A C T A U N I V E R S I T A T I S L O D Z I E N S I S
FOLIA OECONOMICA 255, 2011

Małgorzata Graczyk[*]

BALANCED BIPARTITE WEIGHING DESIGNS LEADING TO REGULAR D-OPTIMAL SPRING BALANCE WEIGHING DESIGNS

Abstract. New construction methods of the regular D-optimal spring balance weighing designs under assumption of nonhomogeneity of variances of errors are presented. The constructions are based on the incidence matrices of the balanced bipartite weighing designs.

Key words: balanced bipartite weighing design, D-optimal design, spring balance weighing design.

I. INTRODUCTION

Assume that using n measurement operation we determine unknown measurements of given number p of objects according to the standard Gauss-Markoff model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where $\mathbf{y}$ is an $n \times 1$ random vector of the recorded results of observations (measurements), $\mathbf{X} = (x_{ij})$ is an $n \times p$ matrix with $x_{ij} = 1, 0$ according to if in the i-th measurement operation the j-th object is included or not, $i = 1,2,...,n,\ j = 1,2,...,p$, $\mathbf{w}$ is a $p \times 1$ vector representing unknown measurements of objects and $\mathbf{e}$ is an $n \times 1$ random vector of errors. We assume $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2 \mathbf{G}$, where $\mathbf{G}$ is the positive definite $n \times n$ known matrix. The squares estimator of the vector representing unknown measurements of objects $\mathbf{w}$ is equal to $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ assuming that $\mathbf{X}$ is of full column rank. The variance matrix of $\hat{\mathbf{w}}$ is given by $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

[*] Ph.D., Department of Mathematical and Statistical Methods, Poznań University of Life Sciences.

D-optimal designs there are designs for which the determinant of $\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is minimal. Particularly, the conditions determining the existence of the D-optimal weighing designs for $\mathbf{G} = \mathbf{I}_n$, where $\mathbf{I}_n$ is an $n \times n$ identity matrix, were studied in Gail, Kiefer (1980), Neubauer et al. (1998). The initiation of the study of the existence and construction of D-optimal spring balance weighing design for a special form of $\mathbf{G}$ is presented in Katulska, Przybył (2007).

In the present paper the construction method of a D-optimal spring balance weighing designs for a special form of the variance matrix of errors given in Katulska, Przybył (2007) is considered.

II. REGULAR D – OPTIMAL DESIGN

For a given n and p, let $\mathbf{X}$ be an $n \times p$ design matrix of rank p of a spring balance weighing design in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \end{bmatrix} \tag{1}$$

where $\mathbf{x}$ is $p \times 1$ vector of elements 1 or 0, $\mathbf{X}_1$ is $(n-1) \times p$ design matrix which satisfied the conditions given in Neubauer et al. (1998)

$$\mathbf{X}_1'\mathbf{X}_1 = \eta\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right), \text{ where } \eta = \begin{cases} \dfrac{(p+1)(n-1)}{4p} & \text{if } p \text{ is odd} \\ \dfrac{(p+2)(n-1)}{4(p+1)} & \text{if } p \text{ is even} \end{cases}, \tag{2}$$

$\mathbf{1}_p$ denotes the $p \times 1$ column vector of ones.

Let the variance matrix of errors be of the form

$$\sigma^2\mathbf{G} = \sigma^2 diag\left(1, 1, ..., 1, g^{-1}\right), \quad g > 0 . \tag{3}$$

The definition of the regular D-optimal spring balance weighing design for the matrix $\mathbf{X}$ in the particularly form and the following Theorem are given in Katulska, Przybył (2007).

Definition 1. Any nonsingular spring balance weighing design $\mathbf{X}$ in the form (1) with the variance matrix of errors in (3) is is called the regular D-optimal if

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}=\begin{cases}(p+1)\left(1+\frac{gp}{n-1}\right)\left(\frac{(p+1)(n-1)}{4p}\right)^{p} & \text{if } p \text{ is odd}\\ (p+1)\left(1+\frac{gp}{n-1}\right)\left(\frac{(p+2)(n-1)}{4(p+1)}\right)^{p} & \text{if } p \text{ is even}\end{cases}.$$

Theorem 1. Any nonsingular spring balance weighing design $\mathbf{X}$ in the form (1) for which (2) holds and with the variance matrix of errors in (3) is a regular D-optimal if

(i) $\mathbf{x}'\mathbf{1}_p=\frac{p+1}{2}$ for odd p,

(ii) $\mathbf{x}'\mathbf{1}_p=\frac{p}{2}$ or $\mathbf{x}'\mathbf{1}_p=\frac{p+2}{2}$ for even p.

Although the construction of a regular D-optimal spring balance weighing design is possible by applying the incidence matrices of the balanced incomplete block designs (See Neubauer et al. (1998)) in the present paper we show the construction for certain incidence matrices of the balanced bipartite weighing designs.

III. BALANCED BIPARTITE WEIGHING DESIGNS

Now, we shall give a brief description and elementary properties of the balanced bipartite weighing designs. In the papers Huang(1976) and Swamy (1982) the balanced bipartite weighing design with the parameters v, k_1, k_2, λ_1 are considered. There is design which describe how to replace v treatments in b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k=k_1+k_2$. Each treatment appears in r blocks. Every pair of treatments from different subblocks appears together in λ_1 blocks and every pair of treatments from the same subblock appears together in λ_2 blocks. The integers $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ are all parameters of the balanced bipartite weighing design. The parameters are not independent and they are related by the following identities

$$vr = bk, \quad b = \frac{\lambda_1 v(v-1)}{2k_1k_2}, \quad r = \frac{\lambda_1 k(v-1)}{2k_1k_2}, \quad \lambda_2 = \frac{\lambda_1(k_1(k_1-1)+k_2(k_2-1))}{2k_1k_2}. \quad (4)$$

Let $\mathbf{N}$ be the incidence matrix of such a design with the elements equal to 0 or 1, then

$$\mathbf{NN}' = (r - \lambda_1 - \lambda_2)\mathbf{I}_v + (\lambda_1 + \lambda_2)\mathbf{1}_v\mathbf{1}_v'. \quad (5)$$

In order to exclude trivial balanced bipartite weighing designs, $k = k_1 + k_2$ is always assumed to be greater than two and without loss of generality we assume that $k_1 \le k_2$ and $k_2 = k_1 + c$, where $c = 0, 1, \ldots$. It is justified because the existence of the balanced bipartite weighing design with the parameters $v, b, r, k_{11}, k_{21}, \lambda_1, \lambda_2$ implies the existence of the balanced bipartite weighing design with the parameters $v, b, r, k_{12} = k_{21}, k_{22} = k_{11}, \lambda_1, \lambda_2$. In the other words, if we change the sizes of subblocks then the other parameters of the design do not change.

A balanced bipartite weighing design for which $k_1 = k_2 + c$ is called also a tournament design, see Bose, Cameron (1967). A resolvable balanced bipartite weighing design with $k_1 = k_2 = 2$ and $\lambda_1 = 1$ is also called a whist-tournament, see Baker, Wilson (1974).

IV. CONSTRUCTION OF REGULAR D-OPTIMAL DESIGNS

Let $\mathbf{N}$ be the $v \times b$ incidence matrix of the balanced bipartite weighing design with the parameters $v, b, r, k_1, k_2, \lambda_1, \lambda_2$. Then

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}' \\ \mathbf{x}' \end{bmatrix} \quad (6)$$

is the matrix of a spring balanced weighing design. In this design we have $p = v$ and $n = b + 1$.

Theorem 2. Let v be odd. If there exists any balanced bipartite weighing design for which $v = 2k - 1$ and $r = 2(\lambda_1 + \lambda_2)$ then a spring balance weighing design $\mathbf{X}$ in (6) with the variance matrix of errors in (3) is regular D-optimal.

Proof. From the condition (i) of Theorem 1 and from the conditions (4) it follows that a spring balance weighing design is regular D-optimal if $r = 2(\lambda_1 + \lambda_2)$ and. Thus we obtain the Theorem.

Theorem 3. Let v be odd. If $\mathbf{N}$ is the incidence matrix of the balanced bipartite weighing design with the parameters

$$v = 4k_1 + 2c - 1, \quad b = \frac{\lambda_1(4k_1 + 2c - 1)(2k_1 + c - 1)}{k_1(k_1 + c)},$$

$$r = \frac{\lambda_1(2k_1 + c)(2k_1 + c - 1)}{k_1(k_1 + c)},$$

$$k_1, k_2 = k_1 + c, \lambda_1, \lambda_2 = \frac{\lambda_1\left(2k_1^2 + 2(c-1)k_1 + c(c-1)\right)}{2k_1(k_1 + c)}$$

then $\mathbf{X}$ in (6) is a regular D-optimal spring balance weighing design with the variance matrix of errors given in (3).

Proof. Applying v and r of the form given in Theorem 2 and $k = 2k_1 + c$ we get $v = 4k_1 + 2c - 1$ and $r = 2(\lambda_1 + \lambda_2)$. Since the parameters $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ of the balanced bipartite weighing design satisfy relations (4) it can be easy verified that

$$b = \frac{\lambda_1(4k_1 + 2c - 1)(2k_1 + c - 1)}{k_1(k_1 + c)},$$

$$v = 4k_1 + 2c - 1, \; b = \frac{\lambda_1(4k_1 + 2c - 1)(2k_1 + c - 1)}{k_1(k_1 + c)}, \; r = \frac{\lambda_1(2k_1 + c)(2k_1 + c - 1)}{k_1(k_1 + c)}$$

$$\text{and } \lambda_2 = \frac{\lambda_1\left(2k_1^2 + 2(c-1)k_1 + c(c-1)\right)}{2k_1(k_1 + c)}.$$

Theorem 4. Any regular D-optimal spring balance weighing design $\mathbf{X}$ given by (6) for even v and with the variance matrix of errors in the form (3) does not exist.

Proof. The conditions (2) and (5) imply that a spring balance weighing design is regular D-optimal if $r = 2(\lambda_1 + \lambda_2)$ and $b(v+1) = 2vr$. Now, using the last relation we obtain square equation $v^2 - 2(k-1)v - 2k = 0$, which has not solution in natural numbers. Hence we obtain the Theorem.

V. THE PARAMETERS OF THE BALANCED BIPARTITE WEIGHING DESIGNS

With the results obtained until now we shall give the parameters of the balanced bipartite weighing design based on which we construct the incidence matrix $\mathbf{N}$ and the regular D-optimal spring balance weighing design $\mathbf{X}$ in the form (6).

Theorem 5. If the conditions

(i) $\lambda_1 \equiv 0 \bmod (k_1 + c)$,

(ii) $\lambda_1 c(c-1) \equiv 0 \bmod (2k_1(k_1 + c))$,

(iii) $\lambda_1(2c-1)(c-1) \equiv 0 \bmod (k_1(k_1 + c))$

are simultaneously satisfied then any balanced bipartite weighing design exists.

Proof. The theorem is the consequence of the equalities

$$b = \frac{\lambda_1(4k_1 + 2c - 1)(2k_1 + c - 1)}{k_1(k_1 + c)} = 8\lambda_1 - \frac{6\lambda_1}{k_1 + c} + \frac{\lambda_1(2c-1)(c-1)}{k_1(k_1 + c)},$$

$$v = 4k_1 + 2c - 1,\ b = \frac{\lambda_1(4k_1 + 2c - 1)(2k_1 + c - 1)}{k_1(k_1 + c)},\ r = \frac{\lambda_1(2k_1 + c)(2k_1 + c - 1)}{k_1(k_1 + c)}$$

$$= 4\lambda_1 - \frac{2\lambda_1}{k_1 + c} + \frac{\lambda_1 c(c-1)}{k_1(k_1 + c)} \text{ and } \lambda_2 = \frac{\lambda_1\left(2k_1^2 + 2(c-1)k_1 + c(c-1)\right)}{2k_1(k_1 + c)}$$

$$= \lambda_1 - \frac{\lambda_1}{k_1 + c} + \frac{\lambda_1 c(c-1)}{2k_1(k_1 + c)}.$$

From Theorems 2 and 3 the following Corollary can be establish.

Corollary 1. If $c = 0$ and $\lambda_1 \equiv 0 \bmod \left(k_1^2\right)$ then the balanced bipartite weighing design with parameters

$$v = 4k_1 - 1,\ b = \frac{\lambda_1(4k_1 - 1)(2k_1 - 1)}{k_1^2},\ r = \frac{2\lambda_1(2k_1 - 1)}{k_1}, k_1, k_2 = k_1, \lambda_1,$$

$\lambda_2 = \dfrac{\lambda_1(k_1 - 1)}{k_1}$ exists.

The condition $\lambda_1 \equiv 0 \bmod (k_1^2)$ is equivalent $\lambda_1 = uk_1^2$, $u = 1,2,\ldots$. If $\lambda_1 = uk_1^2$ then the necessary conditions for the existence of the balanced bipartite weighing designs are $v = 4k_1 - 1, b = u(4k_1 - 1)(2k_1 - 1), r = 2uk_1(2k_1 - 1)$, $k_1,\ k_2 = k_1, \lambda_1 = uk_1^2, \lambda_2 = uk_1(k_1 - 1)$.

Let us consider the special cases.

If $u = 1$ and $k_1 = k_2 = 2$ then $v = 7, b = 21, r = 12, \lambda_1 = 4, \lambda_2 = 2$ (the design N^{o} 20 of Huang (1976)).

If $u = 1$ and $k_1 = k_2 = 3$ then $v = 11, b = 55, r = 30, \lambda_1 = 9, \lambda_2 = 6$ (the design N^{o} 94 of Huang (1976)).

Corollary 2. If $c = 1$ and $\lambda_1 \equiv 0 \bmod (k_1 + 1)$ then the balanced bipartite weighing design with parameters $v = 4k_1 + 1, b = \dfrac{2\lambda_1(4k_1 + 1)}{k_1 + 1}, r = \dfrac{2\lambda_1(2k_1 + 1)}{k_1 + 1}$, $k_1,\ k_2 = k_1 + 1, \lambda_1, \lambda_2 = \dfrac{\lambda_1 k_1}{k_1 + 1}$ exists.

The condition $\lambda_1 \equiv 0 \bmod (k_1 + 1)$ is equivalent $\lambda_1 = u(k_1 + 1), u = 1, 2, \ldots$. If $\lambda_1 = u(k_1 + 1)$ then the necessary conditions for the existence of the balanced bipartite weighing designs are $v = 4k_1 + 1, b = 2u(4k_1 + 1), r = 2u(2k_1 + 1)$, $k_1,\ k_2 = k_1 + 1,\ \lambda_1 = u(k_1 + 1), \lambda_2 = uk_1$.

Also, note the following remark.

If $u = 1$ and $k_1 = 1$ then $v = 5, b = 10, r = 6, k_1 = 1, k_2 = 2, \lambda_1 = 2, \lambda_2 = 1$ (the design N^{o} 5 from Lemma 3.8 given in Huang (1976)).

If $u = 1$ and $k_1 = 2$ then $v = 9, b = 18, r = 10, k_1 = 2, k_2 = 3, \lambda_1 = 3, \lambda_2 = 2$ (the design N^{o} 48 from Section 4 in Huang (1976)).

If $u=1$ and $k_1=3$ then $v=13, b=26, r=14, k_1=3, k_2=4, \lambda_1=4, \lambda_2=3$ (the design N^{o} 161 from Theorem 3.4 in Huang (1976)).

Corollary 3. If $c=2$ and $\lambda_1 \equiv 0 \bmod (k_1(k_1+2))$ then the balanced bipartite weighing design with the parameters

$$v=4k_1+3, b=\frac{\lambda_1(4k_1+3)(2k_1+1)}{k_1(k_1+2)}, \quad r=\frac{2\lambda_1(2k_1+1)(k_1+1)}{k_1(k_1+2)}, k_1,$$

$k_2=k_1+2, \lambda_1, \lambda_2=\dfrac{\lambda_1(k_1^2+k_1+1)}{k_1(k_1+2)}$ exists.

The condition $\lambda_1 \equiv 0 \bmod (k_1(k_1+2))$ is equivalent $\lambda_1=uk_1(k_1+2)$, $u=1,2,\ldots$. If $\lambda_1=uk_1(k_1+2)$ then the necessary conditions for the existence of the balanced bipartite weighing designs are $v=4k_1+3, b=u(4k_1+3)(2k_1+1)$, $r=2u(2k_1+1)(k_1+1), k_1, \ \lambda_1=uk_1(k_1+2), \ \ \lambda_2=u(k_1^2+k_1+1)\ k_2=k_1+2$.

Suppose that $u=1$. One obtains the following results.

(i) $v=7, b=21, r=12, k_1=1, k_2=3, \lambda_1=3, \lambda_2=3$,

(ii) $v=11, b=55, r=30, k_1=2, k_2=4, \lambda_1=8, \lambda_2=7$,

(iii) $v=15, b=105, r=56, k_1=3, k_2=5, \lambda_1=15, \lambda_2=13$

(Theorem 3.3 of Huang (1976)).

Now, if $k_1=0{,}5\cdot c(c-1)$ then from Theorem 3 the following Corollary can be expressed.

Corollary 4. If $\mathbf{N}$ is the incidence matrix of the balanced bipartite weighing design with the parameters $v=2c^2-1, b=\dfrac{4\lambda_1(2c^2-1)}{c^2}, r=4\lambda_1$, $k_1=\dfrac{c(c-1)}{2}, k_2=\dfrac{c(c+1)}{2}, \lambda_1, \lambda_2=\lambda_1, \ c=2,3,\ldots,$ then $\mathbf{X}$ in (6) is the regular D-optimal spring balance weighing design with the variance matrix of errors given by (3).

Then, from Corollary 4 and Theorem 2 given in Ceranka, Graczyk (2005) is also obtained directly the following Corollary.

Corollary 5. If the condition

$$4\lambda_1 \equiv 0 \bmod\left(c^2\right) \tag{7}$$

is satisfied, then any balanced bipartite weighing design exists.

The condition (7) is equivalent $4\lambda_1 = uc^2, \quad u = 1, 2, \ldots$. Hence $u \equiv 0 \bmod(4)$ or c is even. If $c = 2d, \quad d = 1, 2, \ldots$ then the necessary conditions for the existence of the balanced bipartite weighing designs are $v = 8d^2 - 1, b = u\left(8d^2 - 1\right)$, $r = 4ud^2$, $k_1 = d(2d-1)$, $k_2 = d(2d+1)$, $\lambda_1 = ud^2$, $\lambda_2 = ud^2$.

Some special cases can be described as follows.

If $u = 1$ and $d = 1$ then $v = 7, b = 7, r = 4, k_1 = 1, k_2 = 3, \lambda_1 = 1, \lambda_2 = 1$ (the design $\mathrm{N^o}$ 19 given by Huang (1976)).

If $u = 4t, t = 1, 2, \ldots$, then the necessary conditions for the existence of the balanced bipartite weighing designs are $v = 2c^2, b = 4t\left(2c^2 - 1\right), r = 4tc^2$, $k_1 = \frac{c(c-1)}{2}, k_2 = \frac{c(c+1)}{2}, \ \lambda_1 = tc^2, \lambda_2 = tc^2$.

If $c = 2$ and $t = 1$ then design $\mathrm{N^o}$ 19 is repeated 4 times.

VI. EXAMPLE

Let us consider the balanced bipartite weighing design with the parameters $v = 5, b = 10, r = 6, k_1 = 1, k_2 = 2, \lambda_1 = 2, \lambda_2 = 1$ (the design from Corollary 2) given by the incidence matrix

$$\mathbf{N} = \begin{bmatrix} 1_2 & 1_2 & 1_2 & 1_2 & 0 & 0 & 0 & 0 & 1_1 & 1_1 \\ 1_2 & 0 & 1_1 & 1_1 & 1_2 & 1_2 & 1_2 & 0 & 0 & 0 \\ 1_1 & 1_2 & 0 & 0 & 1_2 & 0 & 1_1 & 1_2 & 1_2 & 0 \\ 0 & 1_1 & 1_2 & 0 & 1_1 & 1_2 & 0 & 1_2 & 0 & 1_2 \\ 0 & 0 & 0 & 1_2 & 0 & 1_1 & 1_2 & 1_1 & 1_2 & 1_2 \end{bmatrix},$$

where 1_h denotes the object belonging to the h-th subblock, $h = 1,2$, respectively. Hence we form regular D-optimal spring balance weighing design in (6) as

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

REFERENCES

Baker, R.D., Wilson, R.M. (1974), The whist-tournament problem of E.H. Moore. Notices Amer. Math. Soc., 21.

Bose, R.C., Cameron, J.M. (1967), The bridge tournament problem and calibration designs for comparing pairs of objects. J. Res. Nat. Bur. Standards, 69, 323–332.

Ceranka, B.. Graczyk, M. (2005), About relations between the parameters of the balanced bipartite weighing designs. Proceedings of the Fifth Workshop on Simulation. S.M. Ermakov, V.B. Melas, A.N. Pepelyshev, Eds., Saint Petersburg University Publishers, 197–202.

Huang, Ch. (1976), Balanced bipartite weighing designs. Journal of Combinatorial Theory, 21, 20–34.

Katulska, K., Przybył, K. (2007), On certain D-optimal spring balance weighing designs. Journal of Statistical Theory and Practice, 1, 393–404.

Gail, Z., Kiefer, J. (1980), D-optimum weighing designs. Ann. Statistics, 8, 1293–1306.

Gail, Z., Kiefer, J. (1982), Construction methods D-optimum weighing designs when $n \equiv 3(\mathrm{mod}\,4)$. Ann. Statistics, 10, 502–510.

Neubauer, M.G., Watkins, W., Zeitlin, J. (1998), Notes on D-optimal designs. Linear Algebra and its Applications, 280, 109–127.

Swamy, M.N. (1982), Use of balanced bipartite weighing designs as chemical balance designs. Commun. Statist. - Theor. Meth., 11, (7), 769–785.

Małgorzata Graczyk

DWUDZIELNE UKŁADY BLOKÓW PROWADZĄCE DO REGULARNYCH D-OPTYMALNYCH SPRĘŻYNOWYCH UKŁADÓW WAGOWYCH

W pracy przedstawiono nową metodę konstrukcji regularnego D-optymalnego sprężynowego układu z tarowaniem przy założeniu, że wariancje błędów pomiarów nie są jednorodne. Do konstrukcji macierzy układu wykorzystano macierze incydencji dwudzielnych układów bloków.