J Glob Optim (2015) 61:695–720
DOI 10.1007/s10898-014-0203-1

Sufficient optimality criteria and duality for multiobjective variational control problems with G-type I objective and constraint functions

Tadeusz Antczak

Received: 2 October 2013 / Accepted: 16 May 2014 / Published online: 5 June 2014

Abstract In the paper, we introduce the concepts of G-type I and generalized G-type I functions for a new class of nonconvex multiobjective variational control problems. For such nonconvex vector optimization problems, we prove sufficient optimality conditions for weakly efficiency, efficiency and properly efficiency under assumptions that the functions constituting them are G-type I and/or generalized G-type I objective and constraint functions. Further, for the considered multiobjective variational control problem, its dual multiobjective variational control problem is given and several duality results are established under (generalized) G-type I objective and constraint functions.

Keywords Multiobjective variational problems · Properly efficient solution · G-type I objective and constraint functions · Optimality conditions · Duality

1 Introduction

Multiobjective variational control programming is an interesting subject that appears in many types of optimization problem, for instance, in flight control design, in the control of space structures, in industrial process control, in impulsive control problems, in the control of production and inventory, and other diverse fields. Various types of control programming problems, including multiobjective variational programming problems with equality and inequality restrictions, are applied in various areas of operational research by many authors (see, for instance, [9,10,20,24–26], and others).

On the other hand, investigation of optimality conditions and/or duality has been one of the most attracting topics in the theory of nonlinear programming. In recent years, some numerous generalizations of convex functions have been derived which proved to be useful for extending optimality conditions and some classical duality results, previously restricted to convex programs, to larger classes of nonconvex optimization problems. One of them

T. Antczak (✉)
Faculty of Mathematics, University of Łódź, Banacha 22, 90-238 Lodz, Poland
e-mail: antczak@math.uni.lodz.pl

is invexity notion introduced by Hanson [14]. Later, Hanson and Mond [15] defined two new classes of functions called type I and type II functions, and they established sufficient optimality conditions and duality results for differentiable scalar optimization problems by using these concepts. Furthermore, in the natural way, the definition of type I functions was also extended to the case of differentiable vector-valued functions. Aghezzaf and Hachimi [1,16] introduced classes of generalized type I functions for a differentiable multiobjective programming problem and derived some Mond–Weir type duality results under the generalized type I assumptions. One of a generalization of invexity is the concept of G-invexity introduced by Antczak [2] for scalar optimization problems. In [3,4], Antczak extended the definition of G-invexity to the vectorial case and he used it to prove the necessary and sufficient optimality conditions and duality results for a new class of nonconvex multiobjective programming problems.

The relationship between mathematical programming and classical calculus of variation was explored and extended by Hanson [13]. Thereafter variational control programming problems have attracted some attention in literature. Optimality conditions and duality for multiobjective variational control problems have been of much interest in the recent years, and several contributions have been made to their development (see, for example, [5–7,12,16–18,21–23,27,29], and references here). Bhatia and Mehra [8] extended the concepts of B-type I and generalized B-type I functions to the continuous case and they used these concepts to establish sufficient optimality conditions and duality results for multiobjective variational programming problems. Kim and Kim [19] introduced new classes of generalized V-type I invex functions for variational problems and they proved a number of sufficiency results and duality theorems using Lagrange multiplier conditions under various types of generalized V-type I invexity requirements. Further, under the generalized V-type I invexity assumptions and their generalizations, they obtained duality results for Mond–Weir type duals. Also Hachimi and Aghezzaf [16] obtained several mixed type duality results for multiobjective variational programming problems, but under a new introduced concept of generalized type I functions. In [18], Khazafi et al. introduced the classes of (B, ρ)-type I functions and of generalized (B, ρ)-type I functions and derived a series of sufficient optimality conditions and mixed type duality results for multiobjective control problems. Recently, Khazafi and Rueda [17] extended the concept of V-univexity type I to multiobjective variational programming problems and derived various sufficient optimality conditions and mixed type duality results under generalized V-univexity type I conditions.

In this paper, by taking the motivation from Antczak [3,4] and Aghezzaf and Hachimi [1], we introduce the definition of G-type I objective and constraint functions and various concepts of generalized G-type I objective and constraint functions for a multiobjective variational control programming problem with inequality constraints. The class of G-type I objective and constraint functions is a generalization of the class of G-invex functions introduced by Antczak [2] for differentiable vector optimization problems and type I functions introduced by Aghezzaf and Hachimi [1] to the case of a multiobjective variational control programming problem. Under a variety of G-type I hypotheses, we prove the sufficient optimality conditions for the considered multiobjective variational control programming problem. We also define vector variational control dual problem and we prove various duality results between the considered multiobjective variational control programming problem and its vector variational control dual problem. Furthermore, some incorrectness in definitions of the concepts of G-invexity and generalized G-invexity for a multiobjective programming problems and the sufficient optimality conditions for such a vector optimization problem given in [28] are corrected. Also the sufficient conditions are proved for a larger class of nonconvex multiobjective programming problems than in [28].

2 Multiobjective variational control problem and G-type I functions

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

Let $I = [a, b]$ be a real interval and let $P = \{1, 2, \ldots, p\}$, $J = \{1, 2, \ldots, q\}$.

In this paper, we assume that $x(t)$ is an n-dimensional piecewise smooth function of t, and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in $[a, b]$.

Denote by X the space of piecewise smooth functions $x : I \to R^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by $z = Dx \iff x(t) = x(a) + \int_a^t z(s)\, ds$, where $x(a)$ is a given boundary value. Therefore, $\frac{d}{dt} \equiv D$ except at discontinuities.

Further, denote by U the space of piecewise smooth control functions $u : I \to R^m$ with norm $\|u\|_\infty$.

The multiobjective variational control problem is to choose, under given conditions, a control $u(t)$, such that the state vector $x(t)$ is brought from the specified initial state $x(a) = \alpha$ to some specified final state $x(b) = \beta$ in such a way to minimize a given functional. A more precise mathematical formulation is given in the following multiobjective variational control problem:

$$
\begin{aligned}
&V\text{-Minimize} \int_a^b f\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right) dt \\
&= \left(\int_a^b f^1\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right) dt, \ldots, \int_a^b f^p\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right) dt \right) \\
&\quad \text{subject to } g\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right) \leqq 0, \quad t \in I, \qquad \text{(MCP)} \\
&\quad x(a) = \alpha, \;\; x(b) = \beta,
\end{aligned}
$$

where $f = \left(f^1, \ldots, f^p\right) : I \times R^n \times R^n \times R^m \times R^m \to R^p$ is a p-dimensional function and each its component is a continuously differentiable real scalar function and $g : I \times R^n \times R^n \times R^m \times R^m \to R^q$ is assumed to be a continuously differentiable q-dimensional function.

For notational simplicity, we write $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively. We denote the partial derivatives of f^1 with respect to t, x and $\dot{x}$, respectively, by f_t^1, f_x^1, $f_{\dot{x}}^1$ such that $f_x^1 = \left(\frac{\partial f^1}{\partial x_1}, \ldots, \frac{\partial f^1}{\partial x_n}\right)$ and $f_{\dot{x}}^1 = \left(\frac{\partial f^1}{\partial \dot{x}_1}, \ldots, \frac{\partial f^1}{\partial \dot{x}_n}\right)$. Similarly, the partial derivatives of the vector function g can be written, using matrices with q rows instead of one.

Let Ω denote the set of all feasible points of (MCP), i.e.:

$$\Omega = \{(x, u) : x(t) \in X, u(t) \in U \text{ verifying the constraints of (MCP) for all } t \in I\}.$$

In order to simplify the presentation, in our subsequent theory, we shall set

$$\pi_{xu}(t) = (t, x(t), \dot{x}(t), u(t), \dot{u}(t)), \quad \pi_{\overline{xu}}(t) = (t, \overline{x}(t), \dot{\overline{x}}(t), \overline{u}(t), \dot{\overline{u}}(t)),$$
$$\pi_{xu\overline{xu}}(t) = \left(t, x, \dot{x}, u, \dot{u}, \overline{x}, \dot{\overline{x}}, \overline{u}, \dot{\overline{u}}\right).$$

Definition 1 A solution $(\overline{x}, \overline{u}) \in \Omega$ is said to be weakly efficient of (MCP) if there exists no other $(x, u) \in \Omega$ such that, the following relation is satisfied

$$\int_a^b f(\pi_{xu}(t))\,dt < \int_a^b f(\pi_{\overline{xu}}(t))\,dt.$$

Definition 2 A solution $(\overline{x}, \overline{u}) \in \Omega$ is said to be efficient of (MCP) if there exists no other $(x, u) \in \Omega$ such that, the following relation is satisfied

$$\int_a^b f(\pi_{xu}(t))\,dt \leq \int_a^b f(\pi_{\overline{xu}}(t))\,dt.$$

In multiobjective programming, some efficient solutions presented an undesirable property with respect to the ratio between the marginal profit of an objective function and the loss of some other. To these solutions, Geoffrion [11] introduced the concept of a properly efficient solution.

Definition 3 A solution $(\overline{x}, \overline{u}) \in \Omega$ is said to be properly efficient of (MCP) if there exists a scalar $M > 0$ such that, for each $i = 1, \ldots, p$, the following inequality

$$\int_a^b f^i(\pi_{\overline{xu}}(t))\,dt - \int_a^b f^i(\pi_{xu}(t))\,dt \leqq M\left(\int_a^b f^k(\pi_{xu}(t))\,dt - \int_a^b f^k(\pi_{\overline{xu}}(t))\,dt\right)$$

holds for some k, satisfying $\int_a^b f^k(\pi_{xu}(t))\,dt > \int_a^b f^k(\pi_{\overline{xu}}(t))\,dt$, whenever $(x(t), u(t)) \in \Omega$ and $\int_a^b f^i(\pi_{xu}(t))\,dt < \int_a^b f^i(\pi_{\overline{xu}}(t))\,dt$.

Definition 4 A function $\varphi : R \to R$ is said to be strictly increasing if and only if

$$\forall x, y \in R \;\; x < y \implies \varphi(x) < \varphi(y).$$

In [3], Antczak introduced the following definition of a G-invex vector-valued function.

Definition 5 Let $f = (f_1, \ldots, f_k) : C \to R^k$ be a differentiable vector-valued function defined on a nonempty open set $C \subset R^n$, and $I_{f_i}(C)$, $i = 1, \ldots, k$, be the range of f_i, that is, the image of C under f_i and $u \in C$. If there exist a differentiable vector-valued function $G_f = \left(G_{f_1}, \ldots, G_{f_k}\right) : R \to R^k$ such that any its component $G_{f_i} : I_{f_i}(C) \to R$ is a strictly increasing function on its domain and a vector-valued function $\eta : C \times C \to R^n$ such that, for all $x \in C$ and for any $i = 1, \ldots, k$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - G'_{f_i}(f_i(u))\nabla f_i(u)\eta(x, u) \geqq 0,$$

then f is said to be a G_f-invex vector-valued function at u on X with respect to η. If the above inequalities are satisfied for each $u \in C$, then f is vector G_f-invex on C with respect to η.

Remark 6 In [28], Zhang et al. extended the definition of a G-invex vector-valued function introduced by Antczak [28] for a multiobjective programming problem defined in finite-dimensional Euclidean space to the case of a multiobjective variational control problem and also gave definitions of generalized G-invex functions for such vector optimization problems. Unfortunately, these definitions seem to be wrong. Namely, Zhang et al. [28] assumed in their definition of a (generalized) G-invex vector-valued function $F = \left(F_1, \ldots, F_p\right)$, where $F_i(x(t), u(t)) = \int_a^b f^i\left(t, x, \dot{x}, u, \dot{u}\right) dt$, that functions G_{f^i} are defined on the set $C \subset R^n$. Whereas F_i, as it follows from their definitions, are functions $F_i : X \times U \to R$, that is, they are defined on $X \times U$, not on any subset of R^n. Further, the next wrong part of their definitions of (generalized) G-invex vector-valued functions is the following: if f is defined on $C \subset R^n$, that is, $f = (f_1, \ldots, f_k) : C \to R^k$ and then $I_{f_i}(C)$, $i = 1, \ldots, k$, is the range of f_i (that is, the image of C under f_i) and, therefore, as it follows from the definition of G-invexity introduced by Antczak [3] (see also Definition 5), a function η with respect to which f is G-invex, should be defined as follows $\eta : C \times C \to R^n$. Whereas Zhang et al. [28] defined any component of a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_p})$, that is, $G_{f_i} : I_{f_i}(C) \to R$ as a strictly increasing function on its domain, that is, on the set $C \subset R^n$, nevertheless the function η is defined by $\eta : I \times X \times X \times U \times U \to R^n$ in their definitions. This means that η is defined on the set $I \times X \times X \times U \times U$, not on a set $C \times C$ as it follows from Antczak's definition of G-invexity for a vector-valued function $f = (f_1, \ldots, f_k) : C \to R^k$. At last, also the symbol $I_{f_i}(C)$ defined by Zhang et al. [28] as the range of f_i, that is, the image of C under f_i, is not correct in their definition of G-invexity given for a multiobjective variational control problem. Indeed, the symbol $I_{f_i}(C)$, $i = 1, \ldots, k$, can not be the image of $C \subset R^n$ under f_i, since every f_i is defined on $X \times U$. As it follows from the above, the definition of a G-invex vector-valued function for a multiobjective variational control problem introduced by Zhang et al. [28] is, in some part, the definition of a G-invex vector-valued function introduced by Antczak [3] for a multiobjective programming problem in finite-dimensional Euclidean space.

Furthermore, in their sufficient optimality conditions, Zhang et al. [28] defined functions G_{f_i} as follows: $G_{f_i} : I_{\int_b^a f^i}(X) \to R$, in opposition to the definition of $G_{f_i} : I_{f_i}(C) \to R$, used in their definitions of G-invexity and generalized G-invexity for a multiobjective variational control problem. Also this definition of G_{f_i} seems to be wrong, since functions constituting the multiobjective variational control problem considered by Zhang et al. [28] are not defined on X. However, Zhang et al. [28] proved the sufficient optimality conditions with functions $G_{f_i} : I_{\int_b^a f^i}(X) \to R$, where X is the space of all piecewise smooth functions, under (generalized) G-invexity hypotheses with functions $G_{f_i} : I_{f_i}(C) \to R$, where $C \subset R^n$.

Now, in the natural way, we generalize the definition of a G-invex vector-valued function introduced by Antczak [2] and the definition of differentiable type I multiple objective and constraint functions introduced by Aghezzaf and Hachimi [1] to the case of a multiobjective variational control problem.

Let $I_{\int_b^a f^i}(X \times U)$, $i = 1, \ldots, p$, be the range of $\int_a^b f^i\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right) dt$, where $x(t) \in X$, $u(t) \in U$, and $I_{\int_b^a g^j}(X \times U)$, $j = 1, \ldots, q$, be the range of $\int_a^b g^j\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right) dt$, where $x(t) \in X$, $u(t) \in U$. For notational convenience, we use $f^i\left(t, x, \dot{x}, u, \dot{u}\right)$ for $f^i\left(t, x(t), \dot{x}(t), u(t), \dot{u}(t)\right)$, x for $x(t)$ and $\dot{x}$ for $\dot{x}(t)$.

Definition 7 Let $(\overline{x}, \overline{u}) \in X \times U$. If there exist a differentiable vector-valued function $G_f = \left(G_{f^1}, \ldots, G_{f^p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function

$G_g = \left(G_{g^1}, \ldots, G_{g^q}\right) : R \to R^q$ such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain, $\eta : I \times R^n \times R^n \times R^m \times R^m \to R^n$ with $\eta\left(t, x(t), \overline{x}(t), u(t), \overline{u}(t)\right) = 0$ at t if $x(t) = \overline{x}(t)$ and $\vartheta : I \times R^n \times R^n \times R^m \times R^m \to R^m$ such that the following inequalities

$$\begin{aligned} & G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right) dt\right) - G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \\ & \geqq G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right. \\ & \quad \left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\} dt, \ \ i = 1, \ldots, p \end{aligned} \tag{1}$$

and

$$\begin{aligned} & -G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) \\ & \geqq G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_x^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right. \\ & \quad \left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_u^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\} dt, \quad j = 1, \ldots, q \end{aligned} \tag{2}$$

hold for all $(x, u) \in X \times U$, then (f, g) is said to be G-type I functions at $(\overline{x}, \overline{u}) \in X \times U$ on $X \times U$ (with respect to G_f, G_g, η and ϑ).

If the relations (1) and (2) are satisfied for each $(\overline{x}, \overline{u}) \in X \times U$, then the functional (f, g) is said to be G-type I objective and constraint functions on $X \times U$ with respect to G_f, G_g, η and ϑ.

Definition 8 Let $(\overline{x}, \overline{u}) \in X \times U$. If there exist a differentiable vector-valued function $G_f = \left(G_{f^1}, \ldots, G_{f^p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = \left(G_{g^1}, \ldots, G_{g^q}\right) : R \to R^q$ such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain, $\eta : I \times R^n \times R^n \times R^m \times R^m \to R^n$ with $\eta\left(t, x(t), \overline{x}(t), u(t), \overline{u}(t)\right) = 0$ at t if $x(t) = \overline{x}(t)$ and $\vartheta : I \times R^n \times R^n \times R^m \times R^m \to R^m$ such that the inequalities

$$\begin{aligned} & G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right) dt\right) - G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \\ & > G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right. \\ & \quad \left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\} dt, \quad i = 1, \ldots, p \end{aligned} \tag{3}$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)$$
$$\geqq G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt,\quad j=1,\ldots,q \tag{4}$$

hold for all $(x,u)\in X\times U$, $x\neq u$, then (f,g) is said to be strictly-G-type I objective and constraint functions at $(\overline{x},\overline{u})\in X\times U$ on $X\times U$ with respect to G_f, G_g, η and ϑ.

If the inequalities (3) and (4) are satisfied for each $(\overline{x},\overline{u})\in X\times U$, then the functional (f,g) is said to be strictly-G-type I objective and constraint functions on $X\times U$ with respect to G_f, G_g, η and ϑ.

Definition 9 Let $(\overline{x},\overline{u})\in X\times U$. If there exist a differentiable vector-valued function $G_f=\left(G_{f^1},\ldots,G_{f^p}\right):R\to R^p$ such that every its component $G_{f^i}:I_{\int_b^a f^i}(X\times U)\to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g=\left(G_{g^1},\ldots,G_{g^q}\right):R\to R^q$ such that every its component $G_{g^j}:I_{\int_b^a g^j}(X\times U)\to R$ is a strictly increasing function on its domain, $\eta:I\times R^n\times R^n\times R^m\times R^m\to R^n$ with $\eta\left(t,x\left(t\right),\overline{x}\left(t\right),u\left(t\right),\overline{u}\left(t\right)\right)=0$ at t if $x\left(t\right)=\overline{x}\left(t\right)$ and $\vartheta:I\times R^n\times R^n\times R^m\times R^m\to R^m$ such that the relations

$$G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right)<G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)$$
$$\Longrightarrow G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[f_x^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right]-\frac{d}{dt}f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\}dt<0,\ i=1,\ldots,p \tag{5}$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\leqq 0$$
$$\Longrightarrow G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt\leqq 0,\ j=1,\ldots,q \tag{6}$$

hold for all $(x,u)\in X\times U$, then (f,g) is said to be pseudo-quasi-G-type I objective and constraint functions at $(\overline{x},\overline{u})\in X\times U$ on $X\times U$ (with respect to G_f, G_g, η and ϑ).

If the relations (5) and (6) are satisfied for each $(\overline{x},\overline{u})\in X\times U$, then the functional (f,g) is said to be pseudo-quasi-G-type I objective and constraint functions on $X\times U$ with respect to G_f, G_g, η and ϑ.

Definition 10 Let $(\overline{x}, \overline{u}) \in X \times U$. If there exist a differentiable vector-valued function $G_f = \left(G_{f^1}, \ldots, G_{f^p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = \left(G_{g^1}, \ldots, G_{g^q}\right) : R \to R^q$ such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain, $\eta : I \times R^n \times R^n \times R^m \times R^m \to R^n$ with $\eta\left(t, x(t), \overline{x}(t), u(t), \overline{u}(t)\right) = 0$ at t if $x(t) = \overline{x}(t)$ and $\vartheta : I \times R^n \times R^n \times R^m \times R^m \to R^m$ such that the relations

$$G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right) dt\right) \leqq G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right)$$
$$\Longrightarrow G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\} dt < 0, \; i = 1, \ldots, p \qquad (7)$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) \leqq 0$$
$$\Longrightarrow G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_x^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_u^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\} dt \leqq 0, \quad j = 1, \ldots, q \qquad (8)$$

hold for all $(x, u) \in X \times U$, $x \neq u$, then (f, g) is said to be strictly-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x}, \overline{u}) \in X \times U$ on $X \times U$ with respect to G_f, G_g, η and ϑ.

If the relations (7) and (8) are satisfied for each $(\overline{x}, \overline{u}) \in X \times U$, then the functional (f, g) is said to be strictly-pseudo-quasi-G-type I objective and constraint functions on $X \times U$ with respect to G_f, G_g, η and ϑ.

Definition 11 Let $(\overline{x}, \overline{u}) \in X \times U$. If there exist a differentiable vector-valued function $G_f = \left(G_{f^1}, \ldots, G_{f^p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = \left(G_{g^1}, \ldots, G_{g^q}\right) : R \to R^q$ such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain, $\eta : I \times R^n \times R^n \times R^m \times R^m \to R^n$ with $\eta\left(t, x(t), \overline{x}(t), u(t), \overline{u}(t)\right) = 0$ at t if $x(t) = \overline{x}(t)$ and $\vartheta : I \times R^n \times R^n \times R^m \times R^m \to R^m$ such that the relations

$$G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right) dt\right) < G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right)$$
$$\Longrightarrow \left[\underset{i \in P}{\forall} G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.\right.$$

$$+ \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\Bigg\} dt \leqq 0$$

$$\wedge \underset{i \in P}{\exists} G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \Bigg\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]$$

$$+ \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\Bigg\} dt < 0\Bigg] \tag{9}$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) \leqq 0$$

$$\Longrightarrow G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \Bigg\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_x^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]$$

$$+ \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_u^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\Bigg\} dt \leqq 0, \quad j = 1, \ldots, q \tag{10}$$

hold for all $(x, u) \in X \times U$, $x \neq u$, then (f, g) is said to be weak-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x}, \overline{u}) \in X \times U$ on $X \times U$ with respect to G_f, G_g, η and ϑ.

If the relations (9) and (10) are satisfied for each $(\overline{x}, \overline{u}) \in X \times U$, then the functional (f, g) is said to be weak-pseudo-quasi-G-type I objective and constraint functions on $X \times U$ with respect to G_f, G_g, η and ϑ.

Definition 12 Let $(\overline{x}, \overline{u}) \in X \times U$. If there exist a differentiable vector-valued function $G_f = \left(G_{f^1}, \ldots, G_{f^p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = \left(G_{g^1}, \ldots, G_{g^q}\right) : R \to R^q$ such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain, $\eta : I \times R^n \times R^n \times R^m \times R^m \to R^n$ with $\eta\left(t, x(t), \overline{x}(t), u(t), \overline{u}(t)\right) = 0$ at t if $x(t) = \overline{x}(t)$ and $\vartheta : I \times R^n \times R^n \times R^m \times R^m \to R^m$ such that the relations

$$\Bigg[\underset{i \in P}{\forall} G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right) dt\right) \leqq G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right)$$

$$\wedge \underset{i \in P}{\exists} G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right) dt\right) < G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right)\Bigg]$$

$$\Longrightarrow G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \Bigg\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]$$

$$+ \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\Bigg\} dt \leqq 0, \, i = 1, \ldots, p, \tag{11}$$

$$\left[\underset{i\in P}{\forall} G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right) \leqq G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right) \right.$$

$$\left. \wedge \underset{i\in P}{\exists} G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right) < G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right) \right]$$

$$\Longrightarrow G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b \left\{ \left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right] \right.$$

$$\left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\} dt < 0 \text{ for at least one } i \in P \quad (12)$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right) \leqq 0$$

$$\Longrightarrow G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b \left\{ \left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_x^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right] \right.$$

$$\left. + \left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T \left[g_u^j\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\} dt \leqq 0, \quad j = 1, \ldots, q \quad (13)$$

hold for all $(x, u) \in X \times U$, then (f, g) is said to be strong-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x}, \overline{u}) \in X \times U$ on $X \times U$ with respect to G_f, G_g, η and ϑ.

If the relations (11), (12) and (13) are satisfied for each $(\overline{x}, \overline{u}) \in X \times U$, then the functional (f, g) is said to be strong-pseudo-quasi-G-type I objective and constraint functions on $X \times U$ with respect to G_f, G_g, η and ϑ.

Definition 13 Let $(\overline{x}, \overline{u}) \in X \times U$. If there exist a differentiable vector-valued function $G_f = \left(G_{f^1}, \ldots, G_{f^p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain, a differentiable vector-valued function $G_g = \left(G_{g^1}, \ldots, G_{g^q}\right) : R \to R^q$ such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain, $\eta : I \times R^n \times R^n \times R^m \times R^m \to R^n$ with $\eta\left(t, x(t), \overline{x}(t), u(t), \overline{u}(t)\right) = 0$ at t if $x(t) = \overline{x}(t)$ and $\vartheta : I \times R^n \times R^n \times R^m \times R^m \to R^m$ such that the relations

$$\left[\underset{i\in P}{\forall} G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right) \leqq G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right) \right.$$

$$\left. \wedge \underset{i\in P}{\exists} G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right) < G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right) \right]$$

$$\Longrightarrow G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[f_x^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$

$$\left.+\left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right]-\frac{d}{dt}f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\}dt<0,\ i=1,\ldots,p \tag{14}$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\leqq 0$$

$$\Longrightarrow G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$

$$\left.+\left[\vartheta\left(\pi_{xu\overline{xu}}(t)\right)\right]^T\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt\leqq 0,\quad j=1,\ldots,q \tag{15}$$

hold for all $(x,u)\in X\times U$, then (f,g) is said to be weak-strictly-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x},\overline{u})\in X\times U$ on $X\times U$ with respect to G_f, G_g, η and ϑ.

If the relations (14) and (15) are satisfied for each $(\overline{x},\overline{u})\in X\times U$, then the functional (f,g) is said to be weak-strictly-pseudo-quasi-G-type I objective and constraint functions on $X\times U$ with respect to G_f, G_g, η and ϑ.

3 Optimality conditions

In this section, for the considered multiobjective continuous programming problem (MCP), we prove the sufficient optimality conditions for weakly efficiency, efficiency and properly efficiency under assumptions that the functions constituting it are G-type I and/or generalized G-type I functions.

Theorem 14 *Let $(\overline{x},\overline{u})$ be a feasible solution in the considered multiobjective continuous programming problem (MCP). Assume that there exist $\overline{\lambda}\in R^p$ and a piecewise smooth function $\overline{\xi}(\cdot):I\to R^q$ such that the following conditions*

$$\sum_{i=1}^p\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[f_x^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]$$

$$+\sum_{j=1}^q\overline{\xi}_j G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]=0,\quad t\in I, \tag{16}$$

$$\sum_{i=1}^p\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[f_u^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right]$$

$$+\sum_{j=1}^q\overline{\xi}_j G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]=0,\quad t\in I, \tag{17}$$

$$\overline{\xi}_j(t)\, G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right) dt\right) = 0,\ t \in I, \quad j = 1, \ldots, q, \tag{18}$$

$$\overline{\lambda} \geq 0, \quad \overline{\lambda}^T e = 1, \quad \overline{\xi}(t) \geqq 0 \tag{19}$$

hold, where $G_f = \left(G_{f_1}, \ldots, G_{f_p}\right) : R \to R^p$ *is a differentiable vector-valued function such that every its component* $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ *is a strictly increasing function on its domain,* $G_g = \left(G_{g_1}, \ldots, G_{g_q}\right) : R \to R^q$ *is a differentiable vector-valued function such that every its component* $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ *is a strictly increasing function on its domain. Further, assume that* (f, g) *are strictly-G-type I objective and constraint functions at* $(\overline{x}, \overline{u})$ *on* Ω *with respect to* G_f, G_g, η *and* ϑ. *Then* $(\overline{x}, \overline{u})$ *is an efficient solution in (MCP).*

Proof Suppose, contrary to the result, that $(\overline{x}, \overline{u}) \in \Omega$ is not an efficient solution in (MCP). Hence, there exists $(\widetilde{x}, \widetilde{u}) \in \Omega$ such that

$$\int_a^b f\left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \leq \int_a^b f\left(\pi_{\overline{xu}}(t)\right) dt. \tag{20}$$

This means that

$$\int_a^b f^i\left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \leqq \int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt,\ i = 1, \ldots, p \tag{21}$$

and

$$\int_a^b f^{i^*}\left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt < \int_a^b f^{i^*}\left(\pi_{\overline{xu}}(t)\right) dt \text{ for some } i^* \in P. \tag{22}$$

By assumption, there exist $\overline{\lambda} \in R^p$, a piecewise smooth function $\overline{\xi}(\cdot) : I \to R^q$, a differentiable vector-valued function $G_f = \left(G_{f_1}, \ldots, G_{f_p}\right) : R \to R^p$ such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain and a differentiable vector-valued function $G_g = \left(G_{g_1}, \ldots, G_{g_q}\right) : R \to R^q$ such that any its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain such that the conditions (16)–(19) are satisfied. Since (f, g) are strictly-G-type I objective and constraint functions at $(\overline{x}, \overline{u})$ on Ω with respect to G_f, G_g, η and ϑ, and, moreover, $(\widetilde{x}, \widetilde{u}) \in \Omega$, by Definition 7, the following inequalities are satisfied

$$\begin{aligned} &G_{f^i}\left(\int_a^b f^i\left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt\right) - G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \\ &> G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T \left[f_x^i\left(\pi_{\overline{xu}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right. \\ &\quad \left. + \left[\vartheta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T \left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right] - \frac{d}{dt} f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\} dt, \quad i = 1, \ldots, p, \end{aligned} \tag{23}$$

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)$$
$$\geqq G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[g^j_x\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g^j_{\dot{x}}\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[g^j_u\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g^j_{\dot{u}}\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt,\quad j=1,\ldots,q. \tag{24}$$

Since every $G_{f^i}, i=1,\ldots,p$, is a strictly increasing function on its domain, the inequalities (21) and (22) yield

$$G_{f^i}\left(\int_a^b f^i\left(\pi_{\widetilde{xu}}(t)\right)dt\right)\leqq G_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right),\ i=1,\ldots,p, \tag{25}$$

and

$$G_{f^{i^*}}\left(\int_a^b f^{i^*}\left(\pi_{\widetilde{xu}}(t)\right)dt\right)<G_{f^{i^*}}\left(\int_a^b f^{i^*}\left(\pi_{\overline{xu}}(t)\right)dt\right)\ \text{for some } i^*\in P. \tag{26}$$

Combining (23), (25) and (26), we obtain

$$G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[f^i_x\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f^i_{\dot{x}}\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[f^i_u\left(\pi_{\overline{xu}}(t)\right)\right]-\frac{d}{dt}f^i_{\dot{u}}\left(\pi_{\overline{xu}}(t)\right)\right\}dt<0,\quad i=1,\ldots,p. \tag{27}$$

Multiplying each inequality (27) by the associated Lagrange multiplier $\overline{\lambda}_i, i=1,\ldots,p$, and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^p\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[f^i_x\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f^i_{\dot{x}}\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[f^i_u\left(\pi_{\overline{xu}}(t)\right)\right]-\frac{d}{dt}f^i_{\dot{u}}\left(\pi_{\overline{xu}}(t)\right)\right\}dt<0,\quad i=1,\ldots,p. \tag{28}$$

Multiplying each inequality (24) by $\overline{\xi}_j(t)\geqq 0,\ j=1,\ldots,q$, and then adding both sides of the obtained inequalities, we get

$$-\sum_{j=1}^q\overline{\xi}_j(t)\,G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)$$
$$\geqq\sum_{j=1}^q\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[g^j_x\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g^j_{\dot{x}}\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{xu}\overline{xu}}(t)\right)\right]^T\left[g^j_u\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g^j_{\dot{u}}\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt. \tag{29}$$

By (18) and (29), it follows that

$$\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j(\pi_{\overline{xu}}(t))\,dt\right)\int_a^b\left\{[\eta(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t))]^T\left[g_x^j(\pi_{\overline{xu}}(t))-\frac{d}{dt}g_{\dot{x}}^j(\pi_{\overline{xu}}(t))\right]\right.$$
$$\left.+[\vartheta(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t))]^T\left[g_u^j(\pi_{\overline{xu}}(t))-\frac{d}{dt}g_{\dot{u}}^j(\pi_{\overline{xu}}(t))\right]\right\}dt\leqq 0. \tag{30}$$

Adding both sides of (27) and (30), we get that the following inequality

$$\int_a^b[\eta(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t))]^T\left\{\sum_{i=1}^{p}\overline{\lambda}_iG'_{f^i}\left(\int_a^b f^i(\pi_{\overline{xu}}(t))\,dt\right)\left[f_x^i(\pi_{\overline{xu}}(t))-\frac{d}{dt}f_{\dot{x}}^i(\pi_{\overline{xu}}(t))\right]\right.$$
$$\left.+\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j(\pi_{\overline{xu}}(t))\,dt\right)\left[g_x^j(\pi_{\overline{xu}}(t))-\frac{d}{dt}g_{\dot{x}}^j(\pi_{\overline{xu}}(t))\right]\right\}dt$$
$$+\int_a^b[\vartheta(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t))]^T\left\{\sum_{i=1}^{p}\overline{\lambda}_iG'_{f^i}\left(\int_a^b f^i(\pi_{\overline{xu}}(t))\,dt\right)\left[f_u^i(\pi_{\overline{xu}}(t))-\frac{d}{dt}f_{\dot{u}}^i(\pi_{\overline{xu}}(t))\right]\right.$$
$$\left.+\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j(\pi_{\overline{xu}}(t))\,dt\right)\left[g_u^j(\pi_{\overline{xu}}(t))-\frac{d}{dt}g_{\dot{u}}^j(\pi_{\overline{xu}}(t))\right]\right\}dt<0$$

holds, contradicting (16) and (17). Thus, $(\overline{x},\overline{u})$ is an efficient solution in (MCP) and the proof is completed. □

Theorem 15 *Assume that all hypotheses of Theorem 14 are fulfilled. If* $\overline{\lambda}>0$*, then* $(\overline{x},\overline{u})$ *a properly efficient solution in (MCP).*

Proof Since all hypotheses of Theorem 14 are fulfilled, therefore, $(\overline{x},\overline{u})$ is an efficient solution in problem (MCP).

Now, we prove that $(\overline{x},\overline{u})$ is a properly efficient solution in problem (MCP). Suppose, contrary to the result, that $(\overline{x},\overline{u})$ is not a properly efficient solution in problem (MCP). Then, there exist $(\widetilde{x},\widetilde{u})\in\Omega$ and $i\in P$, such that $\int_a^b f^i(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt<\int_a^b f^i(\pi_{\overline{xu}}(t))\,dt$ and

$$\frac{\int_a^b f^i(\pi_{\overline{xu}}(t))\,dt-\int_a^b f^i(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt}{\int_a^b f^k(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt-\int_a^b f^k(\pi_{\overline{xu}}(t))\,dt}>M \tag{31}$$

for each $k\neq i$ such that $\int_a^b f^k(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt>\int_a^b f^k(\pi_{\overline{xu}}(t))\,dt$. Since, for each $k\in P$, $k\neq i$, $\int_a^b f^k(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt>\int_a^b f^k(\pi_{\overline{xu}}(t))\,dt$ and each function G_{f^k}, $k\in P$, is a strictly increasing function on its domain, we have

$$G_{f^k}\left(\int_a^b f^k(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt\right)>G_{f^k}\left(\int_a^b f^k(\pi_{\overline{xu}}(t))\,dt\right). \tag{32}$$

Thus, by $\overline{\lambda}_k > 0$, $k \in P$, it follows that

$$\sum_{k \in P \setminus \{i\}} \overline{\lambda}_k \left[G_{f^k} \left(\int_a^b f^k \left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \right) - G_{f^k} \left(\int_a^b f^k \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) \right] < 0. \tag{33}$$

Since $\int_a^b f^i \left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt < \int_a^b f^i \left(\pi_{\overline{x}\overline{u}}(t)\right) dt$, using that G_{f^i} is a strictly increasing function on its domain together with $\overline{\lambda}_i > 0$, we obtain

$$\overline{\lambda}_i G_{f^i} \left(\int_a^b f^i \left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \right) < \overline{\lambda}_i G_{f^i} \left(\int_a^b f^i \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right). \tag{34}$$

Combining (33) and (34), we get

$$\overline{\lambda}_i G_{f^i} \left(\int_a^b f^i \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) - \overline{\lambda}_i G_{f^i} \left(\int_a^b f^i \left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \right)$$
$$> \sum_{k \in P \setminus \{i\}} \overline{\lambda}_k \left[G_{f^k} \left(\int_a^b f^k \left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \right) - G_{f^k} \left(\int_a^b f^k \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) \right].$$

Hence, the above inequality gives

$$\sum_{i=1}^{p} \overline{\lambda}_i \left[G_{f^i} \left(\int_a^b f^i \left(\pi_{\widetilde{x}\widetilde{u}}(t)\right) dt \right) - G_{f^i} \left(\int_a^b f^i \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) \right] < 0. \tag{35}$$

By assumption, (f, g) are strictly G-type I objective and constraint functions at $(\overline{x}, \overline{u})$ on Ω with respect to G_f, G_g, η and ϑ. Then, by Definition 7, the inequality (35) implies

$$\sum_{i=1}^{p} \overline{\lambda}_i G'_{f^i} \left(\int_a^b f^i \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) \int_a^b \left\{ \left[\eta \left(\pi_{\widetilde{x}\widetilde{u}\overline{x}\overline{u}}(t)\right)\right]^T \left[f_x^i \left(\pi_{\overline{x}\overline{u}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i \left(\pi_{\overline{x}\overline{u}}(t)\right) \right] \right.$$
$$\left. + \left[\vartheta \left(\pi_{\widetilde{x}\widetilde{u}\overline{x}\overline{u}}(t)\right)\right]^T \left[f_u^i \left(\pi_{\overline{x}\overline{u}}(t)\right) \right] - \frac{d}{dt} f_{\dot{u}}^i \left(\pi_{\overline{x}\overline{u}}(t)\right) \right\} dt < 0. \tag{36}$$

Since $(\overline{x}, \overline{u}) \in \Omega$, $(\widetilde{x}, \widetilde{u}) \in \Omega$ and $\overline{\xi}(t) \geqq 0$, by Definition 7, in the similar manner as in the proof of Theorem 14, we obtain

$$\sum_{j=1}^{q} \overline{\xi}_j(t)\, G'_{g^j} \left(\int_a^b g^j \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) \int_a^b \left\{ \left[\eta \left(\pi_{\widetilde{x}\widetilde{u}\overline{x}\overline{u}}(t)\right)\right]^T \left[g_x^j \left(\pi_{\overline{x}\overline{u}}(t)\right) - \frac{d}{dt} g_{\dot{x}}^j \left(\pi_{\overline{x}\overline{u}}(t)\right) \right] \right.$$
$$\left. + \left[\vartheta \left(\pi_{\widetilde{x}\widetilde{u}\overline{x}\overline{u}}(t)\right)\right]^T \left[g_u^j \left(\pi_{\overline{x}\overline{u}}(t)\right) - \frac{d}{dt} g_{\dot{u}}^j \left(\pi_{\overline{x}\overline{u}}(t)\right) \right] \right\} dt \leqq 0. \tag{37}$$

By (36) and (37), it follows that the following inequality

$$\int_a^b \left[\eta \left(\pi_{\widetilde{x}\widetilde{u}\overline{x}\overline{u}}(t)\right)\right]^T \left\{ \sum_{i=1}^{p} \overline{\lambda}_i G'_{f^i} \left(\int_a^b f^i \left(\pi_{\overline{x}\overline{u}}(t)\right) dt \right) \left[f_x^i \left(\pi_{\overline{x}\overline{u}}(t)\right) - \frac{d}{dt} f_{\dot{x}}^i \left(\pi_{\overline{x}\overline{u}}(t)\right) \right] \right.$$

$$+\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j(\pi_{\overline{x}\overline{u}}(t))\,dt\right)\left[g^j_x(\pi_{\overline{x}\overline{u}}(t))-\frac{d}{dt}g^j_{\dot{x}}(\pi_{\overline{x}\overline{u}}(t))\right]\Bigg\}\,dt$$

$$+\int_a^b[\vartheta(\pi_{\widetilde{x}\widetilde{u}\overline{x}\overline{u}}(t))]^T\left\{\sum_{i=1}^{p}\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i(\pi_{\overline{x}\overline{u}}(t))\,dt\right)\left[f^i_u(\pi_{\overline{x}\overline{u}}(t))-\tfrac{d}{dt}f^i_{\dot{u}}(\pi_{\overline{x}\overline{u}}(t))\right]\right.$$

$$\left.+\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j(\pi_{\overline{x}\overline{u}}(t))\,dt\right)\left[g^j_u(\pi_{\overline{x}\overline{u}}(t))-\frac{d}{dt}g^j_{\dot{u}}(\pi_{\overline{x}\overline{u}}(t))\right]\right\}dt<0$$

holds, contradicting (16) and (17). Thus, $(\overline{x},\overline{u})$ is a properly efficient solution in (MCP) and the proof is completed. □

Now, we prove sufficient optimality for efficiency and properly efficiency in the considered multiobjective variational control problem under assumption that the functions constituting it are generalized G-type I objective and constraint functions.

Theorem 16 *Let $(\overline{x},\overline{u})$ be a feasible solution in the considered multiobjective variational control problem (MCP). Assume that there exist $\overline{\lambda}\in R^p$ and a piecewise smooth function $\overline{\xi}(\cdot):I\to R^r$ such that the conditions (16)–(19) are satisfied with $G_f=\left(G_{f_1},\ldots,G_{f_p}\right):R\to R^p$ being a differentiable vector-valued function such that every its component $G_{f^i}:I_{\int_b^a f^i}(X\times U)\to R$ is a strictly increasing function on its domain and $G_g=\left(G_{g_1},\ldots,G_{g_q}\right):R\to R^q$ being a differentiable vector-valued function such that every its component $G_{g^j}:I_{\int_b^a g^j}(X\times U)\to R$ is a strictly increasing function on its domain. Further, assume that one of the following hypotheses is satisfied:*

a) (f,g) are strictly-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x},\overline{u})$ on Ω with respect to G_f, G_g, η and ϑ,

b) (f,g) are strong-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x},\overline{u})$ on Ω with respect to G_f, G_g, η and ϑ.

Then $(\overline{x},\overline{u})$ is an efficient solution in (MCP). If we assume, moreover, that $\overline{\lambda}>0$, then $(\overline{x},\overline{u})$ is a properly efficient solution in (MCP).

Proof Suppose, contrary to the result, that $(\overline{x},\overline{u})\in\Omega$ is not an efficient solution in (MCP). Hence, there exists $(\widetilde{x},\widetilde{u})\in\Omega$ such that the inequalities (21) and (22) are satisfied. By assumption, there exist $\overline{\lambda}\in R^p$, a piecewise smooth function $\overline{\xi}(\cdot):I\to R^q$, a differentiable vector-valued function $G_f=\left(G_{f_1},\ldots,G_{f_p}\right):R\to R^p$ such that every its component $G_{f^i}:I_{\int_b^a f^i}(X\times U)\to R$ is a strictly increasing function on its domain and a differentiable vector-valued function $G_g=\left(G_{g_1},\ldots,G_{g_q}\right):R\to R^q$ such that any its component $G_{g^j}:I_{\int_b^a g^j}(X\times U)\to R$ is a strictly increasing function on its domain such that the conditions (16)–(19) are satisfied. Since every G_{f^i}, $i=1,\ldots,p$, is a strictly increasing function on its domain, therefore, (21) and (22) yield

$$G_{f^i}\left(\int_a^b f^i(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt\right)\leqq G_{f^i}\left(\int_a^b f^i(\pi_{\overline{x}\overline{u}}(t))\,dt\right),\ i=1,\ldots,p \tag{38}$$

and

$$G_{f^{i^*}}\left(\int_a^b f^{i^*}(\pi_{\widetilde{x}\widetilde{u}}(t))\,dt\right)<G_{f^{i^*}}\left(\int_a^b f^{i^*}(\pi_{\overline{x}\overline{u}}(t))\,dt\right)\ \text{for some } i^*\in P. \tag{39}$$

We now prove this theorem under hypothesis a). Since (f, g) are strictly-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x}, \overline{u})$ on Ω with respect to G_f, G_g, η and ϑ, and, moreover, $(\widetilde{x}, \widetilde{u}) \in \Omega$, by (7) (see Definition 10), the inequalities (38) and (39) imply

$$G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[f_x^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right]-\frac{d}{dt}f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\}dt<0,\quad i=1,\ldots,p. \tag{40}$$

Multiplying (40) by the associated Lagrange multiplier $\overline{\lambda}_i$, $i = 1, \ldots, p$, and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^{p}\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[f_x^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[f_u^i\left(\pi_{\overline{xu}}(t)\right)\right]-\frac{d}{dt}f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right\}dt<0. \tag{41}$$

Since $\overline{\xi}(t) \geqq 0$, by Definition (10) and (18), we obtain

$$\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt\leqq 0. \tag{42}$$

Adding both sides of the inequalities above, we get

$$\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{u}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt\leqq 0. \tag{43}$$

Adding both sides of (41) and (43), we get that the following inequality

$$\int_a^b\left[\eta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left\{\sum_{i=1}^{p}\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[f_x^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{x}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[g_x^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{\dot{x}}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt$$
$$+\int_a^b\left[\vartheta\left(\pi_{\widetilde{x}\widetilde{u}\overline{xu}}(t)\right)\right]^T\left\{\sum_{i=1}^{p}\overline{\lambda}_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[f_u^i\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}f_{\dot{u}}^i\left(\pi_{\overline{xu}}(t)\right)\right]\right.$$
$$\left.+\sum_{j=1}^{q}\overline{\xi}_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{xu}}(t)\right)dt\right)\left[g_u^j\left(\pi_{\overline{xu}}(t)\right)-\frac{d}{dt}g_{u}^j\left(\pi_{\overline{xu}}(t)\right)\right]\right\}dt<0$$

holds, contradicting (16) and (17). Thus, $(\overline{x}, \overline{u})$ is an efficient solution in (MCP). The proof of properly efficiency is similar to the proof of Theorem 15.

Proof of theorem under hypothesis b) is similar and, therefore, it is omitted in the paper. □

In order to prove that a feasible solution satisfying the conditions (16)–(19) is weakly efficient in problem (MCP), we need weaker (generalized) G-type I assumptions imposed on the objective and constraint functions.

Theorem 17 *Let $(\overline{x}, \overline{u})$ be a feasible solution in the considered multiobjective continuous programming problem (MCP). Assume that there exist $\overline{\lambda} \in R^p$ and a piecewise smooth function $\overline{\xi}(\cdot) : I \to R^r$ such that the conditions (16)–(19) are satisfied with $G_f = \left(G_{f_1}, \ldots, G_{f_p}\right) : R \to R^p$ being a differentiable vector-valued function such that every its component $G_{f^i} : I_{\int_b^a f^i}(X \times U) \to R$ is a strictly increasing function on its domain and $G_g = \left(G_{g_1}, \ldots, G_{g_q}\right) : R \to R^q$ being a differentiable vector-valued function such that every its component $G_{g^j} : I_{\int_b^a g^j}(X \times U) \to R$ is a strictly increasing function on its domain. Further, assume that one of the following hypotheses is satisfied:*

a) (f, g) are G-type I objective and constraint functions at $(\overline{x}, \overline{u})$ on Ω with respect to G_f, G_g, η and ϑ,
b) (f, g) are pseudo-quasi-G-type I objective and constraint functions at $(\overline{x}, \overline{u})$ on Ω with respect to G_f, G_g, η and ϑ,
c) (f, g) are weak-pseudo-quasi-G-type I objective and constraint functions at $(\overline{x}, \overline{u})$ on Ω with respect to G_f, G_g, η and ϑ.

Then $(\overline{x}, \overline{u})$ is a weakly efficient solution in (MCP).

Proof Proof of theorem under hypothesis a) is similar to the proof of Theorem 14 and, under hypotheses b) and c), to the proof of Theorem 16. □

4 Duality

In this section, for the considered multiobjective variational control problem (MCP), we define its vector variational control dual problem. Under assumptions that the functions constituting these vector optimization problems are (generalized) G-type I objective and constraint functions, we prove various dual results.

Consider the following vector variational control dual problem in the sense of Mond-Weir:

$$V\text{-Minimize} \int_a^b f\left(\pi_{yv}(t)\right) dt = \left(\int_a^b f^1\left(\pi_{yv}(t)\right) dt, \ldots, \int_a^b f^p\left(\pi_{yv}(t)\right) dt\right)$$

$$\text{s.t.} \sum_{i=1}^{p} \lambda_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right) dt\right)\left[f_y^i\left(\pi_{yv}(t)\right) - \frac{d}{dt} f_{\dot{y}}^i\left(\pi_{yv}(t)\right)\right]$$

$$+\sum_{j=1}^{q} \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right) dt\right)\left[g_y^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_{\dot{y}}^j\left(\pi_{yv}(t)\right)\right] = 0, \quad t \in I,$$

$$\sum_{i=1}^{p} \lambda_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right)dt\right)\left[f_v^i\left(\pi_{yv}(t)\right)-\frac{d}{dt}f_{\dot v}^i\left(\pi_{yv}(t)\right)\right]$$

$$+\sum_{j=1}^{q} \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right)dt\right)\left[g_v^j\left(\pi_{yv}(t)\right)-\frac{d}{dt}g_{\dot v}^j\left(\pi_{yv}(t)\right)\right]=0, \quad t\in I,$$

$$\text{subject to } \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right)dt\right) \geqq 0, \quad t\in I, \qquad \text{(DCP)}$$

$$\lambda \in R^p, \quad \lambda \geq 0, \ \xi(t)\in R^q, \quad \xi(t)\geqq 0, \ y(a)=\alpha, \ y(b)=\beta,$$

where $f=\left(f^1,\ldots,f^p\right): I\times R^n\times R^n\times R^m\times R^m \to R^p$ is a p-dimensional function and each its component is a continuously differentiable real scalar function and $g: I\times R^n\times R^n\times R^m\times R^m \to R^q$ is assumed to be a continuously differentiable q-dimensional function.

Let Q be the set of all feasible solutions in (DCP), that is, the set

$$Q=\{(y,v,\lambda,\xi): y(t)\in X, \quad v(t)\in U \text{ verifying the constraints of (DCP) for all } t\in I\}.$$

Further, we denote by Γ the following set $\Gamma=\Omega\cup pr_{X\times U}Q$.

Theorem 18 *(Weak duality). Let (x,u) and (y,v,λ,ξ) be feasible solutions in the considered multiobjective variational control problem (MCP) and its multiobjective variational control dual problem (DCP), respectively. Further, assume that one of the following hypotheses is satisfied:*

a) (f,g) are strictly G-type I objective and constraint functions at (y,v) on Γ with respect to G_f, G_g, η and ϑ,
b) (f,g) are strictly-pseudo-quasi-G-type I objective and constraint functions at (y,v) on Γ with respect to G_f, G_g, η and ϑ,
c) (f,g) are strong-pseudo-quasi-G-type I objective and constraint functions at (y,v) on Γ with respect to G_f, G_g, η and ϑ.
Then the following relations cannot hold

$$\int_a^b f^i\left(\pi_{xu}(t)\right)dt \leqq \int_a^b f^i\left(\pi_{yv}(t)\right)dt \text{ for each } i\in P \tag{44}$$

and

$$\int_a^b f^{i^*}\left(\pi_{xu}(t)\right)dt < \int_a^b f^{i^*}\left(\pi_{yv}(t)\right)dt \text{ for some } i^*\in P. \tag{45}$$

Proof Let (x,u) and (y,v,λ,ξ) be feasible solutions in the considered multiobjective variational control problem (MCP) and its multiobjective variational control dual problem (DCP), respectively. We proceed by contradiction. Suppose, contrary to the result, that (44) and (45) are satisfied.

We prove this theorem under hypothesis a). Since (f,g) are strictly G-type I objective and constraint functions at (y,v) on Γ with respect to G_f, G_g, η and ϑ, by Definition 7, the following inequalities are satisfied

$$G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right) - G_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right)dt\right)$$
$$> G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xuyv}(t)\right)\right]^T\left[f^i_y\left(\pi_{yv}(t)\right) - \frac{d}{dt}f^i_{\dot{y}}\left(\pi_{yv}(t)\right)\right]\right.$$
$$\left. + \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T\left[f^i_v\left(\pi_{yv}(t)\right)\right] - \frac{d}{dt}f^i_{\dot{v}}\left(\pi_{yv}(t)\right)\right\}dt, \quad i = 1, \ldots, p \tag{46}$$

and

$$-G_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right)dt\right)$$
$$\geqq G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xuyv}(t)\right)\right]^T\left[g^j_y\left(\pi_{yv}(t)\right) - \frac{d}{dt}g^j_{\dot{y}}\left(\pi_{yv}(t)\right)\right]\right.$$
$$\left. + \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T\left[g^j_v\left(\pi_{yv}(t)\right) - \frac{d}{dt}g^j_{\dot{v}}\left(\pi_{yv}(t)\right)\right]\right\}dt, \quad j = 1, \ldots, q. \tag{47}$$

Since every $G_{f^i}, i = 1, \ldots, p$, is a strictly increasing function on its domain, the inequalities (44) and (45) yield

$$G_{f^i}\left(\int_a^b f^i\left(\pi_{xu}(t)\right)dt\right) \leqq G_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right)dt\right), \quad i = 1, \ldots, p, \tag{48}$$

and

$$G_{f^{i^*}}\left(\int_a^b f^{i^*}\left(\pi_{xu}(t)\right)dt\right) < G_{f^{i^*}}\left(\int_a^b f^{i^*}\left(\pi_{yv}(t)\right)dt\right) \text{ for some } i^* \in P. \tag{49}$$

By (46), (48) and (49), it follows that

$$G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xuyv}(t)\right)\right]^T\left[f^i_y\left(\pi_{yv}(t)\right) - \frac{d}{dt}f^i_{\dot{y}}\left(\pi_{yv}(t)\right)\right]\right.$$
$$\left. + \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T\left[f^i_v\left(\pi_{yv}(t)\right)\right] - \frac{d}{dt}f^i_{\dot{v}}\left(\pi_{yv}(t)\right)\right\}dt < 0, \quad i = 1, \ldots, p. \tag{50}$$

Multiplying each inequality (50) by λ_i, $i = 1, \ldots, p$, and then adding both sides of the obtained inequalities, we get

$$\sum_{i=1}^p \lambda_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{xuyv}(t)\right)\right]^T\left[f^i_y\left(\pi_{yv}(t)\right) - \frac{d}{dt}f^i_{\dot{y}}\left(\pi_{yv}(t)\right)\right]\right.$$
$$\left. + \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T\left[f^i_v\left(\pi_{yv}(t)\right)\right] - \frac{d}{dt}f^i_{\dot{v}}\left(\pi_{yv}(t)\right)\right\}dt < 0. \tag{51}$$

Multiplying each inequality (24) by $\xi_j(t) \geqq 0$, $j = 1, \ldots, q$, and then adding both sides of the obtained inequalities, we obtain

$$
\begin{aligned}
&-\sum_{j=1}^{q} \xi_j(t)\, G_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right) dt\right) \\
&\geqq \sum_{j=1}^{q} \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xuyv}(t)\right)\right]^T \left[g_y^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_{\dot{y}}^j\left(\pi_{yv}(t)\right)\right]\right. \\
&\quad \left. + \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T \left[g_v^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_{\dot{v}}^j\left(\pi_{yv}(t)\right)\right]\right\} dt. \qquad (52)
\end{aligned}
$$

Using the feasibility of (y, v, λ, ξ) in (DCP) together with (52), we get

$$
\begin{aligned}
&\sum_{j=1}^{q} \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right) dt\right) \int_a^b \left\{\left[\eta\left(\pi_{xuyv}(t)\right)\right]^T \left[g_y^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_{\dot{y}}^j\left(\pi_{yv}(t)\right)\right]\right. \\
&\quad \left. + \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T \left[g_v^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_{\dot{v}}^j\left(\pi_{yv}(t)\right)\right]\right\} dt \leqq 0. \qquad (53)
\end{aligned}
$$

Adding both sides of (51) and (53), we have that the following inequality

$$
\begin{aligned}
&\int_a^b \left[\eta\left(\pi_{xuyv}(t)\right)\right]^T \left\{\sum_{i=1}^{p} \lambda_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right) dt\right)\left[f_y^i\left(\pi_{yv}(t)\right) - \frac{d}{dt} f_{\dot{y}}^i\left(\pi_{yv}(t)\right)\right]\right. \\
&\quad \left. + \sum_{j=1}^{q} \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right) dt\right)\left[g_y^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_{\dot{y}}^j\left(\pi_{yv}(t)\right)\right]\right\} dt \\
&\quad + \int_a^b \left[\vartheta\left(\pi_{xuyv}(t)\right)\right]^T \left\{\sum_{i=1}^{p} \lambda_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{yv}(t)\right) dt\right)\left[f_v^i\left(\pi_{yv}(t)\right) - \frac{d}{dt} f_{\dot{v}}^i\left(\pi_{yv}(t)\right)\right]\right. \\
&\quad \left. + \sum_{j=1}^{q} \xi_j(t)\, G'_{g^j}\left(\int_a^b g^j\left(\pi_{yv}(t)\right) dt\right)\left[g_v^j\left(\pi_{yv}(t)\right) - \frac{d}{dt} g_v^j\left(\pi_{yv}(t)\right)\right]\right\} dt < 0
\end{aligned}
$$

holds, which is a contradiction to the feasibility of (y, v, λ, ξ) in (DCP). This completes the proof of theorem under hypothesis a). □

If weaker generalized invexity hypotheses are assumed on the objective function, then the weaker result is true:

Theorem 19 (Weak duality) *Let (x, u) and (y, v, λ, ξ) be feasible solutions in the considered multiobjective variational control problem (MCP) and its multiobjective variational control dual problem (DCP), respectively. Further, assume that one of the following hypotheses is satisfied:*

a) (f, g) are G-type I objective and constraint functions at (y, v) on Γ with respect to G_f, G_g, η and ϑ, *b) (f, g) are pseudo-quasi-G-type I objective and constraint functions at (y, v) on Γ with respect to G_f, G_g, η and ϑ,*

c) (f, g) are weak-strictly-pseudo-quasi-G-type I objective and constraint functions at (y, v) on Γ with respect to G_f, G_g, η and ϑ.

Then the following relation cannot hold

$$\int_a^b f^i\left(\pi_{xu}(t)\right) dt < \int_a^b f^i\left(\pi_{yv}(t)\right) dt \text{ for each } i \in P.$$

Theorem 20 (Strong duality) *Let $(\overline{x}, \overline{u})$ be an (weakly efficient) efficient solution in the considered multiobjective variational control problem (MCP) and the conditions (16)–(19) be satisfied at this point. Then, there exist $\overline{\lambda} \in R^p$ and a piecewise smooth function $\overline{\xi}(\cdot) : I \to R^r$ such that $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is feasible in the multiobjective variational control dual problem (DCP). If also weak duality Theorem 18 (Theorem 19) holds between (MCP) and (DCP), then $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is an (weakly efficient) efficient solution in (DCP).*

Theorem 21 (Strong duality) *Let $(\overline{x}, \overline{u})$ be a properly efficient solution in the considered multiobjective variational control problem (MCP) and the conditions (16)–(19) be satisfied at this point. Then, there exist $\overline{\lambda} \in R^p$, $\overline{\lambda} > 0$ and a piecewise smooth function $\overline{\xi}(\cdot) : I \to R^r$ such that $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is feasible in the multiobjective variational control dual problem (DCP). Moreover, $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is a properly efficient solution in (DCP) and the objective values at these points are equal.*

Proof Since $(\overline{x}, \overline{u})$ is a properly efficient solution in the considered multiobjective variational control problem (MCP) and the conditions (16)–(19) are satisfied at this point, there exist $\overline{\lambda} \in R^p$, $\overline{\lambda} > 0$ and a piecewise smooth function $\overline{\xi}(\cdot) : I \to R^r$ such that the conditions (16)–(19) are satisfied. Thus, $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is feasible in the multiobjective variational control dual problem (DCP). Thus, by weak duality (Theorem 18), it follows that $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is an efficient solution in problem (DCP).

We shall prove that $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is a properly efficient solution in (DCP) by the method of contradiction. Suppose that $\left(\overline{x}, \overline{u}, \overline{\lambda}, \overline{\xi}\right)$ is not so. Then, there exists $\left(\widetilde{y}, \widetilde{u}, \widetilde{\lambda}, \widetilde{\xi}\right)$ feasible in (DCP) and $i^* \in P$ such that the following inequality

$$\int_a^b f^{i^*}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right) dt - \int_a^b f^{i^*}\left(\pi_{\overline{x}\overline{u}}(t)\right) dt > M\left(\int_a^b f^k\left(\pi_{\overline{x}\overline{u}}(t)\right) dt - \int_a^b f^k\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right) dt\right) \tag{54}$$

holds for every scalar $M > 0$ and all k satisfying

$$\int_a^b f^k\left(\pi_{\overline{x}\overline{u}}(t)\right) dt > \int_a^b f^k\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right) dt. \tag{55}$$

We divide the index set P and denote by P_1 the set of indexes of objective functions satisfying the inequality (55). By P_2 we denote the set of indexes of objective functions defining as follows $P_2 = P \setminus (P_1 \cup i^*)$. The inequality (54) is satisfied for all $M > 0$. Then, we set $M > \frac{\overline{\lambda}_k}{\overline{\lambda}_{i^*}} |P_1|$, where $|P_1|$ denotes the number of elements in the set P_1. Thus, (54) and (55) yield

$$\overline{\lambda}_{i^*}\left(\int_a^b f^{i^*}\left(\pi_{\overline{xu}}(t)\right)dt-\int_a^b f^{i^*}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right)dt\right)$$
$$>\sum_{k\in P_1}\overline{\lambda}_k\left(\int_a^b f^{k}\left(\pi_{\overline{xu}}(t)\right)dt-\int_a^b f^{k}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right)dt\right). \quad (56)$$

By the definition of the set P_2, (54), (55) and (56), it follows that

$$\sum_{i=1}^{p}\overline{\lambda}_i\int_a^b f^{i}\left(\pi_{\overline{xu}}(t)\right)dt=\overline{\lambda}_{i^*}\int_a^b f^{i^*}\left(\pi_{\overline{xu}}(t)\right)dt+\sum_{k\in P_1}\overline{\lambda}_k\int_a^b f^{k}\left(\pi_{\overline{xu}}(t)\right)dt$$
$$+\sum_{k\in P_2}\overline{\lambda}_k\int_a^b f^{k}\left(\pi_{\overline{xu}}(t)\right)dt<\overline{\lambda}_{i^*}\int_a^b f^{i^*}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right)dt$$
$$+\sum_{k\in P_1}\overline{\lambda}_k\int_a^b f^{k}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right)dt+\sum_{k\in P_2}\overline{\lambda}_k\int_a^b f^{k}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right)dt=\sum_{i=1}^{p}\overline{\lambda}_i\int_a^b f^{i}\left(\pi_{\widetilde{y}\widetilde{v}}(t)\right)dt.$$

This is a contradiction to the weak duality theorem. Hence, $\left(\overline{x},\overline{u},\overline{\lambda},\overline{\xi}\right)$ is a properly efficient solution in the vector Mond–Weir dual problem (VMWD), and the optimal objective function values in the primal and the dual problems are equal. □

Theorem 22 (Strict converse duality) *Let* $(\overline{x},\overline{u})$ *and* $\left(\overline{y},\overline{v},\overline{\lambda},\overline{\xi}\right)$ *be feasible solutions in the vector variational control problems (MCP) and (DCP), respectively, such that*

$$\overline{\lambda}_i G_{f^i}\left(\int_a^b f^{i}\left(\pi_{\overline{xu}}(t)\right)dt\right)=\overline{\lambda}_i G_{f^i}\left(\int_a^b f^{i}\left(\pi_{\overline{yv}}(t)\right)dt\right). \quad (57)$$

Further, assume that (f,g) *are strictly-G-type I objective and constraint functions at* $(\overline{y},\overline{v})$ *on* Γ *with respect to* G_f, G_g, η *and* ϑ. *Then* $(\overline{x},\overline{u})=(\overline{y},\overline{v})$.

Proof Suppose, contrary to the result, that $(\overline{x},\overline{u})\neq(\overline{y},\overline{v})$. By assumption, (f,g) are strictly-G-type I objective and constraint functions at $(\overline{y},\overline{v})$ on Γ with respect to G_f, G_g, η and ϑ. Then, by Definition 7, the following inequalities are satisfied

$$G_{f^i}\left(\int_a^b f^{i}\left(\pi_{\overline{xu}}(t)\right)dt\right)-G_{f^i}\left(\int_a^b f^{i}\left(\pi_{\overline{yv}}(t)\right)dt\right)$$
$$>G'_{f^i}\left(\int_a^b f^{i}\left(t,\pi_{\overline{yv}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[f^i_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}f^i_{\dot{y}}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[f^i_v\left(\pi_{\overline{yv}}(t)\right)\right]-\frac{d}{dt}f^i_{\dot{v}}\left(\pi_{\overline{yv}}(t)\right)\right\}dt,\quad i=1,\ldots,p, \quad (58)$$

$$-G_{g^j}\left(\int_a^b g^{j}\left(\pi_{\overline{yv}}(t)\right)dt\right)$$

$$\geqq G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{yv}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[g^j_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[g^j_v\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot v}\left(\pi_{\overline{yv}}(t)\right)\right]\right\}dt,\quad j=1,\ldots,q. \tag{59}$$

Multiplying each inequality (58) by $\overline{\lambda}_i$, $i=1,\ldots,p$, then (57) gives

$$G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{yv}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[f^i_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}f^i_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[f^i_v\left(\pi_{\overline{yv}}(t)\right)\right]-\frac{d}{dt}f^i_{\dot v}\left(\pi_{\overline{yv}}(t)\right)\right\}dt<0,\quad i=1,\ldots,p.$$

Adding both sides of the above inequalities, we get

$$\sum_{i=1}^p\overline{\lambda}_iG'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{yv}}(t)\right)dt\right)\int_a^b\left\{\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[f^i_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}f^i_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[f^i_v\left(\pi_{\overline{yv}}(t)\right)\right]-\frac{d}{dt}f^i_{\dot v}\left(\pi_{\overline{yv}}(t)\right)\right\}dt<0. \tag{60}$$

Multiplying each inequality (59) by $\overline{\xi}_j(t)\geqq 0$, $j=1,\ldots,q$, and then adding both sides of the obtained inequalities, we obtain

$$-\sum_{j=1}^q\overline{\xi}_j(t)\,G_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{yv}}(t)\right)dt\right)$$
$$\geqq\sum_{j=1}^q G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{yv}}(t)\right)dt\right)\int_a^b\left\{\overline{\xi}_j(t)\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[g^j_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[g^j_v\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot v}\left(\pi_{\overline{yv}}(t)\right)\right]\right\}dt,\quad j=1,\ldots,q. \tag{61}$$

Hence, the feasibility of $\left(\overline{y},\overline{v},\overline{\lambda},\overline{\xi}\right)$ in (DCP) implies

$$\sum_{j=1}^q G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{yv}}(t)\right)dt\right)\int_a^b\left\{\overline{\xi}_j(t)\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[g^j_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left[g^j_v\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot v}\left(\pi_{\overline{yv}}(t)\right)\right]\right\}dt\leqq 0. \tag{62}$$

Adding both sides of (60) and (62), we get that the following inequality

$$\int_a^b\left[\eta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T\left\{\sum_{i=1}^p\lambda_iG'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{yv}}(t)\right)dt\right)\left[f^i_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}f^i_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$
$$\left.+\sum_{j=1}^q\xi_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{yv}}(t)\right)dt\right)\left[g^j_y\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g^j_{\dot y}\left(\pi_{\overline{yv}}(t)\right)\right]\right\}dt$$

$$+\int_a^b \left[\vartheta\left(\pi_{\overline{xuyv}}(t)\right)\right]^T \left\{\sum_{i=1}^{p} \lambda_i G'_{f^i}\left(\int_a^b f^i\left(\pi_{\overline{yv}}(t)\right)dt\right)\left[f_v^i\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}f_{\dot v}^i\left(\pi_{\overline{yv}}(t)\right)\right]\right.$$

$$\left.+\sum_{j=1}^{q}\xi_j(t)\,G'_{g^j}\left(\int_a^b g^j\left(\pi_{\overline{yv}}(t)\right)dt\right)\left[g_v^j\left(\pi_{\overline{yv}}(t)\right)-\frac{d}{dt}g_{\dot v}^j\left(\pi_{\overline{yv}}(t)\right)\right]\right\}dt<0$$

holds, which is a contradiction to the feasibility of $\left(\overline{y}, \overline{v}, \overline{\lambda}, \overline{\xi}\right)$ in (DCP). This completes the proof of theorem. □

5 Conclusion

In the paper, the concept of G-type I objective and constraint functions and its various generalizations have been extended to the continuos case. Thus, a new class of nonconvex multiobjective variational control problems has been considered. The sufficient optimality criteria for such a class of nonconvex multiobjective variational control problems have been studied under hypotheses that the functions constituting such nonconvex multiobjective variational control problems are G-type I objective and constraint functions and/or belong to various classes of generalized G-type I objective and constraint functions. Also various duality results between the considered multiobjective variational control problem and its multiobjective variational control dual problem in the sense of Mond–Weir have also proved under a variety of G-type I hypotheses. We are going to extend the results established in the paper to a larger class of nonconvex multiobjective variational control problems. This will orient the future research of the author.

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