Physics Letters B 706 (2012) 427–430

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

On the triviality of higher-derivative theories

K. Andrzejewski [a,*], J. Gonera [a], P. Machalski [a], K. Bolonek-Lasoń [b]

[a] Department of Theoretical Physics and Computer Science, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland
[b] Department of Statistical Methods, University of Łódź, Rewolucji 1905r 41/43, 90-214 Łódź, Poland

ARTICLE INFO

Article history:
Received 18 October 2011
Received in revised form 10 November 2011
Accepted 11 November 2011
Available online 15 November 2011
Editor: M. Cvetič

ABSTRACT

The higher-derivative theories with degenerate frequencies exhibit BRST symmetry [V.O. Rivelles, Phys. Lett. B 577 (2003) 147]. In the present Letter meaning of BRST-invariance condition is analyzed. The BRST symmetry is related to nondiagonalizability of the Hamiltonian and it is shown that BRST condition singles out the subspace spanned by proper eigenvectors of the Hamiltonian.

1. Introduction

Theories described by Lagrangians containing higher derivatives of basic variables seem to play some role in physics. Originally they were proposed as a method for dealing with ultraviolet divergences [1]; this idea appeared to be quite successful in the case of gravity: the Einstein action supplied by the terms containing higher powers of curvature leads to renormalizable [2] and asymptotically free [3] theory. Other examples of higher-derivative theories include theory of radiation reaction [4], field theory on noncommutative space–time [5], anyons [6] or string theories with extrinsic curvature [7].

Once we get used to the idea that second or higher derivatives can enter the Lagrangian leading to reasonable theory the question arises whether there exists a viable quantum version of such dynamics. Following conservative route, to answer this question we look first for the Hamiltonian formalism. Then we encounter serious difficulty: the Hamiltonian is unbounded from below [8]. Moreover, this property is related to large volumes of phase space so it cannot disappear on the quantum level due to the uncertainty principle (like, for example, it is the case for hydrogen atom). In some situations this frustrating property of Ostrogradski Hamiltonian causes no problem. This is the case for linear systems like the celebrated Pais–Uhlenbeck (PU) one [9], because there are no transitions between the states of different energies. However, linearity is an idealization and any perturbation may cause instability. Another possibility of making the problem harmless arises if there exists some additional integral of motion bounded from below; then one can try to construct an alternative Hamiltonian formalism with the new constant of motion playing the role of Hamiltonian. This is again the case for PU oscillator which is a completely integrable system [10]. In general, however, nothing allows us to believe that there should exist integrals of motion functionally independent of the one generated by time translation invariance. Therefore, we cannot expect that, in general, there exists an alternative Hamiltonian formalism with bounded Hamiltonian.

There is some way out of this situation. In the Ostrogradski formalism one has to introduce additional auxiliary canonical variables in order to make the dimension of phase space coinciding with the number of initial data needed for obtaining unique solution of Lagrange equations. Starting from Ostrogradski variables one can then perform a complex canonical transformation. The only condition one has to impose is that the reality properties of basic Lagrangian variables remain unaffected; the auxiliary variables may become complex. Upon standard quantization their properties under (Hermitean) conjugation will, in general, change. This would change the spectral properties of the Hamiltonian; the eigenvalues may become positive. However, there is some price to be payed: to obtain an agreement between transformation properties of classical and quantum auxiliary variables under conjugation one has to modify the metric in the space of states. The new metric is necessarily indefinite. This is easily seen by referring to the correspondence principle: the classical energies can be negative while the expectation values of the Hamiltonian with positive spectrum become negative only provided there exist states with negative norm.

The above-sketched scheme is encountered in the quantization scheme proposed by Hawking and Hertog [11]. Their starting point is the higher-derivative theory defined by the action which is positive definite in Euclidean region. Then the Euclidean path integral

$$\int \mathcal{D}e^{-S_E[\phi]} \tag{1}$$

makes sense. Hawking and Hertog gave well-defined prescription allowing to compute the Euclidean time transition amplitudes in terms of the above path integral.

* Corresponding author.
E-mail address: k-andrzejewski@uni.lodz.pl (K. Andrzejewski).

0370-2693/$ – see front matter
doi:10.1016/j.physletb.2011.11.024

The real-time counterpart of the Hawking–Hertog construction for PU oscillator has been carefully analyzed in [12]. It appeared that, for certain range of parameter, the Hamiltonian has purely real positive point spectrum; while the metric is in the space of states indefinite and one obtains the scenario sketched above. However, it appeared also that the quantum theory is well-defined only in some subrange of parameters for which the integral (1) makes sense. In particular, if both frequencies of PU oscillator coincide, the Hamiltonian is no longer diagonalizable [12,13]. Additionally, all eigenvectors, expect the ones corresponding to the lowest eigenvalue, have vanishing norm. One can hardly ascribe any physical meaning to the theory with such properties.

In an interesting paper [14] Rivelles proposed an alternative point of view on some higher-derivative theories. He considered the case of equal frequencies (in the limit when the interaction is switched off) and added the Faddeev–Popov ghost term to the Lagrangian. The dynamics of initial fields remains unaffected but the theory acquires BRST symmetry. Imposing the condition of BRST-invariance one can show that the so-called quartet mechanism [15] operates leading to trivial theory. The argument presented in Ref. [14] is very elegant but it is based on the a priori assumption of BRST-invariance of physical states. Contrary to the case of gauge theories, the status of this condition is here slightly unclear. We shall analyze this problem in more detail and show how it is related to the nondiagonalizability of the Hamiltonian. We will consider the free quartic case (i.e. PU oscillator) but we conjecture our results survive in more general context (see below). Moreover, due to the translation invariance we can Fourier transform the spatial variables and reduce the problem to $(1+0)$-dimensional case.

The Lagrangian considered in Ref. [14], with the ghost term neglected, reads ($\phi^*=\phi$, $b^*=b$):

$$\begin{aligned} L &= -b\left(\frac{d^2}{dt^2}+2m^2\right)\phi+\frac{1}{2}b^2 \\ &= \dot b\dot\phi-2m^2 b\phi+\frac{1}{2}b^2+\text{total derivative}, \end{aligned} \tag{2}$$

and yields

$$H = \Pi_b\cdot\Pi_\phi+2m^2 b\cdot\phi-\frac{1}{2}b^2. \tag{3}$$

To make the contact with the model considered in Ref. [12] we perform the canonical transformation

$$\begin{aligned} &\phi=\frac{q_1}{2m}, \qquad b=m(q_1+2ip_2), \\ &\Pi_\phi=m(-iq_2+2p_1), \qquad \Pi_b=\frac{iq_2}{2m}. \end{aligned} \tag{4}$$

Eqs. (4) imply that q_2 and p_2 are purely imaginary. The Hamiltonian, when expressed in new variables, reads

$$H = iq_2p_1+2m^2p_2^2+\frac{m^2}{2}q_1^2+\frac{1}{2}q_2^2. \tag{5}$$

One can show [12] that on the quantum level H becomes Hermitean (while the expectation values of q_2 and p_2 – purely imaginary) only provided the metric of the space of states is indefinite.

To find the spectrum of H the creation and annihilation operators are introduced by putting

$$\begin{aligned} &q_1=\frac{1}{\sqrt[4]{8m^2}}\left(g+g^\star\right), \qquad p_1=-i\sqrt[4]{\frac{m^2}{2}}\left(g-g^\star+\frac{1}{2}\left(d-d^\star\right)\right), \\ &q_2=\sqrt[4]{\frac{m^2}{2}}\left(d-d^\star\right), \qquad p_2=\frac{-i}{\sqrt[4]{8m^2}}\left(d+d^\star+\frac{1}{2}\left(g+g^\star\right)\right), \end{aligned} \tag{6}$$

which yields

$$\left[g,g^\star\right]=1, \qquad \left[d,d^\star\right]=-1, \tag{7}$$

as well as

$$H=-\sqrt{2}m\left(2d^\star d+dg^\star+d^\star g-1\right). \tag{8}$$

The space of states is spanned by the vectors

$$\begin{aligned} &|n_1,n_2\rangle=\frac{1}{\sqrt{n_1!}}\frac{1}{\sqrt{n_2!}}\left(g^\star\right)^{n_1}\left(d^\star\right)^{n_2}|0,0\rangle, \\ &d|0,0\rangle=g|0,0\rangle=0, \end{aligned} \tag{9}$$

obeying

$$\begin{aligned} &\langle n_1,n_2|n_1',n_2'\rangle=(-1)^{n_2}\delta_{n_1n_1'}\delta_{n_2n_2'}, \\ &H|0,0\rangle=m\sqrt{2}|0,0\rangle. \end{aligned} \tag{10}$$

Let

$$N=g^*g-d^*d \tag{11}$$

be the number operator

$$N|n_1,n_2\rangle=(n_1+n_2)|n_1,n_2\rangle. \tag{12}$$

The following important identity holds

$$H-\sqrt{2}m(N+1)=\sqrt{2}m(d+g)\left(d^\star+g^\star\right). \tag{13}$$

Consider the subspace $\mathcal{H}_n$ spanned by the vectors $|n_1,n_2\rangle$ such that $n_1+n_2=n$. Then on $\mathcal{H}_n$, one has (by virtue of Eq. (13))

$$\left(H-\sqrt{2}m(N+1)\right)^{n+1}=0. \tag{14}$$

Therefore on $\mathcal{H}_n$, H acquires a Jordan cell form corresponding to the eigenvalue $\sqrt{2}m(n+1)$. The canonical basis in $\mathcal{H}_n$ is spanned by the vectors $(g^*+d^*)^k(g^*-d^*)^{n-k}|0,0\rangle$, $k=0,1,\ldots,n$; in particular,

$$|\psi_n\rangle=\left(g^*+d^*\right)^n|0,0\rangle \tag{15}$$

is the eigenvector of H, $H|\psi_n\rangle=\sqrt{2}m(n+1)|\psi_n\rangle$. We have also $\langle\psi_n|\psi_n\rangle=\delta_{n0}$ so among the eigenvectors of H only the vacuum has a nonvanishing norm.

We conclude that H is not fully diagonalizable. Instead, it can be put in Jordan block form. The spectrum of H is purely real, positive and discrete, $E_n=\sqrt{2}m(n+1)$, $n=0,1,\ldots$; the Jordan cell corresponding to E_n is $(n+1)$-dimensional. The important point is that, as noted above, the only eigenvector of H with nonvanishing norm is the vacuum $|\psi_0\rangle=|0,0\rangle$.

In the case of nondiagonalizable Hamiltonian the important question arises what is the physical subspace of our theory. Apart from obvious condition that the metric is nonnegative when restricted to physical subspace there should exist further constraints related to the question of proper definition of energy of the system. The natural assumption is that there exist stationary states and any physical state may be represented as their combination. Formally, this means that the physical subspace is spanned by the eigenvectors of H. In our case, all eigenvectors, except vacuum, have vanishing norm. Therefore the only physical state is the vacuum.

Consider now the BRST-symmetry formulation of our model [14]. The starting point is the BRST-extended Lagrangian

$$L=\dot b\dot\phi-2m^2b\phi+\frac{1}{2}b^2\pm i\left(\dot{\bar c}\dot c-2m^2\bar c c\right), \tag{16}$$

where $c, \bar{c}$ are fermionic ghost fields obeying $c^* = c$, $\bar{c}^* = \bar{c}$. Due to the form of L the ghost fields are free so they do not influence the original dynamics. The BRST symmetry reads

$$\delta c = 0, \qquad \delta b = 0, \qquad \delta\phi = \mp i\delta\lambda c, \qquad \delta\bar{c} = \delta\lambda b, \tag{17}$$

where $\delta\lambda$ is anticommuting real parameters, $\delta\lambda^* = \delta\lambda$. To see what is happening let us work in Heisenberg picture. First, we write out the modified Hamiltonian

$$H = \mp i\Pi_{\bar{c}}\Pi_c \pm 2m^2\bar{c}c + \Pi_b\Pi_\phi - \frac{1}{2}b^2 + 2m^2 b\phi. \tag{18}$$

For the bosonic sector the field equations and canonical commutation rules yield

$$b(t) = \sqrt{m}\big(\beta e^{-i\sqrt{2}mt} + \beta^\star e^{i\sqrt{2}mt}\big),$$
$$\phi(t) = \frac{1}{2\sqrt{2m^3}}\big((\alpha + imt\beta)e^{-i\sqrt{2}mt} + \big(\alpha^\star - imt\beta^\star\big)e^{i\sqrt{2}mt}\big), \tag{19}$$

with

$$\big[\alpha, \beta^\star\big] = 1, \qquad \big[\beta, \alpha^\star\big] = 1, \qquad \big[\alpha, \alpha^\star\big] = \frac{1}{\sqrt{2}}, \tag{20}$$

and the remaining commutators vanishing. As far as the fermionic sector is concerned one obtains

$$c(t) = \frac{1}{2}\sqrt{\frac{\sqrt{2}}{m}}\big(ue^{-i\sqrt{2}mt} + u^\star e^{i\sqrt{2}mt}\big),$$
$$\bar{c}(t) = \pm\frac{i}{2}\sqrt{\frac{\sqrt{2}}{m}}\big(ve^{-i\sqrt{2}mt} - v^\star e^{i\sqrt{2}mt}\big),$$
$$\big\{u, v^\star\big\} = 1, \qquad \big\{v, u^\star\big\} = 1. \tag{21}$$

The BRST and ghost charges

$$Q_B = \mp(b\dot{c} - c\dot{b}),$$
$$Q_C = \mp i(\dot{\bar{c}}c - \bar{c}\dot{c} \mp 1) \tag{22}$$

take the forms (up to numerical factors)

$$Q_B = \beta^* u - \beta u^*, \qquad Q_c = v^* u - u^* v. \tag{23}$$

To recover the original dynamics we have to get rid of ghosts. To this end we impose first the condition

$$Q_C|\text{phys}\rangle = 0. \tag{24}$$

There are four fermionic states: $|0\rangle_f$ ($u|0\rangle_f = 0 = v|0\rangle_f$), $|u\rangle_f \equiv u^\star|0\rangle_f$, $|v\rangle_f \equiv v^\star|0\rangle_f$ and $|uv\rangle_f \equiv u^\star v^\star|0\rangle_f$. Eq. (24) immediately yields

$$|\text{phys}\rangle = |\text{phys}\rangle_b \otimes |0\rangle_f. \tag{25}$$

Now, one demands the BRST condition to be fulfilled

$$Q_B|\text{phys}\rangle = 0. \tag{26}$$

Contrary to the case of theories invariant under some gauge group there exists here no a priori reason to impose (26). However, once we do this the following constraint emerges

$$\beta|\text{phys}\rangle_b = 0. \tag{27}$$

To find the meaning of this condition we consider again the transformation (4) viewed as the one relating both sets of variables at (say) $t = 0$. One easily finds that

$$\beta = \sqrt[4]{2}(d + g). \tag{28}$$

Referring to the analysis performed at the beginning we conclude that the BRST-invariance singles out the subspace spanned by the eigenvectors of the Hamiltonian. As we have shown above the only state of nonvanishing norm is the vacuum, in full agreement with quartet mechanism [14,15].

Now the question arises what is happening if the quartic oscillator is quantized by the method of Pais and Uhlenbeck [9]. To this end let us note that the PU Hamiltonian is obtained from Eq. (3) by the transformation (4) with iq_2 replaced by q_2 and ip_2 by $-p_2$. Then q_2 and p_2 are Hermitean so no modification of metric is necessary. The space of states becomes a Hilbert space while the Hamiltonian is unbounded from below. In fact, it becomes a difference of two commuting terms: the angular momentum operator and radial coordinate squared [9]. This can be directly shown by considering the canonical transformation

$$b = -\sqrt{2}Q_1, \qquad \phi = -\frac{1}{4\sqrt{2}m^2}Q_1 + \frac{1}{2m}P_2,$$
$$\Pi_b = \frac{1}{4m}Q_2 - \frac{1}{\sqrt{2}}P_1, \qquad \Pi_\phi = -2mQ_2, \tag{29}$$

which converts H, Eq. (3), into

$$H = \sqrt{2}m(Q_2P_1 - Q_1P_2) - \frac{1}{2}\big(Q_1^2 + Q_2^2\big). \tag{30}$$

The two pieces of H commute. The first part has purely discrete unbounded spectrum while the second – purely continuous nonpositive one. In terms of α and β variables, H reads

$$H = \sqrt{2}m\big(\alpha^*\beta + \beta^*\alpha + 1\big) - 2m\beta^*\beta$$
$$= \big(\sqrt{2}m\big(\alpha^*\beta + \beta^*\alpha + 1\big) - m\beta^*\beta\big) - m\beta^*\beta. \tag{31}$$

On the level of these variables the Ostrogradski approach differs by the choice of the representation of the algebra (20).

Namely, one defines the new operators by

$$\alpha = \frac{1}{\sqrt[4]{2}}a_+, \qquad \beta = \sqrt[4]{2}\big(a_+ + a_-^*\big). \tag{32}$$

They obey standard commutation rules for creation and annihilation operators

$$\big[a_\varepsilon, a_{\varepsilon'}^*\big] = \delta_{\varepsilon\varepsilon'}. \tag{33}$$

Then the space of states becomes an ordinary Hilbert space and

$$H = \sqrt{2}m\big(a_+^*a_+ - a_-^*a_-\big) - \sqrt{2}m\big(a_+^* + a_-\big)\big(a_+ + a_-^*\big). \tag{34}$$

One can pose the question concerning the status of BRST-invariance condition (27). It is easy to see that it has no normalizable solutions. Also, the quartet mechanism doesn't work due to the redefinition of creation and annihilation operators (this is obvious as the bosonic sector has positive definite metric).

We have considered only the simplest $(1 + 0)$-dimensional model. However, as we already mentioned above the results are valid for field-theoretical model in $1 + 3$ dimensions. We also restricted ourselves to the quartic case. However, the relation between BRST symmetry and nondiagonalizability of the Hamiltonian seems to be general. It has been noted in Ref. [14] that the BRST symmetry of higher-derivative theory is nilpotent provided the frequencies coincide. On the other hand, on the classical level the solutions to the equations with degenerate frequencies contain, apart from exponential also polynomial (in time) pieces. Assuming the existence of matrix elements of field operator between the states belonging to the domain of Hamiltonian (which is not always the case [9,16]) one can use the correspondence principle to conclude that the Hamiltonian is not diagonalizable.

Let us comment finally on the interacting case. To preserve the BRST symmetry the interaction is introduced through the prepotential $U(\phi)$ [14]. Except U is chosen to be linear in ϕ the bosonic and fermionic sectors do interact. For linear U we are still dealing with free case with degenerate frequencies; the previous conclusions remain valid.

For more general prepotentials the analytic solution is not available. However, one can conjecture that the Hamiltonian, when reduced to the eigenspace of ghost charge operator corresponding to the eigenvalue zero, is not diagonalizable and the BRST-invariance condition singles out the subspace spanned by its eigenvectors.

Acknowledgements

The discussions with C. Gonera, P. Kosiński and P. Maślanka are gratefully acknowledged. This work is supported in part by MNiSzW grant No. N202331139 (J.G.), as well as from the earmarked subsidies MNiSzW for Young Scientists grant No. 545/199 (K.A.).

References

[1] W. Thiring, Phys. Rev. 77 (1950) 570.
[2] K.S. Stelle, Phys. Rev. D 16 (1977) 953.
[3] E.S. Fradkin, A.A. Tseytlin, Nucl. Phys. B 201 (1982) 469.
[4] A. Mironov, A. Morozov, Theor. Math. Phys. 156 (2008) 1209;
A. Mironov, A. Morozov, Int. J. Mod. Phys. A 23 (2008) 4686;
D. Galakhov, JETP Lett. 87 (2008) 452.
[5] M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977;
R.J. Szabo, Phys. Rep. 378 (2003) 207.
[6] M.S. Plyushchay, Mod. Phys. Lett. A 3 (1988) 1299;
M.S. Plyushchay, Nucl. Phys. B 362 (1991) 54;
M.S. Plyushchay, Phys. Lett. B 262 (1991) 71;
M.S. Plyushchay, Mod. Phys. Lett. A 4 (1989) 837;
M.S. Plyushchay, Phys. Lett. B 243 (1990) 383.
[7] A.M. Polyakov, Nucl. Phys. B 268 (1986) 406.
[8] M. Ostrogradski, Mem. Acad. St. Petersburg 4 (1850) 385;
T. Govaerts, M.S. Rashid, hep-th/9403009;
T. Govaerts, M.S. Rashid, T. Nakamura, S. Hamamoto, Prog. Theor. Phys. 95 (1996) 409.
[9] A. Pais, G.E. Uhlenbeck, Phys. Rev. 79 (1950) 145.
[10] K. Bolonek, P. Kosiński, Acta. Phys. Polon. B 36 (2005) 2115;
A. Mostafazadeh, Phys. Lett. A 375 (2010) 93.
[11] S. Hawking, T. Hertog, Phys. Rev. D 65 (2002) 103515.
[12] K. Andrzejewski, J. Gonera, P. Maślanka, Prog. Theor. Phys. 125 (2011) 247.
[13] K. Bolonek, P. Kosiński, quant-ph/0612009.
[14] V.O. Rivelles, Phys. Lett. B 577 (2003) 147;
See also M. Faizal, M. Khan, Eur. Phys. J. C 71 (2011) 1603.
[15] T. Kugo, Phys. Lett. B 83 (1979) 93.
[16] K. Bolonek, P. Kosiński, J. Phys. A 10 (2007) 11561.