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MULTIVARIATE MULTIVALUED RANDOM VARIABLE

ABSTRACT. Given a probability measure space (Ω, A, P), random variable in classical definition is a mapping from Ω to **R**. Multivalued random variable is a mapping from Ω to all subset of X. For a real separable Banach space X with dual space X^*, let L^p (Ω, A), for $1 \le p \le \infty$, denote the X – valued L^p – space. In this paper we present the integral for multifunction and some property of multivalued random variables in multivariate case. The theory of multivalued random variables has been established for Banach space-valued and for Bochner-integrable function. The main purpose of this paper is to present a theory of multivalued random variables as a generalisation of point-valued cases.

Key words: multivariate random variable, multivalued random variables.

I. INTRODUCTION

We shall give some properties of the integration of multivalued function, introduced by Aumann (1965). In this paper we present the integral for multifunction and some property of multivalued random variables in multivariate case. We shall establish the existence of the multivalued conditional expectation of multivalued random variables, and present a number of properties analogous to those of the usual conditional expectation. The theory of conditional expectation has been established for Banach space-valued and for Bochner-integrable function.

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II. MULTIVALUED RANDOM VARIABLE

Given a probability measure space (Ω, A, μ) random variable in classical definition is a mapping from Ω to **R**. Multivalued random variable is a mapping from Ω to all closed subset of X.

We have a real Banach space X with metric d. For any nonempty and closed sets $A, B \subset X$ we define the Hausdorff distance $h(A, B)$ of A and B.

Definition 1. The excess for two nonempty and closed sets be defined by

$$e(A, B) = \sup_{x \in A} d(x, B), \text{ where } d(x, B) = \inf_{y \in B} | x - y |$$

the Hausdorff distance of A and B is given by

$$h(\mathrm{A}, \mathrm{B}) = \max \{(eA, B), e(B, A)\},$$

the norm $\| A \|$ of set A we get as

$$\| \mathrm{A} \| = h(A, \{0\}) = \sup_{x \in A} \| x \|.$$

The set of all nonempty and closed subsets of X is a metric space with the Hausdorff distance. The set of all nonempty and compact subsets of X is a complete, separable metric space with the metric h.

Definition 2. A multivalued function φ: $\Omega \to 2^X$ with nonempty and closed values, is said to be (weakly) measurable if φ satisfies the following equivalent conditions:

a) $\varphi^{-1}(C) = \{\omega \in \Omega : \varphi(\omega) \cap C \neq \emptyset\} \in A$ for every C open subset of X,

b) $d(x, \varphi(\omega))$ is measurable in ω for every $x \in X$,

c) there exists a sequence $\{f_n\}$ of measurable functions f_n: $\Omega \to X$ such that

$$\varphi(\omega) = cl\{f_n(\omega)\} \text{ for all } \omega \in \Omega.$$

Definition 3. A measurable multivalued function $\varphi : \Omega \to 2^X$ with nonempty and closed values is called a multivalued random variable.

A multivalued function φ is called strongly measurable, if there exists a sequence $\{\varphi_n\}$ of simple functions (measurable functions having a finite number of values in 2^X), such that $h(\varphi_n(\omega), \varphi(\omega)) \to 0$ a.s.

Since set of all nonempty and compact (or convex and compact) subsets of X is a complete separable metric space with the metric h, so multifunction $\varphi : \Omega \to 2^X$ is measurable if and only if is strongly measurable. This is equivalent to the Borel measurability of φ.

Let $K(X)$ denote all nonempty and closed subsets of X. As the σ- field on $K(X)$, we get the σ- field generated by $\varphi^{-1}(C) = \{\omega \in \Omega: \varphi(\omega) \cap C \neq \varnothing\}$, for every open subset C of X. The smallest σ-algebra containing these $\varphi^{-1}(C)$ were denoted by $A\varphi$. Two multifunctions φ and ψ are independent if $A\varphi$ and A_ψ are independent. Two multifunctions ϕ and φ are identically distributed if $\mu(\varphi^{-1}(C)) = \mu(\psi^{-1}(C))$ for all closed $C \subset X$.

Definition 4. We say that a sequence of multivalued random variables $\varphi_n: \Omega \to 2^{K(X)}$ is independent if so is $\{\varphi_n\}$ considered as measurable functions from (Ω, A, μ) to $(K(X), G)$.

Definition 5. Two multivalued random variables $\varphi, \psi : \Omega \to 2^{K(X)}$ are identically distributed if $\varphi(\omega) = \psi(\omega)$ a.s.

Particularly for φ_n with compact values independence (identical distributedness) of $\{\varphi_n\}$ coincides with that considered as Borel measurable functions to all nonempty, compact subsets of X.

Definition 6. A selection of the measurable multifunction $\varphi : \Omega \to 2^X$ is a measurable function $f: \Omega \to X$, such that $f(\omega) \in \varphi(\omega)$ for all $\omega \in \Omega$.

Let $\varphi, \psi : \Omega \to 2^{K(X)}$ be two multivalued random variables, we define the following operation (C a s t a i n g, V a l a d i e r 1977):

1) $(\varphi \cup \psi)(\omega) = cl(\varphi(\omega) + \psi(\omega)), \omega \in \Omega$.

2) for a measurable real-valued function g

$$(g\varphi)(\omega) = g(\omega)\varphi(\omega), \qquad \omega \in \Omega.$$

3) $(\overline{co}\,\varphi)(\omega) = \overline{co}\,\varphi(\omega), \qquad \omega \in \Omega,$

($\overline{co}$ -denote the closed convex hull).

III. MEAN OF MULTIVALUED RANDOM VARIABLE

Let $L^p(\Omega, A)$, for $1 \le p \le \infty$, denote the X – valued L^p – space. We introduce the multivalued $\mathbf{L}^p$ space.

Definition 7. The multivalued space $\mathbf{L}^p\ [\Omega, K(X)]$, for $1 \le p \le \infty$ denote the space of all measurable multivalued functions $\varphi : \Omega \to 2^{K(X)}$, such that $\|\varphi\| = \|\varphi(\cdot)\|$ is in $\mathbf{L}^p$.

Then $\mathbf{L}^p\,[\Omega, K(X)]$ becomes a complete metric space with the metric H_p given by

$$H_p(\varphi, \psi) = \{ \int_\Omega h(\varphi(\omega),\psi(\omega))^p d\mu \}^{1/p}, \text{ for } 1 \le p \le \infty$$

$$H_\infty(\varphi, \psi) = \underset{\omega\in\Omega}{ess\ \sup}\, \mathrm{h}(\varphi(\omega), \psi(\omega)),$$

where φ and ψ are considered to be identical if $\varphi(\omega) = \psi(\omega)$ a.s.

We can define similarly other $\mathbf{L}^p$ space for set of different subsets of X (convex and closed, weakly compact or compact). We denote by $\mathbf{L}^p\,[\Omega, K(X)]$ the space of all strongly measurable functions in $\mathbf{L}^p\,[\Omega, K(X)]$. Then this space is complete metric space with the metric H_p.

Definition 8. The mean $E(\varphi)$, for a multivalued random variables $\varphi : \Omega \to 2^{K(X)}$ is given as the integral $\int_\Omega \varphi d\mu$ of φ defined by

$$E(\varphi) = \int_\Omega \varphi d\mu = \{ \int_\Omega f d\mu : f \in S(\varphi)\},$$

where

$$S(\varphi) = \{f \in L^1 [\Omega, X]: f(\omega) \in \varphi(\omega) \text{ a.s.}\}$$

The mean $E(\varphi)$ exists if $S(\varphi)$ is nonempty. Multifunction ϕ is an integrable, if $\|\phi(\omega)\|$ is an integrable. If ϕ have an integral, then $E(\phi)$ is compact. If μ is atomless, then $E(\phi)$ is convex. If ϕ have an integral and $E(\phi)$ is nonempty, then co $E(\phi) = E(\text{co}\phi)$, (co-denote convex hull of the set).

This multivalued integral was introduced by A u m a n n (1965). For detailed arguments concerning the measurability and integration of multifunction we refer to B e r g e (1966); C a s t a i n g, V a l a d i e r (1977); D e b r e u (1967). Now we present some properties of mean of multivalued random variables.

Let $\varphi, \psi : \Omega \to 2^{K(X)}$ be two multivalued random variables with nonempty $S(\varphi)$ and $S(\psi)$ then:

1) $cl\, E(\varphi \cup \psi) = cl\,(E(\varphi) + E(\psi))$, where $(\varphi \cup \psi)(\omega) = cl\,(\varphi(\omega) + \psi(\omega))$.

2) $cl\, E(\overline{co}\,\varphi) = \overline{co}\, E(\varphi)$, where $(\overline{co}\,\varphi)(\omega) = \overline{co}\,\varphi(\omega)$, the closed convex hull.

3) $h(cl\, E(\varphi), cl\, E(\psi)) = H_1(\varphi, \psi)$.

Lemma 1. (B e r g e 1966) Let $\varphi : \Omega \to 2^{K(X)}$ and $1 \le p \le \infty$.
If $S^p(\varphi) = \{f \in L^p[\Omega, X]: f(\omega) \in \varphi(\omega) \text{ a.s.}\}$ then exists a sequence $\{f_n\}$ contained in $S^p(\varphi)$ such that $\varphi(\omega) = cl\,\{f_n(\omega)\}$ for all $\omega \in \Omega$.

Lemma 2. (B e r g e 1966) Let $\varphi, \psi : \Omega \to 2^{K(X)}$ and $1 \le p \le \infty$.
If $S^p(\varphi) = S^p(\psi) \ne \varnothing$ then $\varphi(\omega) = \psi(\omega)$ a.s.

These properties of mean of multivalued random variables are in fact the properties of the multivalued Aumann's integral.

IV. CONDITIONAL EXPECTATION OF MULTIVALUED RANDOM VARIABLES

Given a probability measure space (Ω, A, μ) we assume that it is a finite measure and we get B as a sub- σ- field of A. For $\varphi \in \mathbf{L}^1 [\Omega, B, \mu, X]$ we define:

$$S_B(\varphi) = \{f \in L^1 [\Omega, B, \mu, X]: f(\omega) \in \varphi(\omega) \text{ a.s.}\}$$

The integral of φ on (Ω, B, μ) is defined as

$$\int_{\Omega}^{B} \varphi d\mu = \{ \int_{\Omega} f d\mu : f \in S_B(\varphi) \}$$

Definition 9. For $f \in L^1[\Omega, X]$ the conditional expectation $E(f/B)$ of f relative to B is defined as a function $E(f/B) \in L^1[\Omega, B, \mu, X]$ such that

$$\int_A E(f / B) d\mu = \int_A f d\mu , \; A \in B$$

When X is a Banach space, it is known that conditional expectation $E(f/\mathrm{B})$ exists uniquely for any $L^1[\Omega, X]$. We have some well-known properties of conditional expectation. Now we define the multivalued conditional expectation and next we present properties of our new multivalued random variable.

Definition 10. Let $\varphi \in \mathbf{L}^1[\Omega, X]$, the multivalued function $\phi \in \mathbf{L}^1[\Omega, B, \mu, \mathrm{X}]$ which satisfying

$S_B(\phi) = cl\{E(f/\mathrm{B}) : f \in S(\varphi)\}$,the closure is taken with respect to $\mathbf{L}^1[\Omega, X]$ we call multivalued conditional expectation of φ relative to B, we notice $\phi = E(\varphi / B)$.

Theorem 1. Let $\varphi \in \mathbf{L}^1[\Omega, X]$, then there exists a unique $E(\varphi/B) \in \mathbf{L}^1[\Omega, B, \mu, X]$

There exists a unique ϕ which is equal to the closure of the set of the conditional expectation for all integrable selections of φ. If B is trivial $B = \{\varnothing, \Omega\}$ then $E(\varphi/B) = [\mu(\Omega)]^{-1} \int_{\Omega} \varphi d\mu$. We recall some basic properties of multivalued conditional expectation, analogous to those of the usual conditional expectation (T r z p i o t 1996, 1999).

Theorem 2. Let $\varphi, \psi : \Omega \to 2^{K(X)}$ be two multivalued random variables with nonempty $S(\varphi)$ and $S(\psi)$, then the conditional expectation $E(\varphi / B)$ of φ relative to B have the following properties:

1) $cl\, E(\varphi \cup \psi/B) = cl\, (E(\varphi/B) \cup E(\psi/B))$,

2) $E(g\varphi/B) = gE(\varphi/B)$, where g is measurable real-valued function,

3) $E(\overline{co}\, \varphi/B) = \overline{co}\, \mathrm{E}(\varphi/B)$.

Theorem 3.

1) If $\varphi \in \mathbf{L}^1[\Omega, B, \mu, X]$, $\varphi(\omega)$ is convex and g is nonnegative real L^∞ function, then conditional expectation $E(g\varphi/B) = E(g/B)\varphi$, in particular $E(\varphi/B) = \varphi$.

2) If $B_1 \subset B \subset A$ and $\varphi \in \mathbf{L}^1[\Omega, B, \mu, X]$, $\varphi(\omega)$ is convex then $E(\varphi/B_1)$ taken on the base space (Ω, A, μ) is equal to the conditional expectation of φ relative to B_1 taken on the base space (Ω, B, μ).

3) $E(E(\varphi/B)/B_1) = E(\varphi/B_1)$ for $B_1 \subset B \subset A$.

We can add that both theorems were proved directly from properties of integrals of set-valued functions.

V. CONVERGENCE OF MULTIVALUED CONDITIONAL EXPECTATION

We establish convergence theorem for multivalued conditional expectation (particularly for multivalued integrals). Let B be a fixed sub-σ-filed on A and $\{\varphi_n\}$ a sequence of multivalued random variables with nonempty and closed value. We have the monotone convergence theorem.

Theorem 4. Suppose that $\varphi_1(\omega) \subset \varphi_2(\omega) \subset .....$ a.s. with $S(\varphi_1) \neq \varnothing$ and let $\varphi(\omega) = cl\{\bigcup_{n=1}^{\infty} \varphi_n(\omega)\}$ $\omega \in \Omega$. Then φ has nonempty and closed value and

$$E(\varphi/B)(\omega) = cl\{\bigcup_{n=1}^{\infty} E(\varphi_n/B)(\omega)\} \text{ a.s.}$$

Proof. Let $\psi = cl\{\bigcup_{n=1}^{\infty} E(\varphi_n/B)(\omega)\}$, $\omega \in \Omega$. Then φ and ψ have nonempty and closed value and ψ is B – measurable. Obviously

$$S(\varphi_1) \subset S(\varphi_2) \subset .... \subset S(\varphi),$$

$$S(\varphi_1/B) \subset S(\varphi_2/B) \subset .... \subset S(\varphi/B).$$

For any $f \in S(\varphi)$, we have

$$\inf_{g \in S(\varphi_n)} \|f - g\| = E(d(f(\cdot), \varphi(\cdot))) \to 0$$

since $d(f(\cdot), \varphi(\cdot)) \in L^1$ and $d(f(\omega), \varphi(\omega)) \to 0$ a.s.

Hence $S(\varphi) = cl\,(\bigcup_{n=1}^{\infty} S\,((\varphi_n))$ and similarly $S(\psi/B) = \mathrm{cl}(\bigcup_{n=1}^{\infty} S\,(\varphi_n/B))$.

Thus $S(\varphi/B) = cl(\bigcup_{n=1}^{\infty}\{E(f/B)\colon f \in S(\varphi_n)\} = S(\psi/B)$, which implies $E(\varphi/B)(\omega) =$ $= \psi(\omega)$ a.s.

REFERENCES

Artstein Z., Vitale R. A. (1975), *A Strong Law of Large Numbers for Random Compact Sets*, „Annals of Probability", **3**, 879–882.

Auman R. J. (1965), *Integrals of Set-valued Functions*, „Journal of Mathematical Analysis and Application", **12**, 1, 1–12.

Berge C. (1966), *Espaces Topologiques*, Dunod, Paris.

Borowkow A. (1977), *Rachunek prawdopodobieństwa*, PWN, Warszawa.

Castaing C., Valadier M. (1977), *Convex Analysis and Measurable Multifunctions*, „Lectures Notes of Mathematics", **580**, Springer Verlag, Berlin.

Debreu G. (1967), *Integration of Correspondens*, „Proceedings 5th Berkeley Symposium on Mathematics, Statistics and Probabilistics", **1**, 2, 351–372.

Engelking R. (1975), *Topologia ogólna*, PWN, Warszawa.

Hausdorff F. (1957), *Set Theory*, Chelsea, New Jork.

Hess C. (1991), *Convergence of Conditional Expectations for Unbonded Random Sets, Integrands, and Integral Functionals*, „Mathematics of Operations Research", **16**, 3, 627–649.

Rockefellar R. T. (1976), *Integral Functionals, Normal Integrands, Mesurable Selections*, „Lectures Notes of Mathematics", 543, 157-207.

Salinetti G., Wets R. (1979), *On the Convergence of Sequences of Convex Sets in Finite Dimensions*, „SIAM Review", **21**, 1.

Saporta G. (1990), *Probabilités, analyse des données et statistique*, Edition Technip, Paris.

Trzpiot G. (1994), *Pewne własności całki funkcji wielowartościowych (agregacja zbiorów w modelach decyzyjnych)*, „Prace Naukowe AE Wrocław", **683**, 55–61.

Trzpiot G. (1995), *Multivalued Limit Laws Applied to Stochastic Optimization*, „Random Operators and Stochastic Equations", **3**, 4, 309–314.

Trzpiot G. (1995), *O selektorach projekcji metrycznej*, „Zeszyty Naukowe AE Katowice", **131**, 23–29.

Trzpiot G. (1995). *Twierdzenia graniczne dla wielowartościowych zmiennych losowych*. „Przegląd Statystyczny", **42**, 2, 249–256.

Trzpiot G. (1996), *Conditional Expectation of Multivalued Random Variables*, [in:] *Proceedings of 15th International Conference on Multivariate Statistical Analysis*, Absolwent, Łódź, 31–42.

Trzpiot G. (1997), *Limit Law for Multivalued Random Variable*, „Acta Universitatis Lodziensis", Folia Oeconomica, **141**, 129–136.

Trzpiot G. (1997), *Wielowartościowe aproksymacje stochastyczne*. [w:] *Proceedings of 15th International Conference on Multivariate Statistical Analysis*, Absolwent, Łódź, 224–236.

Trzpiot G. (1999), *Wielowartościowe zmienne losowe w badaniach ekonomicznych*, AE Katowice.

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WIELOWYMIAROWA WIELOWARTOŚCIOWA ZMIENNA LOSOWA

Mając przestrzeń probabilistyczną (Ω, A, P), zmienna losowa jest odwzorowaniem z Ω w **R**. Wielowymiarowa zmienna losowa jest odwzorowaniem z Ω w zbiór wszystkich podzbiorów X. Dla rzeczywistej separowalnej przestrzeni Banacha X z dualną przestrzenią X^*, niech $L^p(\Omega, A)$, dla $1 \leq p \leq \infty$, oznacza X – wartościową przestrzeń L^p. Artykuł zawiera własności całki wielowartościowych odwzorowań w ujęciu wielowymiarowym. Definiujemy warunkowe średnie wraz z własnościami o zbieżności. Podstawowym celem jest ujęcie teorii wielowartościowych zmiennych losowych jako uogólnienia klasycznej teorii.