Analytic and Algebraic Geometry 4
Łódź University Press 2022, 83–93
DOI: https://doi.org/10.18778/8331-092-3.08

REALIZABILITY OF SOME BÖRÖCZKY ARRANGEMENTS OVER THE RATIONAL NUMBERS

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Streszczenie. In this paper, we study the parameter spaces for Böröczky arrangements B_n of n lines, where $n < 12$. We prove that up to $n = 12$, there exist only one arrangement nonrealizable over the rational numbers, that is B_{11}.

1. Introduction

Recently, some considerations about realizability of Böröczky configurations over the rational numbers have shown up, especially in algebra and combinatorics. An excuse for such research is the problem of containment relations between the symbolic and ordinary powers of homogeneous ideals. The Böröczky arrangement of 12 lines was the first counterexample for some hypothesis in this area over the reals. In [9], using the parameter space, it was shown that this arrangement is relizable over the rational numbers and also that 12 lines is the minimal number of Böröczky lines, where intersection points give a similar counterexample.

In this context, some new results appeared with references to the higher number of lines. The aim of this paper is to complete the picture for number of lines between 3 and 11 in Böröczky arrangements and to establish the realizability of these configurations over the rational numbers.

According to [2], the Böröczky configurations were originally introduced in connection with the orchard problem. Böröczky described his construction to some mathematicians but he never published this results. In [2], these configurations are concidered in a relation to the celebrated Sylvester-Gallai Theorem. In [7, 8] they appear as configurations with a large number of ordinary lines.

2010 *Mathematics Subject Classification.* 52C30, 14N20, 05B30.
Key words and phrases. line arrangement, parameter space, singular points, Böröczky configuration.

The interest was recently renewed with a linkaye to the containment problem studied in commutative algebra (see details in [3] and [5]). Our research are inspired by papers [9] and [6], where the parameter spaces of some Böröczky arrangements were considered.

Let us denote by B_n the configuration of n lines arranged with Böröczky construction. Up to now, there were published such results for configurations B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{18} and B_{24}. The Böröczky arrangement of 12 lines is up to now the only known Böröczky configuration realizable over the rational numbers. We mean by this that there exists a configuration of 12 lines with the same incidences between the lines and the intersection points, which all the points have coordinates being the rational numbers. Since, in connection with the containment problem, there were considered only arrangements with at least 12 lines, we fill the gap in picture for $3 \leqslant n \leqslant 11$.

The Böröczky configurations B_n were described in [7]. Following this, we present here the construction.

Consider an $2n$-gon inscribed in a circle. Let us fix one of the $2n$ points and denote it by Q_0. By Q_α we mean the point arising by the rotation of Q_0 around the center of a circle by some angle α.

We take the following n lines:

$$B_n = \left\{ Q_\alpha Q_{\pi-2\alpha},\ \text{where } \alpha = \frac{2k\pi}{n} \text{ for } k = 0, 1, \ldots, n-1 \right\}.$$

If $\alpha \equiv (\pi - 2\alpha)(mod\ 2\pi)$, then $Q_\alpha Q_{\pi-2\alpha}$ is the tangent to the circle at the point Q_α.

These configurations have the maximal numbers of triple intersection points estimated in [8], with reference to n, namely

$$t_3 = 1 + \left\lfloor \frac{n(n-3)}{6} \right\rfloor.$$

2. Realization of line configurations

By a configuration we mean an ordered pair $A = (S, L)$, where a set L is a finite family of lines and by S we denote the set of all their intersection points. The realizability problem for configurations is intensively studied during the last few decades. Sturmfels in [10] establishes a connection between the realizability of projective configurations and some polynomial identity, so called *final polynomial*. Instead of this, we consider a system of equations, which are the generators of some standard basis connected with the configuration.

Following [10], we recall some basic notions necessary in the future considerations.

Let $\mathbb{K}$ be an arbitrary field of characteristic 0 and let $\varphi\colon S \longrightarrow \mathbb{K}^3$ be a mapping such that

$$s \longmapsto r_s = (r_{s,1}, r_{s,2}, r_{s,3})^T.$$

We call φ a *realization* of a configuration A over a field $\mathbb{K}$ if the following conditions are equivalent:

- $det(r_i, r_j, r_k) = 0$,
- $i, j, k \in S$ are contained in some line of A.

If $|S| = n$, then every realization of A can be though as a $3 \times n$ matrix, which columns are the coordinates of points of S. We call such matrix as *points matrix of* A.

Directly from definition, the 3×3 minors of points matrix are 0 iff their collumns are the cordinates of collinear points. Hence the realizations of A correspond to labeled subsets of the projective plane $\mathbb{P}^2(\mathbb{K})$ which satisfy the given incidence structure. The subset $\mathbb{F} \subset \mathbb{K}$ corresponding to a realization of configuration A (i.e. entries of matrix are the elements of $\mathbb{F}$) is called the *realization space of* A.

Realizability of configuration can be expressed in the language of polynomials.

Theorem 2.1 ([10], Theorem 3.2)**.** *The following problems are polynomially equivalent:*

- *Do the polynomials of the set* $\{f_1, \ldots, f_m\}$ *have the common zero in* $\mathbb{K}^n$*?*
- *Is a configuration* A *realizable over* $\mathbb{K}$*?*

A parametrization of the realization space can be found by an analysis of polynomials of the standard basis for some polynomials connected with the configuration.

Let M_A be a points matrix of A and let $\{f_1, \ldots, f_m\}$ be a subset of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ with no common zero in $\overline{\mathbb{K}^n}$, where f_i are minors of collinear points of A. We define the auxiliary polynomials

$$\widehat{f_i}(x_1, \ldots, x_n, t_1, \ldots, t_m) := f_i(x_1, \ldots, x_n) - t_i$$

with the slack variables t_i. Thus $t_i = 0$ if and only if the proper points are collinear. Let $\widehat{G}$ be a Gröbner basis of the set $\{\widehat{f_1}, \ldots, \widehat{f_m}\}$ with pure lexicographic order

$$t_1 < \cdots < t_m < x_1 < \cdots < x_n. \tag{1}$$

Then the generators of $\widehat{G}$ designate the realization space of A (compare to [10], Theorem 6.2). Order (1) assures that the variables $x_1, \ldots, x_n$ appears in generators of $\widehat{G}$ with relatively low powers, comparing to variables $t_1, \ldots, t_m$. It is the main reason why we introduce these additional variables. Taking into consideration that finally $t_i = 0$ for collinear points, we obtain emphatically simpler conditions involving coordinates x_i, than computing a Gröbner basis of the set $\{f_1, \ldots, f_m\}$ directly.

It leads to the explicit algorithm allowing us to conclude a realizability of some configurations. An algorithm is based on general ideas of Sturmfels [10] combined with methods established in [9].

We carry out the construction in the following way:

Step 1 : We fix matrix $M_A = (r_1, r_2, \ldots, r_s) \in M_{3\times s}$ of triple points of the configuration.

Step 2 : We establish the family of equations $\mathfrak{F} = \{f_1, f_2, \ldots, f_k\}$, where f_i are the 3×3 minors of M_A with 3 collinear points as the columns.

Step 3 : We define the family of auxiliary equations $\hat{\mathfrak{F}} = \{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_k\}$ with slack variables.

Step 4 : We compute a Gröbner basis $\widehat{G}$ of $\hat{\mathfrak{F}}$ in the following way. We divide the set $\hat{\mathfrak{F}}$ into finite number of subsets (not necessary disjoint), which sum is all $\hat{\mathfrak{F}}$. We take the ideals of these sets and compute their sum. Finally, the basis of sum of ideal is the basis of $\hat{\mathfrak{F}}$. We substitute $t_i := 0$.

Step 5 : (Optionally) We use one of conditions determined by the elements of $\widehat{G}$ (with no variables t_i) to eliminate some of variables $x_1, \ldots, x_n$. After such substitution we repeat **Steps** $1-4$ for matrix M_A with reduced number of variables.

Step 6 : We repeat all algorithm step by step until we obtain condition clearly designating the realization space of configuration (or eventually we obtain condition excluding realization of configuration over some taken field).

3. Realizability of Böröczky configurations over the rationals

Below we present detailed algorithm for Böröczky configurations B_8 and B_{11}. We establish in this way, which of them are realizable over the rationals.

From now on, if there is no additional informations about fixed point, we assume $P_i = (x_i : y_i : z_i)$. General idea in the first step of algorithm is to introduce as many parameters as necessary and reduce considerably necessary parameters, using some obvious incidences.

Example 3.1. (Configuration of 8 lines)

Step 1:

We start with finding the matrix M_A. We fix the first four appropriate points in the arrangement as the fundamental points: $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$ and $P_4 = (1 : 1 : 1)$. They give as the beginning five lines of the construction, namely P_1P_2, P_2P_3, P_1P_3, P_3P_4 and P_2P_4 (lines distinguished with bold solid line in the Figure 1).

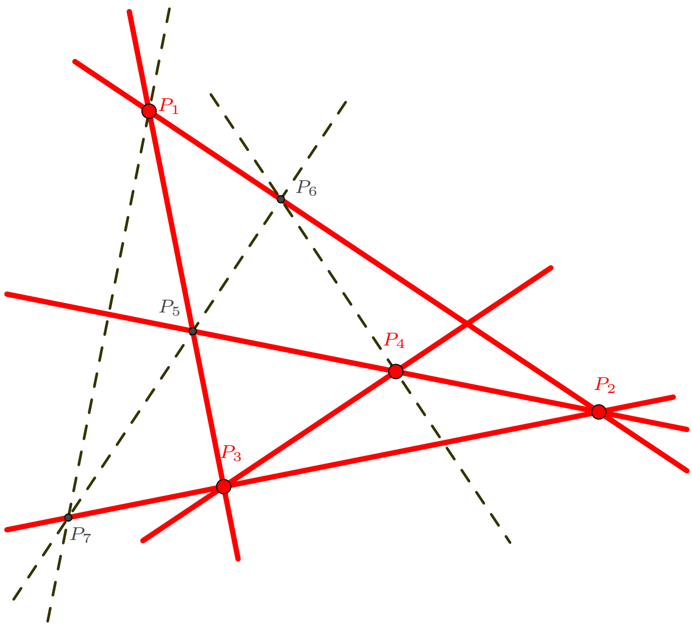

Figure 1

Automatically we obtain one more point:

$$P_5 = P_1P_3 \cap P_2P_4 = (1 : 0 : 1).$$

The last two points of the configuration are taken as some free points on the fixed lines and they are expressed with parameters:

$$P_6 = (x_6 : 1 : 0) \in P_1P_2,$$
$$P_7 = (0 : y_7 : 1) \in P_2P_3.$$

Thus the points matrix of the configuration is the following

$$M_A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & x_6 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & y_7 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The remaining three lines of the construction are P_4P_6, P_1P_7, P_5P_6 (distinguished as the dashed lines).

Step 2:

The lines P_3P_4, P_1P_7, P_4P_6 contain only two from points $\{P_1, \dots, P_7\}$. Remaining five lines contain exactly three of them. The points are grouped on the lines as follows:

$\{P_1, P_2, P_6\}$, $\{P_2, P_3, P_7\}$, $\{P_2, P_4, P_5\}$, $\{P_5, P_6, P_7\}$, $\{P_1, P_3, P_5\}$.

The only collinearity demanding to check is for points P_5, P_6, P_7. The rest of them are automatically satisfied. Thus

$$\mathcal{F} = \{det(P_5, P_6, P_7)\}.$$

Step 3:

We have only one auxiliary equation with slack variable t_1

$$\hat{f}_1 = x_6y_7 + 1 - t_1.$$

Step 4:

The basis of an ideal $< x_6y_7 + 1 - t_1 >$ with $t_1 = 0$ is

$$\widehat{G} = \{x_6y_7 + 1\}.$$

Step 5:

Not applicable.

Step 6:

Since condition $x_6y_7 + 1 = 0$ may be fulfilled by infinitely many pairs of rational numbers (x_6, y_7), the configuration B_8 can be realized over rationals.

Analogously we may easily check, that all remaining Böröczky configurations B_n with $3 \leqslant n \leqslant 10$ are realizable over the rational numbers. In [9], there was proved that also B_{12} may be realized over rationals.

In fact for $n \leqslant 12$ there exist only one configuration in this family, which can not be obtained over the field of rational numbers, namely B_{11}. We prove it in Example 3.2 by showing the resulting of algorithm in this case.

Example 3.2. (Configuration of 11 lines)

Step 1:

We start with finding the matrix M_A. As a core of configuration, we fix the first four appropriate points in the arrangement as the fundamental points:

$$P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1), \quad P_4 = (1 : 1 : 1).$$

They give us the beginning five lines of the construction, namely P_1P_2, P_2P_3, P_1P_3, P_3P_4 and P_2P_4 (distinguished with bold solid lines in the Figure 2). Automatically we obtain two more points:

$$P_5 = P_1P_3 \cap P_2P_4 = (1 : 0 : 1),$$

$$P_{15} = P_1P_2 \cap P_3P_4 = (1 : 1 : 0).$$

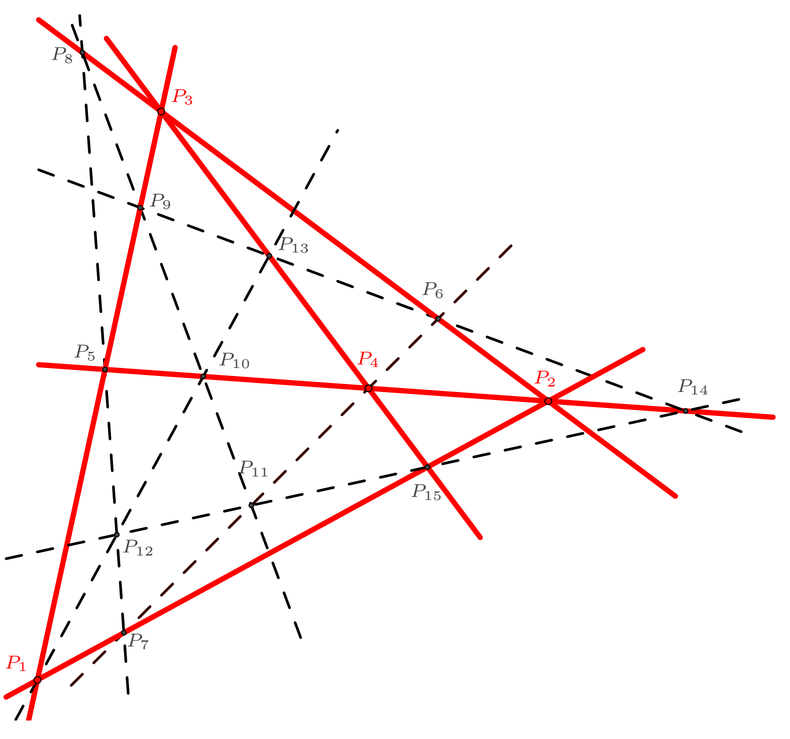

FIGURE 2

Remaining points of the configuration are taken as some free points on the fixed lines and they are expressed with parametres:

$$\begin{aligned}
P_6 &= (0 : y_6 : 1) \in P_2P_3,\\
P_7 &= (1 : 1 - y_6 : 0) \in P_1P_2 \cap P_4P_6,\\
P_8 &= (0 : y_6 - 1 : 1) \in P_2P_3 \cap P_5P_7,\\
P_9 &= (x_9 : 0 : 1) \in P_1P_3,\\
P_{10} &= (1 : y_{10} : 1) \in P_2P_4,\\
P_{11} &= (1 : y_6 \cdot z_{11} - y_6 + 1 : z_{11}) \in P_4P_6,\\
P_{12} &= (1 : 1 - y_6 + z_{12}(y_6 - 1) : z_{12}) \in P_5P_7,\\
P_{13} &= (1 : 1 : z_{13}) \in P_3P_4,\\
P_{14} &= (1 : y_{14} : 1) \in P_2P_4.
\end{aligned}$$

Thus the points matrix of configuration in this case is the following

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & x_9 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & y_6 & 1-y_6 & y_6-1 & 0 & y_{10} & y_6\cdot(z_{11}-1)+1 & (z_{12}-1)(y_6-1) & 1 & y_{14} & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & z_{11} & z_{12} & z_{13} & 1 & 0 \end{pmatrix}.$$

The remaining six lines of the construction are P_4P_6, P_5P_7, P_8P_9, P_1P_{10}, P_6P_9 and $P_{11}P_{15}$.

Step 2:

Triple points $P_1, \ldots, P_{15}$ are grouped on the lines in the following sets (compare with Figure 2):

$\{P_1, P_2, P_7, P_{15}\}$, $\{P_2, P_3, P_6, P_8\}$, $\{P_2, P_4, P_5, P_{10}, P_{14}\}$,
$\{P_3, P_4, P_{13}, P_{15}\}$, $\{P_1, P_3, P_5, P_9\}$, $\{P_1, P_{10}, P_{12}, P_{13}\}$,
$\{P_8, P_9, P_{10}, P_{11}\}$, $\{P_5, P_7, P_8, P_{12}\}$, $\{P_4, P_6, P_7, P_{11}\}$,
$\{P_6, P_9, P_{13}, P_{14}\}$, $\{P_{11}, P_{12}, P_{14}, P_{15}\}$.

Some of these collinearities results directly from the construction (for example P_6 is taken as a point on the line P_2P_3). Remaining collinearities generate the family of polynomials $\mathcal{F}$, where the polynomials are the following determinants:

$$\begin{aligned}
f_1 &= det(P_8, P_9, P_{10}), & f_2 &= det(P_8, P_9, P_{11}),\\
f_3 &= det(P_1, P_{10}, P_{12}), & f_4 &= det(P_1, P_{10}, P_{13}),\\
f_5 &= det(P_6, P_9, P_{13}), & f_6 &= det(P_6, P_9, P_{14}),\\
f_7 &= det(P_{12}, P_{14}, P_{15}), & f_8 &= det(P_{11}, P_{14}, P_{15}).
\end{aligned}$$

Step 3:

We introduce the auxiliary variables $t_1, \ldots, t_8$ and we define the family of equations $\hat{\mathfrak{F}} = \{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_8\}$, where

$$\hat{f}_i = f_i - t_i.$$

Step 4:

We consider the ideals $I =< \hat{f}_1, \ldots \hat{f}_7 >$ and $J =< \hat{f}_4, \ldots, \hat{f}_8 >$. We take $I + J$ and computing with Singular [4] we obtain its basis. Substituting $t_i := 0$ we have

$$\widehat{G} = \{z_{12}^2 \cdot z_{13}^2 - z_{12} \cdot z_{13}^2 - z_{12} + z_{13}, z_{11} \cdot z_{13} - 1, z_{11} \cdot z_{12} - z_{12}^2 \cdot z_{13} + z_{12} \cdot z_{13} - 1,$$
$$y_{14} - z_{12} \cdot z_{13} + z_{13} - 1, y_{10} - z_{11}, y_6 \cdot z_{13} - y_6 + z_{12} \cdot z_{13} - z_{13},$$
$$y_6 \cdot z_{12} - y_6 - z_{12}^2 \cdot z_{13} + z_{12} \cdot z_{13} - z_{12}, y_6 \cdot z_{11} - y_6 - z_{12} + 1, x_9 - z_{12}\}.$$

Step 5:

We make substitution using condition $x_9 - z_{12} = 0$. We repeat all algorithm for matrix M_1 without variable z_{12}. We obtain a new Gröbner basis

$$\widehat{G}_1 = \{z_{11} \cdot z_{13} - 1, y_{14}^2 - y_{14} \cdot z_{11} + y_{14} \cdot z_{13} - 2 \cdot y_{14} + z_{11}, y_{10} - z_{11}, y_6 \cdot z_{13} - y_6 + y_{14} - 1,$$
$$y_6 \cdot z_{11} - y_6 - y_{14} \cdot z_{11} + z_{11}, y_6 \cdot y_{14} - y_6 - y_{14} \cdot z_{11} - y_{14} + z_{11}, x_9 - y_{14} \cdot z_{11} + z_{11} - 1\}.$$

We make new substitution using condition $y_{10} - z_{11} = 0$. We obtain the following basis, independent of variable z_{11}:

$$\widehat{G}_2 = \{y_{14}^2 \cdot z_{13} + y_{14} \cdot z_{13}^2 - 2 \cdot y_{14} \cdot z_{13} - y_{14} + 1, y_{10} \cdot z_{13} - 1,$$
$$y_{10} \cdot y_{14} - y_{10} - y_{14}^2 - y_{14} \cdot z_{13} + 2 \cdot y_{14}, y_6 \cdot z_{13} - y_6 + y_{14} - 1, y_6 \cdot y_{14} - y_6 - y_{14}^2 - y_{14} \cdot z_{13} + y_{14},$$
$$y_6 \cdot y_{10} - y_6 - y_{14}^2 - y_{14} \cdot z_{13} + 2y_{14}, x_9 - y_{14}^2 - y_{14} \cdot z_{13} + 2y_{14} - 1\}.$$

Step 6:

Let us focus on the condition:

$$y_{14}^2 \cdot z_{13} + y_{14} \cdot z_{13}^2 - 2 \cdot y_{14} \cdot z_{13} - y_{14} + 1 = 0.$$

It is a plane cubic in variables y_{14} and z_{13}. To make further considerations more transparent, we substitute $y_{13} := u$ and $z_{14} := v$. Thus we have curve

$$C : u^2 v + uv^2 - 2uv - u + 1 = 0.$$

By homogenization we obtain plane projective cubic

$$\widetilde{C} : u^2 v + uv^2 - 2uvw - uw^2 + w^3 = 0.$$

Using *Magma* computations ([1]), we verify that $\widetilde{C}$ is an elliptic curve with only five rational points, namely

$$(1 : 1 : 1), \quad (1 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0), \quad (-1 : 1 : 0).$$

Only first two of these points can be applied to the curve C. Remaining points are the points at infinity, while C is an affine plane cubic. But if $y_{14} = 1$, the configuration is degenerated. More precisely, $P_4 = P_{14}$. Analogously, when $z_{13} = 1$

or $z_{13} = 0$. Thus $P_{13} = P_4$ or $P_{13} = P_{15}$. We conclude that configuration of 11 Böröczky lines can not be realizable over the rational numbers.

Corollary 3.3. *The configurations B_n for $n \leqslant 12$ can be realizable over the rationals, except the case of $n = 11$.*

The proof of case $n = 12$ reader can find in [9].

Remark 3.4. We believe that, among all Böröczky arrangements B_n with $n > 10$, arrangement B_{12} is the only one realizable over the rationals. In [9], the authors consider B_{12} and B_{15} arrangements and they explain why B_{12} arrangement can be realized over the rationals. Furthermore, in [6], another set of authors consider cases with $n \in \{13, 14, 16, 18, 24\}$. In all these cases, it is directly proved that B_ns are not realizable over the rationals, or there is no evidence that any realization over rationals would not degenerate the whole construction, i.e., available tools do not allow us to decide the existence of another such realizations. We want to reveal additionally here some additional unpublished results for other values of $n < 30$. In these cases, arrangements are not realizable over the rationals.

Our aim is to understand in deep the case $n = 12$ in order to find some combinatorial features that can potentially give some evidence about the speciality of B_{12}. We hope to come back to such a discussion in a forthcoming article soon.

Acknowledgements. We would like to thank Grzegorz Malara and Piotr Pokora for helpful remarks on this text and for assistance with programming of some symbolic computations.

We also thank the knowledgeable referee for helpful remarks, which improved our paper.

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