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LEFSCHETZ NUMBERS AND ASYMPTOTIC PERIODS

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Streszczenie. In this note we prove several results linking Lefschetz numbers with asymptotic behaviour of the orbit in flows. With the aid of the Lefschetz fixed point theorem and the presence of a non-trivial limit set we prove the existence of asymptotically non-periodic orbits.

1. Introduction

The study of dynamical systems is divided into the variety of categories. In this article we want to utilize classic topological methods, going back to Lefschetz [8] and his well–known fixed point theorem.

The Lefschetz fixed point theorem has many applications in mathematics [2, 4], especially in the fixed point theory, but also, surprisingly, in digital topology [3]. The Lefschetz formula and the Euler characteristic are another tolls that have a wide application in algebraic topology and dynamical systems.

In this article, we link the Lefschetz numbers with the so called G-asymptotic period. In Section 3 we, among others, prove that if the limit set of some point x has non-zero Euler characteristic, then x cannot be G-asymptotically periodic. We also provide several examples of flows that justify the assumptions of our results.

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2. Preliminaries

Let us start by introducing fundamental definitions used in the entire paper.

2.1. **Dynamical systems.** Let (X, d) be a metric space. A *dynamical system* (a flow) ϕ is a continuous mapping $\phi\colon \mathbb{R} \times X \to X$ such that $\phi(0, x) = x$ and for any x, s and t we have $\phi(t, \phi(s, x)) = \phi(t + s, x)$. We call X a *phase space* of ϕ. A *motion through* x is the mapping $t \mapsto \phi(t, x)$. We will identify properties of the motion through x with properties of x. Given dynamical system ϕ and $x \in X$, the set $o(x) = \phi(\mathbb{R}, x)$ is the *orbit* of x and $o^+(x) = \phi([0, +\infty), x)$ is the *positive orbit* of x. A point x is *stationary* if $x = \phi(t, x)$ for any $t \in \mathbb{R}$. If for some $T > 0$ we have $x = \phi(T, x)$ and x is not stationary, then x is *periodic*. If $T > 0$ is the smallest such that $x = \phi(T, x)$, then we say that x is *T-periodic* and we call T the *period* of x. The *ω-limit set* $\omega(x)$ consists of all points $y \in X$ such that there exists a strictly increasing and diverging to $+\infty$ sequence $(t_n)_{n\in\mathbb{N}}$ of times with the property: $\phi(t_n, x) \to y$. For more definitions and properties related to dynamical systems see [1, 13].

The following notion is a generalization of periodicity and it relies on the asymptotic behaviour of the orbit outside of a small neighbourhood of a point belonging to the positive orbit of x. This idea was introduced in [5]. We briefly introduce the necessary notation.

Let ϕ be a flow on X. Fix $x \in X$ and $\varepsilon > 0$, and define

$$A(x, \varepsilon) := \{t \geqslant 0 \mid d(\phi(t, x), x) > \varepsilon\}.$$

This set is the union of at most countably many pairwise disjoint and open intervals denoted by (q_i, r_i). Define

$$w_{x,\varepsilon}(t) := \begin{cases} 0, & t \notin A(x, \varepsilon), \\ \operatorname{diam}(q_i, r_i), & t \in (q_i, r_i). \end{cases}$$

The set $W_{x,\varepsilon} := \{w_{x,\varepsilon}(t) \mid t \geqslant 0\}$ contains at most countably many different non-negative real numbers, including $+\infty$ if necessary. We call the elements of that sequence *return times*. Set

$$W(x, \varepsilon) := \limsup_{t\to+\infty} w_{x,\varepsilon}(t).$$

Definition 2.1. The *G-asymptotic period* of x (of the orbit of x) is defined as

$$\text{G-AP}(x) := \lim_{\varepsilon\to 0} \limsup_{t\to+\infty} W(\phi(t, x), \varepsilon).$$

If $\text{G-AP}(x) = 0$, then x is called *G-asymptotically fixed.* If x has a finite asymptotic period, then it is called *G-asymptotically periodic.* If $\text{G-AP}(x) = +\infty$, then x is called *G-asymptotically non-periodic.*

See also [5, 6, 7] for more properties of G-asymptotically periodic orbits.

2.2. **Homotopies and ENRs.** Let X be a topological space. For any mapping $f\colon X \to X$, we say that f has the *fixed point property* if f has a fixed point, i.e., there exists $x_0 \in X$ such that $f(x_0) = x_0$. Define the set

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}.$$

Suppose $f\colon X \to X$ and $g\colon X \to X$ are continuous functions. We say that f is *homotopic to* g and we denote this relation by $f \sim g$, if there is a continuous mapping $h\colon [0,1] \times X \to X$ such that $h(0,\cdot) = f$ and $h(1,\cdot) = g$.

We say that X has the *weak fixed point property* if for any $f\colon X \to X$ which is homotopic to Id_X (the identity function on X) we have $\mathrm{Fix}(f) \neq \emptyset$.

We call the space X *euclidean neighborhood retract* (ENR) if there exists an open set $V \subset \mathbb{R}^n$ and continuous functions $r\colon V \to X$ and $s\colon X \to V$ such that $r \circ s = \mathrm{Id}_X$.

2.3. **Lefschetz numbers.** Let X be a compact ENR and let $f\colon X \to X$ be continuous. Let H denote the singular homology functor with rational coefficients. Let $H(f)\colon H(X) \to H(X)$ be the induced homomorphism.

Definition 2.2. The number

$$L(f) = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{tr} H_n(f) \in \mathbb{Z}$$

is called the *Lefschetz number* of f. Here, $\mathrm{tr}H_n(f)$ is the trace of the endomorphism $H_n(f)\colon H_n(X) \to H_n(X)$.

If $f = \mathrm{Id}_X$, then $\chi(X) = L(\mathrm{Id}_X)$ is called the *Euler characteristic* of X. It can also be defined as

$$\chi(X) = \sum_{n=0}^{+\infty} (-1)^n \dim H_n(X).$$

The above definitions are well-defined since it is well-known that compact ENRs have only finitely many non-zero homologies $H_n(f)$ and they are all of finite dimension. It is also well-known, that if $f \sim g$, then $L(f) = L(g)$. See [2] for more information related to the topic.

3. Main results

We shall use the following lemma. It is a variation of Proposition III 4.8 in [2]. See also [12].

Lemma 3.1. *If ϕ is a flow on a compact metric space (X,d) and X has the weak fixed point property, then ϕ has a stationary point.*

Dowód. For each $t \in \mathbb{R}$ we let ϕ_t denote the map $X \ni x \mapsto \phi(t,x) \in X$. Then $\phi_t \sim \mathrm{Id}_X$; the homotopy is defined via relation

$$h(s,x) = \phi(st,x).$$

Each ϕ_t has a fixed point by the weak fixed point property. We define the sets

$$A_n = \{x \in X \mid \phi(2^{-n}, x) = x\}.$$

Each of the sets A_n is not empty, closed and therefore compact. Furthermore, since

$$x = \phi(2^{-(n+1)}, x) = \phi(2^{-(n+1)}, \phi(2^{-(n+1)}, x)) = \phi(2^{-n}, x)$$

for any $x \in A_{n+1}$, we have

$$A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots .$$

Since X is compact and the family $\{A_n\}_{n\in\mathbb{N}}$ has a finite intersection property, we can take the set

$$A = \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset.$$

Take any $z \in A$. Then $\phi(2^{-n}, z) = z$ for all n. We claim that for all n and all integers m we also have

$$\phi(m \cdot 2^{-n}, z) = z.$$

Since $z \in A_0$, we have for any natural number k,

$$\phi(k, z) = \phi(k-1, \phi(1, z)) = \phi(k-1, z) = \cdots = \phi(1, z) = z,$$

$$\phi(-k, z) = \phi(-k, \phi(1, z)) = \phi(-k+1, z) = \cdots = \phi(-k+k, z) = \phi(0, z) = z,$$

thus for any $m \in \mathbb{Z}$,

$$\phi(m \cdot 2^{-n}, z) = \phi(m \cdot 2^{-n} \bmod 1, z)$$

and it is enough to prove the claim in the case $0 < m \cdot 2^{-n} < 1$.

Suppose $0 < m \cdot 2^{-n} < 1$ and let $m = \sum_{i=0}^{M} m_i \cdot 2^i$ be the binary representation of m. Then $i - n \leqslant 0$ for each $i = 0, \ldots, M$. Note that $z \in A_n$ and $\phi(2^{-n}, z) = z$, hence

$$\begin{aligned}\phi(m \cdot 2^{-n}, z) &= \phi\Big(\sum_{i=0}^{M} m_i \cdot 2^{i-n}, z\Big) = \phi\Big(\sum_{i=1}^{M} m_i \cdot 2^{i-n}, \phi(m_0 \cdot 2^{-n}, z)\Big)\\ &= \phi\Big(\sum_{i=1}^{M} m_i \cdot 2^{i-n}, z\Big)\end{aligned}$$

(if $m_0 = 0$, then $\phi(0, z) = z$, otherwise $\phi(m_0 \cdot 2^{-n}, z) = \phi(2^{-n}, z) = z$). The claim now follows from the induction on i (note that the induction terminates after finitely many steps for any m).

Since the set $\{m2^{-n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$ and ϕ is continuous, this implies that $\phi(t, z) = z$ for all $t \in \mathbb{R}$. □

A great example of a space with the weak fixed point property is a connected polyhedron.

Lemma 3.2 (See Proposition III 4.6 in [2])**.** *Any connected polyhedron K with $\chi(K) \neq 0$ has the weak fixed point property. Any flow on such polyhedron has a stationary point.*

The following lemma shows that if the limit set of the orbit has a stationary point and at least one other point, then the G-asymptotic period need to be infinite. Recall that a metric space is *proper* if all closed balls are compact sets.

Lemma 3.3 (see also [5]). *Assume that* (X, d) *is a proper metric space and* ϕ *is a flow on* X. *If* $x \in X$ *has* $\#\omega(x) > 1$ *and* $\omega(x)$ *contains a stationary point, then* $\text{G-AP}(x) = +\infty$.

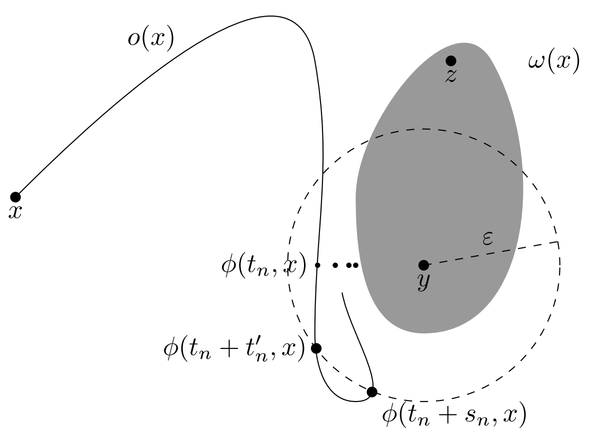
Dowód. Suppose $y \in \omega(x)$ is stationary. It is sufficient to show that the return times of x to $B(y, \varepsilon)$ cannot be bounded, and hence $\text{G-AP}(x) = +\infty$. Indeed, if that were the case, then take $\varepsilon' < \varepsilon$ and t' such that $B(\phi(t', x), \varepsilon') \subset B(y, \varepsilon)$. Then since the return times in the former case are not bounded, they are not bounded in the latter case, thus implying $\text{G-AP}(x) = +\infty$.

Suppose the opposite is true and let K be the bound. Pick $z \in \omega(x) \setminus \{y\}$ and $\varepsilon > 0$ so that $d(y, z) > \varepsilon$. There is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\phi(t_n, x) \to y$ and $d(\phi(t_n, x), y) < \varepsilon$ for all n.

Let t'_n be the infimum of all $u > 0$ such that $\phi(t_n + u, x) \notin B(y, \varepsilon)$. Such an u exists since z is an element of $\omega(x)$ and $d(y, z) > \varepsilon$. Let s_n be the infimum of all $v > t'_n$ such that $\phi(t_n + v, x) \in B(y, \varepsilon)$ (see Figure 1). The sequence s_n is bounded by K. We can assume without loos of generality that it is convergent. Let $s = \lim_{n\to+\infty} s_n$. Then, since the space is proper, $\phi(t_n + s_n, x) \to w$ for some $w \in X$ and $w \notin B(y, \varepsilon)$ On the other hand,

$$\phi(t_n + s_n, x) = \phi(s_n, \phi(t_n, x)) \to \phi(s, y) = y$$

which is a contradiction. □



Rysunek 1. Sketch of the proof of Lemma 3.3.

With the aid of the above lemmas, we can formulate the following theorem.

Theorem 3.4. *Suppose ϕ is a flow on a proper metric space (X, d). Let $x \in X$ be such that $\omega(x) = S$ is a compact ENR with the weak fixed point property. If $\#S > 1$, then* G-AP$(x) = +\infty$.

Dowód. The set S is compact, therefore by Lemma 3.1 there is a stationary point in S. Then, by Lemma 3.3 we have G-AP$(x) = +\infty$. □

The assumption that the limit set S is an ENR is actually not needed for the proof, however it was added since the later results require the set to be an ENR.

Recall the famous Lefschetz fixed point theorem [8, 9, 10, 11].

Theorem 3.5. *Suppose X is a compact ENR and $f \colon X \to X$ is continuous. If $L(f) \neq 0$, then* Fix$(f) \neq \emptyset$.

We have the immediate.

Corollary 3.6. *If X is a compact ENR with $\chi(X) \neq 0$, then any flow ϕ on X has a stationary point.*

Dowód. Indeed, since the Lefschetz numbers are homotopy invariant,

$$\chi(X) = L(\mathrm{Id}_X) = L(\phi(t, \cdot))$$

for any t. Thus by Lefschetz fixed point theorem, each map $x \mapsto \phi(t, x)$ has a fixed point. The rest follows from the proof of Lemma 3.1. □

Example 3.7. Consider n-dimensional spheres. Then

$$\chi(\mathbb{S}^{2k}) = 2, \qquad \chi(\mathbb{S}^{2k+1}) = 0.$$

It is now clear that any flow on $\mathbb{S}^{2k}$ must have a stationary point. On the other hand, each odd-dimensional sphere $\mathbb{S}^{2k+1}$ admits a flow with no stationary points.

Indeed, let $z = (z_1, \ldots, z_{k+1}) \in \mathbb{S}^{2k+1}$ with $z_i \in \mathbb{C}$. Then the function

$$\phi(t, z) = ze^{it} = (z_1 e^{it}, \ldots, z_{k+1} e^{it})$$

defines a flow on $\mathbb{S}^{2k+1}$ with no stationary point.

A variation of Theorem 3.4 is presented below.

Theorem 3.8. *Suppose ϕ is a flow on a proper metric space (X, d) and $\omega(x) = S$ is a compact ENR (or a connected polyhedron) for some $x \in X$. If $\#S > 1$ and $\chi(S) \neq 0$, then* G-AP$(x) = +\infty$.

Example 3.9. If we take $S = \mathbb{S}^{2k}$ in Theorem 3.8, then G-AP$(x) = +\infty$. In particular, even-dimensional sphere cannot be a limit set of G-asymptotically periodic point. On the other hand, the unit circle $\mathbb{S}^1$ is the limit set of all points in $\mathbb{R}^2 \setminus \{(0,0)\}$ of the flow in $\mathbb{R}^2$ generated by the equations

$$\begin{cases} r' = r(1 - r), \\ t' = 1. \end{cases}$$

This in turn implies that the assumption about the Euler characteristic cannot be relaxed.

Finally, in view of Theorem 3.8, by constructing a flow which has $\omega(x) = \mathbb{T}$ (the two-dimensional surface of the torus - one such construction was provided in [5]), we can show that the condition G-AP$(x) = +\infty$ does not imply $\chi(S) \neq 0$,

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