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SWITCHING REGRESSION MODELS WITH NON-NORMAL ERRORS

Abstract. In this paper two forms of switching regression models with non-normal errors are considered. The pseudo maximum likelihood method is proposed for the estimation of their parameters.

Monte Carlo experiments results are presented for a special switching regression model, too. In this research there are compared distributions of parameters estimators for different distributions of errors. The error distributions are as follows: normal, Student's or Laplace's. The maximum likelihood method (for the normal errors) is applied to the estimation. In most of the cases the estimators distributions do not differ significantly.

Key words: switching regression models, maximum likelihood method, pseudo maximum likelihood method.

1. INTRODUCTION

The switching regression models are special cases of models with random coefficients. In this paper two forms of these models are considered. The pseudo maximum likelihood method is proposed for the estimation. This method is an alternative to the maximum likelihood method.

Monte Carlo experiments results are presented for a special switching regression model too. In this research there are compared the distributions of the parameters estimators for different distributions of errors. The error distributions are the following: normal, Student's or Laplace's. The maximum likelihood method and the pseudo maximum likelihood method are applied to the estimation.

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2. THE SWITCHING REGRESSION MODEL

The switching regression is a method of describing the dependence of a certain variable on two or more sets of variables, when the probability of determining the value of a variable explained by a defined group of explanatory variables is either known or unknown.

In this paper we shall deal with a particular case of the form of a switching regression model (see, e.g. Quandt 1972, Kiefer 1978, Charemza 1981, p. 94–97, Tomaszewicz 1985, p. 442–446, Pruska 1987):

$$y_t = \begin{cases} x'_{1t}\alpha_1 + \varepsilon_{1t} & \text{with probability } \lambda \\ x'_{2t}\alpha_2 + \varepsilon_{2t} & \text{with probability } 1 - \lambda \end{cases} \tag{2.1}$$

or

$$\begin{aligned} d_t &= x'_{1t}\alpha_1 + \varepsilon_{1t}, \\ s_t &= x'_{2t}\alpha_2 + \varepsilon_{2t}, \\ y_t &= \min\{d_t, s_t\}, \end{aligned} \tag{2.2}$$

where $t = 1, ..., T$ (T – the sample size), $0 < \lambda < 1$ and λ can be either known or unknown. Other symbols are as follows:

y_t, d_t, s_t – variables explained by the model (d_t, s_t can be observable or not, y_t is observable);

x_{1t}, x_{2t} – column vectors of the explanatory variables;

α_1, α_2 – column vectors of the model's structural parameters;

ε_{1t}, ε_{2t} – errors; random components of the model; random variables with null expectation and variances and σ_1^2 and σ_2^2, respectively, such that $\text{cov}(\varepsilon_{1t}, \varepsilon_{2t}) = 0$, $\text{cov}(\varepsilon_{1t}, \varepsilon_{1\tau}) = 0$, $\text{cov}(\varepsilon_{1t}, \varepsilon_{2\tau}) = 0$, $\text{cov}(\varepsilon_{1t}, \varepsilon_{2t}) = 0$, for $t \neq \tau$ and $t, \tau = \{1, ..., T\}$.

The parameters of the models (2.1) and (2.2), are usually estimated by the maximum likelihood method.

3. PSEUDO MAXIMUM LIKELIHOOD METHOD

The pseudo maximum likelihood (PML) method is an alternative to the maximum likelihood (ML) method. The idea of the PML method is the following (see Gourieroux et al. 1984).

Let us consider a model of the form:

$$y_t = g(x_t, \theta_0) + \varepsilon_t, \tag{3.1}$$

where:

y_t – k-dimensional random vector of the variables explained by the model;
x_t – p-dimensional vector of the explanatory variables;
θ_0 – p-dimensional parameters vector;
ε_t – k-dimensional random vector of errors with zero expectation

and k, p are positive integers, $t = 1, \ldots, T$, T is the sample size, g is a non-linear function.

For the PML estimation of the parameters θ_0 such values of θ are taken (θ is a p-dimensional vector) for which function:

$$L(y_t, x_t, \theta) = \sum_{t=1}^{T} \ln f[(y_t, g(x_t, \theta)] \tag{3.2}$$

called the pseudo maximum likelihood function reaches its maximum, where f is the density function (for continous distributions) or the probability function (for discrete distributions) of distribution which belongs to the linear exponential family.

A family of k-dimensional probability distributions, indexed by parameter $m (m \in R^k)$, is called linear exponential if every element of the family has the density function (or the probability function), which can be written as

$$f(u, m) = \exp\{A(m) + B(u) + C(m)u\} \tag{3.3}$$

where $u \in R^k$, $A(m)$, $B(u)$ are scalars, $C(m)$ is a row k-dimensional vector, and m is expectation of the distribution determined by $f(u, m)$ (see Gourieroux at al. 1984).

Distributions such as binomial, negative binomial, Poisson's, gamma, normal and multidimensional normal belong to this family.

The PML method leads to consistent and asymptotically normal estimators of parameters of investigated random variable (see Gourieroux et al. 1984).

4. APPLICATION OF THE PML METHOD TO THE ESTIMATION OF SWITCHING REGRESSION MODEL

We can apply the PML method to the estimation of model (2.1), when we do not make any assumptions about the class of error distributions. We do not assume that errors are normal.

If $f(u, m)$ in the formula (3.2) is the density function of the normal distribution with expectation m and variance 1, then the PML function for model (2.1.) has its maximum, when the followig function has its minimum:

$$L(y_t; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \lambda) = \sum_{t=1}^{T} [y_t - E(y_t)]^2 \tag{4.1}$$

Let us observe that the density function of y_t has the form:

$$h(y_t) = \lambda f_1(y_t) + (1-\lambda) f_2(y_t) \tag{4.2}$$

where f_1 is the density of y_t with probability λ and f_2 is the density of y_t with probability $1-\lambda$. Next we can write:

$$E(y_t) = \lambda x'_{1t}\alpha_1 + (1-\lambda) x'_{2t}\alpha_2 \tag{4.3}$$

And so we have:

$$L(y_t; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \lambda) = \sum_{t=1}^{T} [y_t - (\lambda x'_{1t}\alpha_1 + (1-\lambda) x'_{2t}\alpha_2))]^2 \tag{4.4}$$

In this case the PML method is reduced to the least squares non-linear method.

If we want to apply the PML method to the estimation of model (2.2), we have to write this model in another form (see Pruska 1992). Let us consider the model:

$$\begin{aligned} d_t &= x'_{1t}\alpha_1 + \sigma_1 u_{1t}, \\ s_t &= x'_{2t}\alpha_2 + \sigma_2 u_{2t}, \\ y_t &= E(\min\{d_t, s_t\}) + \varepsilon_t \end{aligned} \tag{4.5}$$

where u_{1t} and u_{2t} are independent random variables with normal distributions, the expectation of which is equal to zero and the variance is equal to one, ε_t has unknown distribution with the expectation equal to zero. Now, we can minimize the function

$$L(y_t; \alpha_1, \alpha_2, \sigma_1, \sigma_2, \lambda) = \sum_{t=1}^{T} [y_t - E(y_t)]^2 \tag{4.6}$$

where

$$E(y_t) = E(\min\{x'_{1t}\alpha_1 + \sigma_1 u_{1t};\ x'_{2t}\alpha_2 + \sigma_2 u_{2t}\}) \tag{4.7}$$

In this minimizing we can apply numerical approximation of expectation $E(y_t)$ using Monte Carlo methods (see Laroque, Salanie 1989).

5. MONTE CARLO EXPERIMENTS

Monte Carlo experiments have, as their purpose, comparisons of estimators' distributions for different distributions of errors, when we apply the maximum likelihood method (for the normal errors) to the estimation of model (2.2). We consider three distributions of errors: normal, Student's and Laplace's.

For all the cases the model has the form:

$$\begin{aligned} d_t &= x'_{1t}\alpha_1 + \beta_1 + \varepsilon_{1t}, \\ s_t &= x'_{2t}\alpha_2 + \beta_2 + \varepsilon_{2t}, \\ y_t &= \min\{d_t, s_t\} \end{aligned} \tag{5.1}$$

with $\alpha_1 = 1$, $\beta_1 = -5$, $\alpha_2 = 4$, $\beta_2 = 5$, $\sigma_1^2 = 3$, $\sigma_2^2 = 2$, $x_{1t} \sim N(90; 20)$; $x_{2t} \sim N(20; 5)$; symbol $x \sim N(\mu, \sigma)$ means that x has the normal distribution with expectation μ and variance σ^2.

We generate samples for ε_{1t} and ε_{2t} with different distributions. The variances of these distributions are equal: σ_1^2 for ε_{1t} and σ_2^2 for ε_{2t}. We compute the values of d_t, s_t, y_t, and we determine ML-estimates of parameters α_1, α_2, β_1, β_2, σ_1, σ_2 for the normal errors. Next we apply the same algorithm to estimating the parameters of model (5.1) with Student's errors and Laplace's errors. We generate samples with size $n = 20$, 30, 50, 100. We repeat it ten times.

We set three hypotheses:

– parameter estimator distributions do not differ if errors have the normal distribution and the Student's distribution;

– parameter estimator distributions do not differ if errors have the normal distribution and the Laplace's distribution;

– parameter estimator distributions do not differ if errors have the Student's distribution and the Laplace's distribution.

We apply the run test. In this test the critical value is $k = 6$ for significance level $\alpha = 0.05$ and for the test parameters $n_1 = n_2 = 10$.

Values k of the run test statistics are in the Tab. 1. In five cases we reject some hypotheses ($k \leqslant k_\alpha$). In other cases we cannot reject hypotheses ($k > k_\alpha$). These results do not allow to notice significant differences between the parameter estimators' distributions for different errors distributions.

6. FINAL REMARKS

In this paper the pseudo maximum likelihood method for the estimation of two forms of switching regression models with non-normal errors is presented. This method leads to consistent and asymptotically normal estimators of model parameters. These properties are like properties of the maximum likelihood estimators but we do not have to assume that model errors are normal. It seems that the PML method can find wide application

Table 1

Number k of runs in the run test sequences of estimates of switching regression models parametres

Size of sample	Compared distribution of errors	Parameter					
		α_1	β_1	α_2	β_2	σ_1	σ_2
20	normal d. and Student's d.	15	11	14	13	9	12
	normal d. and Laplace's d.	7	5	9	11	13	11
	Student's d. and Laplace's d.	7	7	8	7	13	13
30	normal d. and Student's d.	9	11	12	14	7	10
	normal d. and Laplace's d.	12	13	14	15	9	10
	Student's d. and Laplace's d.	12	12	13	10	8	4
50	normal d. and Student's d.	11	13	13	13	7	12
	normal d. and Laplace's d.	10	13	8	4	5	8
	Student's d. and Laplace's d.	10	14	10	10	13	12
100	normal d. and Student's d.	11	11	8	10	7	9
	normal d. and Laplace's d.	13	7	7	12	7	6
	Student's d. and Laplace's d.	10	8	7	8	14	8

in switching regression analysis. Monte Carlo experiments do not allow to notice significant differences between the ML estimator distributions for different errors distributions. These experiments were performed for one model only. We cannot generalize their results. In the future it is necessary to perform more experiments for different models and for different errors distributions.

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MODELE REGRESJI PRZEŁĄCZNIKOWEJ ZE SKŁADNIKAMI LOSOWYMI O ROZKŁADACH RÓŻNYCH OD NORMALNEGO

W pracy tej rozważane są dwie postacie modeli regresji przełącznikowej ze składnikami losowymi o rozkładach różnych od normalnego. Do estymacji parametrów tych modeli zaproponowana jest metoda największej pseudowiarygodności.

Przedstawione są tu także wyniki eksperymentów Monte Carlo dla szczególnego modelu regresji przełącznikowej, dotyczące porównania rozkładów estymatorów parametrów przy różnych rozkładach składników losowych (rozkład normalny, Studenta i Laplace'a). W większości przypadków nie zauważa się statystycznie istotnych różnic między rozkładami estymatorów, gdy stosujemy do estymacji ten sam algorytm estymacyjny, wynikający z metody największej wiarygodności przy założeniu, że badana zmienna ma rozkład normalny.