P}^3$ with the property that there is a curve of degree d and genus g passing through them, and by $H_{d,g}$ the Hilbert scheme of connected curves C of degree d and genus g in $\mathbb{P}^3$, then we have

$$m(d, g) \leqslant \left[\frac{1}{2}\dim H_{d,g}\right].$$

However, this bound is not sharp in general. If we take $g = (d-1)(d-2)/2$, then curves in $H_{d,g}$ are planar, so $m(d, g) = 3 < \frac{1}{2}H_{d,g}$. In order to cope with this problem, Perrin introduced h^0-stability for locally free sheaves of rank 2 – a model example here is the normal sheaf N_C of a curve $C \subset \mathbb{P}^3$. In this case, we say that N_C is h^0-semi-stable if and only if for every invertible subsheaf L of N_C one has $h^0(L) \leqslant \frac{1}{2}h^0(N_C)$. Building upon the work of Perrin [25], Vogt in [28] has shown recently that there exists a Brill-Noether curve C of degree d and genus g in $\mathbb{P}^3$ passing through a maximum of $2d$ general points apart of a short list of exceptions. Let us recall that a Brill-Noether curve is a member of a unique irreducible component of the Kontsevich space $M_g(\mathbb{P}^3, d)$ which dominates M_g and whose general member is a non-degenerate immersion of a smooth curve. This shows that there is still a room for improvements, see [2].

The postulation problem for codimension 2 subvarieties is directly related to the geometry of Veneroni maps. These are certain birational transformations of higher dimensional projective spaces whose base loci consist of unions of general codimension 2 linear subspaces. Thus the whole machinery of Cremona groups becomes relevant. A special feature of codimension 2 flats is that this approach is neatly connected to free resolutions via the Hilbert-Burch Theorem.

Volume of divisors played a crucial role in Witt-Nyström [29] approach to the duality, postulated by Boucksom-Demailly-Păun-Peternell [6], between the cones $\overline{\mathrm{Eff}}(X)$ of pseudo-effective divisors and the cone of movable curves $\mathrm{Mov}(X)$, under the assumption that X is a projective variety. The key idea was to introduce and use the following transcendental Morse inequality.

Theorem 4.3 (Morse inequality)**.** *Let α and β be two nef classes on a projective manifold X of dimension n, then*

$$\mathrm{vol}(\alpha - \beta) \geqslant \alpha^n - n(\alpha^{n-1}\beta).$$

If we focus on big divisors on a projective variety X, then Khovanskii-Teissier inequality asserts that for big and nef divisors A, B and a movable curve class β one has

$$n(A \cdot B^{n-1})(B \cdot \beta) \geqslant B^n (A \cdot \beta).$$

It is expected that inequalities of this kind hold for cycles of higher codimension. It seems that various generalizations of the Morse-type inequalities to codimension 2-cycles are possible. Towards this direction, one can be guided by ideas and methods introduced by Lehmann and Xiao in [21]. They stated some very natural generalization of the Morse inequalities to convex bodies and support functions. This provides another way of thinking about the volume function based on purely combinatorial and convex geometry methods. This example manifests a natural bridging that can be observed very recently in literature of the subject: highly non-trivial results in algebraic geometry are generalized to the framework of the combinatorial world. It is reasonable to expect that such links are lurking behind in the world of cycles of higher codimension and the mobility function.

Finally, we expect that the mobility function flagged in Section 1, should shed new light on relations expected for codimension 2 cycles. A particularly nice case, we have in mind, is that of surfaces in 4-dimensional projective space (or more generally in 4-dimensional varieties). The study of arrangements of planes in $\mathbb{P}^4$ should be viewed as the degenerate case of this situation. An important, additional tool which one has here at the disposal is the self-intersection formula.

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