Opuscula Math. 34, no. 4 (2014), 837–849
http://dx.doi.org/10.7494/OpMath.2014.34.4.837 Opuscula Mathematica

This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

A NOTE ON THE DEPENDENCE OF SOLUTIONS ON FUNCTIONAL PARAMETERS FOR NONLINEAR STURM-LIOUVILLE PROBLEMS

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Communicated by Marek Galewski

Abstract. We deal with the existence and the continuous dependence of solutions on functional parameters for boundary valued problems containing the Sturm-Liouville equation. We apply these result to prove the existence of at least one solution for a certain class of optimal control problems.

Keywords: positive solution, continuous dependence of solutions on functional parameters, Sturm-Liouville equation.

Mathematics Subject Classification: 34B08, 34B15.

1. INTRODUCTION

Let us consider the following nonlinear Sturm-Liouville boundary value problem:

$$\frac{1}{p(t)}\left(p(t)u'(t)\right)' + f(t,u(t),v(t)) = 0 \quad \text{for} \quad t \in (0,1), \tag{1.1}$$

$$\alpha u(0) - \beta \lim_{t\to 0^+}\left(p(t)u'(t)\right) = 0, \tag{1.2}$$

$$\gamma u(1) + \delta \lim_{t\to 1^-}\left(p(t)u'(t)\right) = 0, \tag{1.3}$$

where $\alpha,\beta,\gamma,\delta \geq 0$, $v \in V \subset \{w \in L^q(0,1);\, w:(0,1)\to(-b,b)\}$ $(V \neq \varnothing)$, $q>1$. Many problems modeled by (1.1) arise in various areas of applied mathematics, in biological, chemical or physical phenomena. In the wide literature devoted to BVPs similar to (1.1)–(1.3) (see e.g. [4–13,17–21] and references therein) the authors investigate mainly the existence of solutions for (1.1) under a variety of boundary conditions. Moreover, in the last fifty years, we could observe increasing interest in investigating sufficient conditions for the oscillation or nonoscillation of solutions of various classes of ODEs ([1–3,5–9], and references therein). We have to also recall results due to G. Vidossich who considered the continuous dependence of solutions for general boundary value problems (see [20, Theorem 1]). The author assumed, among others, that the limit problem possesses at most one solution. The assumption concerning the uniqueness of solutions can be met also in papers due to P. Eloe and J. Henderson (see e.g. [5] and references therein). Moreover, we can note that usually these results are based on global conditions concerning the nonlinearity. Here we consider the case when we control the behavior of the nonlinearity f only in a certain bounded set. Precisely, throughout this paper we adopt the following assumptions:

(A1) $\omega := \beta\gamma + \alpha\gamma B(0,1) + \alpha\delta > 0$, where $B(t,s) = \int_t^s \frac{dr}{p(r)}$,
(A2) $p \in C^1([0,1])$ and $p_{\min} := \min_{t\in[0,1]} p(t) > 0$,
(A3) $f : (0,1)\times(-a,a)\times(-b,b) \to [0,+\infty)$ is continuous, where $a, b > 0$, $t \mapsto f(t,0,0)$ is not identically equal to 0 in $(0,1)$.

We pay our special attention to properties of solutions. We analyze intervals of the monotonicity of solutions and characterize the set of their stationary points. However, the main goal of this paper is the continuous dependence of solutions on functional parameters. We have to emphasize that we do not assume the uniqueness of solutions for our problem. In the first step we fix a parameter and prove the existence of positive and bounded solution for (1.1)–(1.3). In the proof of this fact the main tool is Schauder's fixed point theorem. Next, we show that if a sequence of parameters tends to a certain v_0 a.e. in $(0,1)$, then a sequence of solutions is uniformly convergent (up to a subsequence) to a certain u_0. The properties of the sequences of parameters and solutions allow us to apply the du Bois-Reymond Lemma and infer that u_0 is a solution for our problem with parameter v_0. As an application of the continuous dependence of solutions on functional parameters we obtain the existence of an optimal pair for optimal control problems with constraints given by (1.1)–(1.3).

1.1. PROPERTIES OF SOLUTIONS

We start with the following definition.

Definition 1.1. For given parameter $v \in V$ as a solution of (1.1)–(1.3) we understand function $u \in C([0,1]) \cap C^1(0,1)$ such that $p(\cdot)u'(\cdot) \in W^{1,2}(0,1)$ and u satisfies (1.1)–(1.3).

Taking into account assumptions (A1)–(A3) we can derive some properties of nonnegative solutions.

Proposition 1.2. *Let $u \in C([0,1]) \cap C^1(0,1)$ be a nontrivial and nonnegative solution of* (1.1) *with boundary condition* (1.2) *and* (1.3) *such that $u(t) \in (-a,a)$ for all $t \in [0,1]$. Then:*

1. *$S := \{t \in (0,1), u'(t) = 0\}$ is a nonempty and closed interval; precisely, there exist $t_{\min}, t_{\max} \in (0,1)$, $t_{\min} \le t_{\max}$, such that $S = [t_{\min}, t_{\max}]$,*
2. *u is increasing in $(0, t_{\min})$, u is decreasing in $(t_{\max}, 1)$ and $u(t_0) = \max_{t\in[0,1]} u(t)$ for all $t_0 \in S$,*
3. *$u(t) > 0$ for all $t \in [0,1]$.*

Proof. We start with the observations that the auxiliary continuous function $k(t) := p(t)u'(t)$ for all $t \in (0,1)$ is nonincreasing in $(0,1)$. It is due to the fact that, by (1.1), $k'(t) = -p(t)f(t,u(t)) \le 0$ for each $t \in (0,1)$. Let us introduce notations:

$$\begin{aligned} k(0^+) &:= \lim_{t\to 0^+} k(t) = \lim_{t\to 0^+} (p(t)u'(t)) = \frac{\alpha}{\beta}u(0) \ge 0, \\ k(1^-) &:= \lim_{t\to 1^-} k(t) = \lim_{t\to 1^-} (p(t)u'(t)) = -\frac{\gamma}{\delta}u(1) \le 0. \end{aligned}$$

It is clear that for all $t \in (0,1)$,

$$k(1^-) \le k(t) \le k(0^+). \tag{1.4}$$

We start with the proof of two assertions:

$$u(0) \ne 0 \quad \text{and} \quad u(1) \ne 0. \tag{1.5}$$

To this effect we assume otherwise and suppose that $u(0) = 0$. Then, by (1.4), for all $t \in (0,1)$ we get $k(t) \le k(0^+) = 0$ and further $u'(t) \le 0$ in $(0,1)$. This gives $u(t) \le u(0) = 0$ in $(0,1)$ and finally $u \equiv 0$, which is contrary to the fact that u is nontrivial. Analogously, one obtains $u(1) \ne 0$. Finally, we have shown (1.5). Since k is continuous in $(0,1)$ and $k(1^-) < 0 < k(0^+)$ we obtain the existence of $\bar{t}_0 \in (0,1)$ such that $k(\bar{t}_0) = 0$, which implies that $\bar{t}_0 \in S$. Thus $S \ne \varnothing$.

Let us consider the case when there exist at least two elements $t_1, t_2 \in S$ and $t_1 < t_2$. Then for all $t \in [t_1, t_2]$ we have $0 = k(t_2) \le k(t) \le k(t_1) = 0$ which gives $u'(t) = 0$ for all $t \in [t_1, t_2]$, namely $[t_1, t_2] \subset S$. Our task is now to show that $\overline{S} = S$. Let $\{t_n\}_{n\in\mathbb{N}} \subset S$ and $\lim_{n\to\infty} t_n = t_0$. Then we have $k(t_n) = 0$ for all $n \in \mathbb{N}$. It is easy to note that $t_0 \notin \{0,1\}$. Indeed, if $t_0 = 0$, then $0 = \lim_{n\to\infty} k(t_n) = \frac{\alpha}{\beta}u(0)$, which is impossible (see (1.5)). Analogously, one can prove that $t_0 \ne 1$. Thus we state that $t_0 \in (0,1)$. Taking into account the continuity of u' at t_0 we get $t_0 \in S$.

To prove the second part it suffices to note that for all $t \in (0, t_{\min})$, $k(t) > k(t_{\min}) = 0$ which is equivalent to the inequality $u'(t) > 0$ in $(0, t_{\min})$. Thus we can infer that u is increasing in $(0, t_{\min})$. Analogously, we infer that u is decreasing in $(t_{\max}, 1)$. Consequently, if $t_0 \in S$, then for each $t \in [0,1]$, we get

$$u'(t) \ge 0 \text{ if } 0 < t \le t_0 \quad \text{and} \quad u'(t) \le 0 \text{ if } 1 > t \ge t_0$$

and further

$$u(t_0) \ge u(t) \text{ for } 0 < t \le t_0 \quad \text{and} \quad u(t_0) \ge u(t) \text{ for } 1 > t \ge t_0.$$

Finally, for all $t \in [0,1]$, $u(t_0) \ge u(t)$ what we have claimed.

Coming to the last part of the proof we assume otherwise and suppose that there exists $t_0 \in (0,1)$ such that $u(t_0) = 0$. Since u is nonnegative, t_0 is a global minimum of u and $t_0 \in S$. Taking into account part 2 we have $u(t_0) \ge u(t)$ for all $t \in [0,1]$. Summarizing $u(t_0) = u(t)$ for all $t \in [0,1]$ and further $u'(t) = 0$ for all $t \in [0,1]$ and $S = [0,1]$ which is contrary to part 1.

Let us note that if $S = \{\bar{t}_0\}$, then conclusion 1 is obvious. Moreover, taking into account the monotonicity of k we state that u is increasing in $(0, \bar{t}_0)$ and u is decreasing in $(\bar{t}_0, 1)$. Finally, we get $u(\bar{t}_0) = \max_{t\in[0,1]} u(t)$. Applying the similar reasoning as in the previous case we obtain also conclusion 3 for S being a singleton. $\square$

1.2. THE NONEXISTENCE AND EXISTENCE RESULTS

We start with the nonexistence result which is a consequence of Proposition 1.2. Taking into account the characterization of the set of stationary points of the solutions (Proposition 1.2, part 1) we can state that oscillations for the solutions of (1.1)–(1.3) are not permitted in the case described by assumptions (A1)–(A3).

Corollary 1.3. *If* (A1),(A2) *and* (A3) *are satisfied, then problem* (1.1)–(1.3) *does not possess positive and bounded* (*by a given in* (A3)) *solutions with oscillations.*

Now we formulate an additional condition on the nonlinearity which allows us to show that for each parameter $v \in V$ there exists at least one positive solution of our problem.

(A4) There exists $c \in (0, a)$ such that for all $v \in (-b, b)$,

$$\int_0^1 \max_{u\in[0,c]} f(t, u, v)dt \leq \omega \left(\beta + \alpha B(0, 1)\right) \left(\delta + \gamma B(0, 1)\right)^{-1} c$$

(with $B(t, s) = \int_t^s \frac{dr}{p(r)}$, $\omega = \alpha\delta + \alpha\gamma B(0, 1) + \beta\gamma$).

Owing to the Schauder's fixed point theorem, we will obtain the following result.

Theorem 1.4. *If conditions* (A1)–(A4) *hold, then for all* $v \in V$, *there exists a solution* $u \in U$ *of* (1.1)–(1.3), *where*

$$U := \{u \in C([0, 1]) : 0 \leq u(t) \leq c \ \text{in}\ [0, 1]\}.$$

Proof. Fix $v \in V$. Let us recall Green's function (see e.g. [17])

$$G(t, s) = \frac{1}{\omega} \begin{cases} \left(\beta + \alpha B(0, s)\right) \left(\delta + \gamma B(t, 1)\right) & \text{for } 0 \leq s \leq t \leq 1, \\ \left(\beta + \alpha B(0, t)\right) \left(\delta + \gamma B(s, 1)\right) & \text{for } 0 \leq t \leq s \leq 1 \end{cases}$$

for the following homogeneous problem:

$$\frac{1}{p(t)} \left(p(t)u'(t)\right)' = 0 \quad \text{for} \quad t \in (0, 1)$$

$$\alpha u(0) - \beta \lim_{t\to 0^+} (p(t)u'(t)) = 0,$$
$$\gamma u(1) + \delta \lim_{t\to 1^-} (p(t)u'(t)) = 0.$$

Then we consider (1.1)–(1.3) as a fixed point problem for the operator A defined as follows:

$$Au(t) = \int_0^1 G(t,s)\overline{f}(s,u(s),v(s))ds,$$

where for all $s \in (0,1)$ and $w \in (-b,b)$,

$$\overline{f}(s,u,w) = \begin{cases} f(s,0,w) & \text{for } u < 0, \\ f(s,u,w) & \text{for } u \in [0,c], \\ f(s,c,w) & \text{for } u > c. \end{cases}$$

It is clear that A is well-defined in $C([0,1])$. One can prove that $AU \subset U$. To this end, it suffices to note that for each $u \in U$, $Au \in C([0,1])$ and

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f(s,u(s),v(s))ds \\ &\le \frac{1}{\omega}\left(\beta + \alpha B(0,1)\right)\left(\delta + \gamma B(0,1)\right)\int_0^1 \max_{u\in[0,c]} f(s,u,v(s))ds \le c. \end{aligned}$$

Our task is now to show that A is completely continuous in $C([0,1])$. We prove this fact applying standard reasoning. We start with the continuity of A. Fix $u_0 \in C([0,1])$ and consider a sequence $(u_n)_{n=1}^{\infty} \subset C([0,1])$ converging to u_0 in the sup-norm $\|u\|_C := \max_{t\in[0,1]} |u(t)|$. Then

$$\begin{aligned} &\|Au_n - Au_0\|_C \\ &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\overline{f}(s,u_n(s),v(s))ds - \int_0^1 G(t,s)\overline{f}(s,u_0(s),v(s))ds \right| \\ &\le \frac{1}{\omega}\left(\beta + \alpha B(0,1)\right)\left(\delta + \gamma B(0,1)\right)\int_0^1 |\overline{f}(s,u_n(s),v(s)) - \overline{f}(s,u_0(s),v(s))|ds. \end{aligned}$$

Moreover, for all $s \in (0,1)$,

$$\lim_{n\to\infty} |\overline{f}(s,u_n(s),v(s)) - \overline{f}(s,u_0(s),v(s))| = 0$$

and

$$|\overline{f}(s,u_n(s),v(s)) - \overline{f}(s,u_0(s),v(s))| \le 2 \max_{u\in[0,c]} f(s,u,v(s))$$

with $\max_{u\in[0,c]} f(\cdot,u,v(\cdot)) \in L(0,1)$.

Therefore, the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty}\int_0^1 |\overline{f}(s,u_n(s),v(s))-\overline{f}(s,u_0(s),v(s))|ds=0.$$

We obtain $\lim\limits_{n\to\infty}\|Au_n-Au_0\|_C=0$. Finally, we infer the continuity of A.

Now we investigate the compactness of A. Let us consider a bounded set $B\subset C([0,1])$. Applying the Ascoli-Arzelà theorem we will prove that $A(B)\subset C([0,1])$ is relatively compact. Taking into account (A4) one can see that for all $Au\in A(B)$

$$\max_{t\in[0,1]}|Au(t)|\le\frac{1}{\omega}\left(\beta+\alpha B(0,1)\right)\left(\delta+\gamma B(0,1)\right)\int_0^1\max_{z\in[0,c]}f(s,z,v(s))ds<+\infty.$$

Thus $A(B)$ is equibounded. To show that $A(B)$ is equicontinuous we take any $\varepsilon>0$. Since G is uniformly continuous on $[0,1]\times[0,1]$, we state the existence of $\delta>0$ such that for all $s\in[0,1]$ and all $t_1,t_2\in[0,1]$ satisfying the condition $|t_1-t_2|<\delta$, the following inequality holds:

$$|G(t_1,s)-G(t_2,s)|\le\frac{\varepsilon}{M}$$

with

$$M:=\omega\left(\beta+\alpha B(0,1)\right)\left(\delta+\gamma B(0,1)\right)^{-1}c.$$

Therefore, by (A4), we obtain for all $Au\in A(B)$,

$$\begin{aligned}
&|Au(t_1)-Au(t_2)|\\
&=\left|\int_0^1 G(t_1,s)\overline{f}(s,u(s),v(s))ds-\int_0^1 G(t_2,s)\overline{f}(s,u(s),v(s))ds\right|\\
&\le\int_0^1|G(t_1,s)-G(t_2,s)|\max_{z\in[0,c]}f(s,z,v(s))ds\le\varepsilon.
\end{aligned}$$

Finally, $A(B)$ is equibounded and equicontinuous. With the Ascoli-Arzelà theorem in mind, we state that $A(B)$ is relatively compact in $C([0,1])$. Therefore, we get the compactness of A.

Summarizing, we have proved that the completely continuous operator A maps the convex, closed and nonempty set $U\subset C([0,1])$ into U. Thus the Schauder's fixed point theorem leads to the existence of at least one solution of (1.1)–(1.3) in the set U. □

2. CONTINUOUS DEPENDENCE OF SOLUTIONS ON FUNCTIONAL PARAMETERS

In this section our main result is presented. We will describe the continuous dependence of solutions on functional parameters in the sense presented, among others, in

[15] and [16]. We will prove that if a sequence of parameters $(v_m)_{m\in\mathbb{N}}$ tends to v_0 a.e. in $(0,1)$, then a sequence of solutions $(u_m)_{m\in\mathbb{N}}$ (corresponding to $(v_m)_{m\in\mathbb{N}}$) possesses a subsequence uniformly convergent to u_0. Moreover, u_0 is a solution of the limit problem, namely u_0 is a solution of (1.1)–(1.3) with parameter v_0. For this purpose, we formulate an additional condition:

(A5) there exists $\varphi \in L^2(0,1)$ such that for all $w \in (-b,b)$,

$$\max_{u\in[0,c]} f(t,u,w) \le \varphi(t) \quad \text{a.e. in } (0,1)$$

(with c given in (A4)).

Theorem 2.1. *Suppose that* (A1)–(A5) *hold. Assume that the sequence of parameters* $(v_m)_{m\in\mathbb{N}} \in V$ *converges to* $v_0 \in V$ *a.e. in* $(0,1)$. *For each* $m \in \mathbb{N}$, *let us denote by* $u_m \in U$ *a solution of* (1.1)–(1.3) *with* $v = v_m$. *Then the sequence of solutions* $(u_m)_{m\in\mathbb{N}}$ *tends uniformly* (*up to a subsequence*) *to a certain* $u_0 \in U$ *such that*

$$\begin{cases} \frac{1}{p(t)}\left(p(t)u_0'(t)\right)' + f(t,u_0(t),v_0(t)) = 0 \quad \textit{for} \quad t\in(0,1), \\ \alpha u_0(0) - \beta \lim_{t\to 0^+}\left(p_0(t)u_0'(t)\right) = 0, \\ \gamma u_0(1) + \delta \lim_{t\to 1^-}\left(p_0(t)u_0'(t)\right) = 0. \end{cases}$$

Proof. Since $u_m \in U$ denotes a solution of (1.1)–(1.3) for given v_m, we have

$$-\left(p(t)u_m'(t)\right)' = p(t)f(t,u_m(t),v_m(t)) \quad \text{for} \quad t\in(0,1). \tag{2.1}$$

Thus one obtains the following chain of assertions:

$$\begin{aligned}
&\int_0^1 \left|u_m'(t)\right|^2 dt \\
&\le \frac{1}{p_{\min}} \int_0^1 p(t)\left|u_m'(t)\right|^2 dt \\
&\le \frac{cp_{\max}}{p_{\min}}\omega\left(\beta+\alpha B(0,1)\right)\left(\delta+\gamma B(0,1)\right)^{-1} c + \frac{c}{p_{\min}}\left(\frac{\alpha}{\beta}u_m(0)+\frac{\gamma}{\delta}u_m(1)\right) \\
&\le \frac{c^2}{p_{\min}}\left(p_{\max}\omega\left(\beta+\alpha B(0,1)\right)\left(\delta+\gamma B(0,1)\right)^{-1} + \left(\frac{\alpha}{\beta}+\frac{\gamma}{\delta}\right)\right)
\end{aligned} \tag{2.2}$$

for all $m \in \mathbb{N}$, where $p_{\min} := \min_{t\in[0,1]} p(t)$, $p_{\max} := \max_{t\in[0,1]} p(t)$. Taking into account (2.2) and the boundedness of $(u_m)_{m\in\mathbb{N}}$ in $[0,1]$, we infer the boundedness of $(u_m)_{m\in\mathbb{N}}$ in $W^{1,2}(0,1)$ and further, we state the existence of a subsequence of $(u_m)_{m\in\mathbb{N}}$ (still denoted by $(u_m)_{m\in\mathbb{N}}$) weakly convergent in $W^{1,2}(0,1)$ to a certain $u_0 \in W^{1,2}(0,1)$. The Rellich-Kondrashov theorem ([12]) yields the uniform convergence of $(u_m)_{m\in\mathbb{N}}$ in $[0,1]$. Consequently, we get $0 \le u_0 \le c$ in $[0,1]$. Now we consider the auxiliary sequence

$$k_m(t) = p(t)u_m'(t) \quad \text{in} \quad (0,1).$$

By (2.1),

$$-k'_m(t) = p(t)f(t, u_m(t), v_m(t)) \quad \text{in} \quad (0,1). \tag{2.3}$$

The above assertions and the properties of the sequence $(u_m)_{m\in\mathbb{N}}$ guarantee that $(k_m)_{m\in\mathbb{N}}$ is bounded in $W^{1,2}(0,1)$ and further, it is weakly convergent (up to a subsequence) to $k_0 \in W^{1,2}(0,1)$. Finally $(k_m)_{m\in\mathbb{N}}$ is uniformly convergent to k_0 and k_0 is continuous. Therefore, we have

$$\lim_{t\to 0^+} k_0(t) = \lim_{m\to\infty}\lim_{t\to 0^+} k_m(t) = \lim_{m\to\infty} \frac{\alpha}{\beta} u_m(0) = \frac{\alpha}{\beta} u_0(0)$$

and analogously we get

$$\lim_{t\to 1^-} k_0(t) = -\frac{\gamma}{\delta} u_0(1).$$

Moreover, by the uniqueness of the weak limit, we infer $k_0(t) = p(t)u'_0(t)$ in $(0,1)$, which gives

$$\lim_{t\to 0^+} p(t)u'_0(t) = \frac{\alpha}{\beta} u_0(0),$$
$$\lim_{t\to 1^-} p(t)u'_0(t) = -\frac{\gamma}{\delta} u_0(1).$$

By (2.3), we state that for all $h \in W_0^{1,2}(0,1)$, the following chain of equalities holds:

$$\begin{aligned}\int_0^1 p(t)u'_0(t)h'(t)dt &= \lim_{m\to\infty}\int_0^1 p(t)u'_m(t)h'(t)dt \\ &= \lim_{m\to\infty}\int_0^1 p(t)f(t,u_m(t),v_m(t))h(t)dt \\ &= \int_0^1 p(t)f(t,u_0(t),v_0(t))h(t)dt,\end{aligned}$$

where the last equality is due to the Lebesgue dominated convergence theorem. Applying the du Bois-Reymond lemma ([14]) we infer that $(p_0(t)u'_0(t))'$ exists almost everywhere and

$$-\left(p(t)u'_0(t)\right)' = p(t)f(t,u_0(t),v_0(t)) \text{ for a.a. } t \in (0,1).$$

□

Example 2.2. Let us consider the following problem:

$$\frac{1}{t+1}\left((t+1)u'(t)\right)' + d\left(\frac{1}{\sqrt[4]{t}}\frac{(u(t))^5}{(4-u(t))} + e^{u(t)} + \left[\frac{t}{1+m^2t^2}\right]^2\right) = 0 \text{ for } t \in (0,1), \tag{2.4}$$

$$\alpha u(0) - \beta \lim_{t\to 0^+} ((t+1)u'(t)) = 0, \tag{2.5}$$

$$\gamma u(1) + \delta \lim_{t\to 1^-} ((t+1)u'(t)) = 0, \tag{2.6}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ satisfy

$$\omega := \beta\gamma + \alpha\gamma B(0,1) + \alpha\delta > 0,$$

with

$$B(t,s) = \ln(s+1) - \ln(t+1), \quad s,t \in [0,1].$$

Since $B(0,1) = \ln 2$, we have

$$\omega := \beta\gamma + \alpha\gamma \ln 2 + \alpha\delta.$$

If

$$0 < d \leq \frac{\omega}{5(\beta + \alpha \ln 2))(\delta + \gamma \ln 2)},$$

then for each $m \in \mathbb{N}$, (2.4)–(2.6) possesses at least one positive solution $u_m \in U := \{u \in C([0,1]) : 0 \leq u(t) \leq 1 \ in \ [0,1]\}$. Moreover, the sequence $\{u_m\}_{m\in\mathbb{N}}$ tends uniformly (up to a subsequence) to $u_0 \in U$ being a solution of the equation

$$\frac{1}{t+1}((t+1)u'(t))' + d\left(\frac{1}{\sqrt[4]{t}}\frac{(u(t))^5}{(4-u(t))} + e^{u(t)}\right) = 0, \text{ for } t \in (0,1), \tag{2.7}$$

with boundary conditions (2.5)–(2.6).

Let us note that in our case $p(t) = t+1$ and

$$f(t,u,w) = \frac{1}{\sqrt[4]{t}}\frac{u^5}{(4-u)} + e^u + w^2$$

with $w \in [0,1)$ satisfy assumptions (A2) and (A3) with $c = 1$. Moreover, for each $v \in V \subset \{w \in L^q(0,1) : w : (0,1) \to (-1,1)\}$,

$$\begin{aligned}\int_0^1 \max_{u\in[0,c]} f(t,u,v)dt &= d\int_0^1 \max_{u\in[0,c]}\left(\frac{1}{\sqrt[4]{t}}\frac{u^5}{(4-u)} + e^u + v^2\right)dt \\ &\leq d\left(\frac{1}{3}\int_0^1 \frac{1}{\sqrt[4]{t}}dt + e + 1\right) \leq 4,2d \\ &\leq \omega(\beta + \alpha\ln 2)(\delta + \gamma \ln 2)^{-1},\end{aligned}$$

thus (A4) holds. Finally, for each $m \in \mathbb{N}$, we can apply Theorem 1.4, which gives the existence of at least one positive solution for (2.4)–(2.5)–(2.6) in the set U.
Since

$$\max_{u\in[0,c]} f(t,u,v(t)) \leq \varphi(t) \text{ a.e. in } (0,1)$$

with

$$\varphi(t) := d\left(\frac{1}{3}\frac{1}{\sqrt[4]{t}} + e + 1\right),$$

(A5) is also fulfilled. Now we consider the sequence $\{v_m\}_{m\in\mathbb{N}}$ with

$$v_m := \frac{t}{1+m^2t^2}.$$

It is clear that $\{v_m\}_{m\in\mathbb{N}}$ tends uniformly to $v_0 = 0$ in $[0,1]$. Therefore, Theorem 2.1 leads to the conclusion that there exists a subsequence (still denoted by $\{u_m\}_{m\in\mathbb{N}}$) tending uniformly to $u_0 \in U$ being a solution of (2.7)–(2.5)–(2.6).

3. OPTIMAL CONTROL PROBLEMS

As an application of the continuous dependence of solutions on functional parameters we prove the existence of an optimal pair for a class of optimal control problems. In this section we discuss sufficient conditions for the optimal control problem governed by

$$\begin{cases} \frac{1}{p(t)}\left(p(t)u'(t)\right)' + f(t,u(t),v(t)) = 0 \text{ for } t \in (0,1), \\ \alpha u(0) - \beta \lim_{t\to 0^+}\left(p(t)u'(t)\right) = 0 \\ \gamma u(1) + \delta \lim_{t\to 1^-}\left(p(t)u'(t)\right) = 0, \end{cases} \tag{3.1}$$

with the following integral cost functional

$$J(u,v) = \int_0^1 F(t,u(t),v(t))dt \to \min \tag{3.2}$$

defined in UV given by

$$UV := \{(u,v) \in U \times V : \ u \text{ is a solution of (3.1) corresponding to } v\},$$

where

$$U := \{u \in C([0,1]) : 0 \le u(t) \le c \text{ in } [0,1]\},$$

$$V := \{v : [0,1] \to D : v \text{ satisfies the Lipschitz condition with a fixed constant } L > 0\}, \tag{3.3}$$

c is given in (A4) and D is a compact subset of R. The main goal of this section is to prove the existence of at least one optimal pair $(u_0, v_0) \in UV$. To this effect we consider the cost functional satisfying the following assumptions:

(F1) $F : (0,1) \times (-a,a) \times (-b,b) \to R$ is measurable with respect to the first variable for all $(u,v) \in (-a,a) \times (-b,b)$ and $F(t,\cdot,\cdot)$ is continuous in $(-a,a) \times (-b,b)$ for a.a. $t \in (0,1)$, with $a, b > 0$ and such that $D \subset (-b,b)$,

(F2) there exists $\psi \in L^1(\Omega, R_+)$ such that for all $(u,v) \in UV$,

$$|F(t,u(t),v(t))| \le \psi(t) \text{ a.e. in } (0,1).$$

Theorem 3.1. *Assume that* (A1)–(A5) *hold, with V given by* (3.3)*, and F satisfies conditions* (F1)–(F2)*. Then there exists $(u_0, v_0) \in UV$ such that*

$$J(u_0, v_0) = \min_{(u,v)\in UV} J(u, v). \tag{3.4}$$

Proof. Let us consider the sequence $\{(u_m, v_m)\}_{m\in\mathbb{N}} \subset UV$ minimizing J on UV. Taking into account the facts that $\{v_m(t)\}_{m\in\mathbb{N}} \subset D$ for all $t \in [0, 1]$ and that v_m, $m = 1, 2, \ldots$, are Lipschitz functions with common constant $L > 0$ one can state that $\{v_m\}_{m\in\mathbb{N}}$ is equibounded and equicontinuous. Thus the Ascoli-Arzelà theorem leads to the existence of a subsequence $\{v_{m_l}\}_{l\in\mathbb{N}}$ convergent uniformly to a certain v_0 in $[0, 1]$. It is clear that for all $t \in [0, 1]$, $v_0(t) \in D$ and satisfies the Lipschitz condition with the same constant L. Thus $v_0 \in V$. Now Theorem 2.1 guarantees that there exists a subsequence of solutions $\{u_{m_l}\}_{l\in N} \subset U$ (corresponding to the subsequence of parameters $\{v_{m_l}\}_{l\in\mathbb{N}}$) tending to $u_0 \in U$ and u_0 satisfies (3.1) with $v = v_0$. It suffices to prove that the pair (u_0, v_0) is optimal. For this purpose, we have to note that for all $t \in (0, 1)$,

$$\lim_{l\to\infty} F(t, u_{m_l}(t), v_{m_l}(t)) = F(t, u_0(t), v_0(t))$$

which follows from (F1). Moreover, by (F2), we have for all $l \in \mathbb{N}$,

$$|F(t, u_{m_l}(t), v_{m_l}(t))| \leq \psi(t) \text{ a.e. in } (0, 1).$$

Finally, by the Lebesgue dominated convergence theorem, we infer that

$$\lim_{l\to\infty} \int_0^1 F(t, u_{m_l}(t), v_{m_l}(t))dt = \int_0^1 F(t, u_0(t), v_0(t))dt$$

which gives (3.4). □

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Received: November 18, 2013.
Revised: June 30, 2014.
Accepted: September 9, 2014.