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LIMIT PROPERTY OF A COMPOUND OF THE GENERALIZED NEGATIVE BINOMIAL AND BETA DISTRIBUTIONS

Abstract

In Central European Journal of Mathematics (CEJM) 2(4) 2004, 527–537 T. Gerstenkorn published a probability distribution as a result of compounding of the generalized negative binominal distribution with the generalized beta distribution. Assuming that a parameter of that distribution (w) tends to infinity one obtains a new limit distribution, interesting also in some special cases.

Key words: generalized negative binomial distribution, generalized beta distribution, compound distribution, limit theorems of the compound distribution.

1. Generalized probability distributions

In "Central European Journal of Mathematics" (CEJM) 2(4) 2004, 527–537 T. Gerstenkorn published the paper *A compound of the generalized negative binomial distribution with the generalized beta distribution.*

In 1971 G. C. Jain and P. C. Consul published a generalized negative binomial distribution. After a correction of W. Dyczka in 1978 it can be written in the form

$$GNBD(x;n,p,\beta)=P_\beta(x;n,p)=\frac{n}{n+\beta x}\binom{n+\beta x}{x}p^x(1-p)^{n+\beta x-x},\quad x=0,1,2,\ldots$$

where:

$$0\le p<1,\ \ n>0,\ \ \beta p<1\ \text{ and }\ \beta\ge 1 \tag{1}$$

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or

$$0 \le p \le 1,\ n \in N,\ \beta = 0 \tag{2}$$

If $\beta = 1$ one obtains the negative binomial distribution. In the case (2) one obtains the binomial distribution and for $n \in N$, $\beta = 1$ the Pascal distribution.

The generalized beta distribution (GBD) is defined as follows:

$$GBD(y; a,b,w,r) = \begin{cases} \dfrac{ay^{r-1}}{(bw)^{r/a} B(r/a, w)} \left(1 - \dfrac{y^a}{bw}\right)^{w-1} & \text{if } 0 < y < (bw)^{1/a}, \\ 0 & \text{if } y \le 0 \text{ or } y \ge (bw)^{1/a} \end{cases}$$

where a, b, r, $w > 0$ and $B(r/a, w)$ is a beta function.

This distribution is a special case of a Bessel distribution investigated by T. Śródka (1973). It was also analysed by J. Seweryn (1986) and W. Ogiński (1979) and applicated in reliability theory.

2. The compound distribution

By compounding these two distributions in respect of Y, that is

$$GNBD \underset{Y}{\wedge} GBD$$

one obtains the distribution

$$P_\beta GB(x) = D \sum_{k=0}^{\infty} (-c)^k \binom{n + \beta x - x}{k} (bw)^{\frac{k}{a}} B\left(\frac{x + r + k}{a}, w\right),$$

where:

$$D = \frac{n}{n + \beta x} \frac{\binom{n + \beta x}{x} c^x (bw)^{\frac{x}{a}}}{B(r/a, w)}, \quad x = 0, 1, 2, \ldots$$

while

$$a, b, w, r > 0, \quad \beta c y < 1, \quad 0 < cy < 1, \quad n > 0, \quad \beta \ge 1.$$

If $\beta = 0$ (binomial distribution) one obtains a distribution given in 1982 by T. G e r s t e n k o r n, i.e.

$$P_0GB(x) = \frac{\binom{n}{x} c^x (bw)^{x/a}}{B(r/a, w)} \sum_{k=0}^{n-x} (-c)^k \binom{n-x}{k} (bw)^{\frac{k}{a}} B\left(\frac{x+r+k}{a}, w\right), x = 0, 1, 2...$$

When $\beta = 1$ and n = 1, 2, ..., one obtains from $P_\beta GB(x)$ a compound distribution Pascal $\underset{Y}{\wedge}$ generalized beta in the form

$$P_1GB(x) = \binom{n+x-1}{x} \frac{c^x (bw)^{\frac{x}{a}}}{B(r/a, w)} \sum_{k=0}^{n} (-c)^k \binom{n}{k} (bw)^{\frac{k}{a}} B\left(\frac{x+r+k}{a}, w\right)$$

from which one also can obtain interesting special cases, as it was with the binomial distribution.

In the paper treated by T. G e r s t e n k o r n (2004) there were also given other special cases. In that paper there is also to find (Theorem 4.2, formula (15)) a formula for the factorial moment of order l of the compound distribution of the negative binomial distribution with the generalized beta one, i.e.

$$m_{[l]}^{NB-GB} = \frac{n^{[l,-1]} c^l (bw)^{\frac{l}{a}}}{B(r/a, w)} \sum_{k=0}^{\infty} \binom{-l}{k} (-c)^k (bw)^{\frac{k}{a}} B\left(\frac{l+r+k}{a}, w\right),$$

where

$$n^{[l,-1]} = n(n+1)(n+2)...(n+l-1).$$

3. Limit distributions

For the presented compound distribution $P_\beta GB(x)$ we obtain an interesting limit distribution. Namely, we have

Theorem 1. If $w \to \infty$, then

$$\lim_{w\to\infty} P_\beta GB(x) = LP_\beta GB(x) = D_1 \sum_{k=0}^{\infty} b^{\frac{k}{a}} (-c)^k \binom{n+\beta x - x}{k} \Gamma\left(\frac{x+r+k}{a}\right)$$

or, if n, $\beta x \in N$, then

$$LP_\beta GB(x) = D_1 \sum_{k=0}^{n+\beta x - x} b^{\frac{k}{a}} (-c)^k \binom{n+\beta x - x}{k} \Gamma\left(\frac{x+r+k}{a}\right),$$

where

$$D_1 = \frac{nc^x b^{\frac{x}{a}} \binom{n+\beta x}{x}}{(n+\beta x)\Gamma\left(\frac{r}{a}\right)}.$$

Proof. Taking in account the following limit properties

$$\lim_{w\to\infty} \frac{\Gamma(y+w)}{\Gamma(w)\cdot w^y} = 1, \quad \lim_{w\to\infty} w^y \cdot B(y,w) = \Gamma(y)$$

(see, e.g. J. A n t o n i e w i c z (1969, p. 433); formula 8.35.4), we have

$$\lim_{w\to\infty} P_\beta GB(x) = \frac{nc^x b^{\frac{x}{a}} \binom{n+\beta x}{x}}{(n+\beta x)\cdot \Gamma\left(\frac{r}{a}\right)} \sum_{k=0}^{\infty} b^{\frac{k}{a}} (-c)^k \binom{n+\beta x - x}{k}.$$

$$\cdot \lim_{w\to\infty} \frac{\Gamma\left(\frac{r}{a}+w\right)}{\Gamma(w)\cdot w^{\frac{r}{a}}} \cdot w^{\frac{x+r+k}{a}} \cdot B\left(\frac{x+r+k}{a}, w\right) = LP_\beta GB(x).$$

Regarding the limit process ($w \to \infty$) we also obtain the following

Theorem 2. The factorial moment of order l of the limit distribution is expressed by

$$m_{[l]}^{L(NB-GB)} = \frac{n^{[l,-1]} c^l b^{\frac{l}{a}}}{\Gamma\left(\frac{r}{a}\right)} \sum_{k=0}^{\infty} \binom{-l}{k} (-c)^k b^{\frac{k}{a}} \cdot \Gamma\left(\frac{l+r+k}{a}\right).$$

The proof is a similar one as in the case of Theorem 1.

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Własność graniczna złożonego rozkładu uogólnionego ujemnego dwumianowego z uogólnionym beta

W czasopiśmie „Central European Journal of Mathematics (CEJM)", 2(4) 2004, 527–537 T. Gerstenkorn podał rozkład prawdopodobieństwa, który jest wynikiem złożenia uogólnionego ujemnego rozkładu dwumianowego z uogólnionym beta.

Zakładając, że jeden z parametrów rozkładu (w) dąży do nieskończoności, otrzymuje się nowy rozkład graniczny, ciekawy także w przypadkach szczególnych.