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ON σ–IDEALS WITHOUT MAXIMAL EXTENSIONS

We characterize those σ-ideals in a Boolean σ-algebra which have no maximal extensions in this algebra. We show some applications.

It is well known that every ideal of a Boolean algebra is included in some maximal ideal of that algebra. It seems interesting to verify if the above fact has its analogue for σ-ideals.

Let us observe first that, if Δ is a maximal σ-ideal in the class of all σ-ideals of a given σ-algebra, then it is also maximal in the class of all ideals of that σ-algebra.

Really, if Δ is not maximal in the class of all ideals, then there exists a maximal ideal Δ' such that $\Delta \subsetneq \Delta'$. Let a be an element of our σ-algebra such that $a \in \Delta' \setminus \Delta$. Then the σ-ideal generated by Δ and a, being a proper extension of the σ-ideal Δ, cannot be a proper ideal. This yields that there exist elements $a_1, a_2, \ldots$ of the σ-ideal Δ such that $\mathbf{1} := a \vee \sup a_i$. Now we conclude that $-a \leq \sup a_i \in \Delta$ and $-a \in \Delta \subseteq \Delta'$. Thus we obtain that $a \in \Delta'$ and $-a \in \Delta'$. It is impossible.

Now we introduce the notion of an essential ideal. We shall say that a proper σ-ideal Δ of a given σ-algebra is *essential* if and only if for each maximal ideal Δ' including Δ there exists a sequence $(a_i)_{i=1}^{\infty}$ of elements of that σ-algebra fulfilling the conditions:

(1) $a_i \notin \Delta'$ for $i = 1, 2, 3 \ldots$,
(2) $\inf a_i \in \Delta$.

Theorem 1. *A proper σ-ideal of some σ-algebra is essential if and only if it is not included in any maximal σ-ideal of this σ-algebra.*

Proof. $\Rightarrow$ Suppose that an essential σ-ideal Δ of a σ-algebra $\mathcal{A}$ is included in some maximal σ-ideal Δ' which is also a maximal ideal of $\mathcal{A}$ as we have noticed above. According to the definition of an essential ideal there exists a sequence $(a_i)_{i=1}^{\infty}$ fulfilling both conditions (1) and (2). Since $a_i \notin \Delta'$, therefore the σ-ideal Δ'_{a_i} generated by Δ' and a_i is essentially larger then Δ'. Thus $\mathbf{1}: \in \Delta'_{a_i}$ for every i because Δ' is a maximal σ-ideal. It enables us to conclude that for every i there exists $b_i \in \Delta'$ such that $\mathbf{1}: = a_i \vee b_i$. As a result of taking (2) into account we obtain $\mathbf{1}: = \inf(a_i \vee b_i) \leq (\inf a_i) \vee (\sup b_i) \in \Delta'$. It is impossible since Δ' is proper as a maximal ideal.

$\Leftarrow$ Assume that a proper σ-ideal Δ is not included in any maximal σ-ideal. Let Δ' denote a maximal ideal containing Δ. Since the σ-ideal generated by Δ' is not proper, there exists a sequence $(a_i)_{i=1}^{\infty}$ of elements of Δ' such that $\sup a_i =: \mathbf{1}$. This enables us to conclude that $-a_i \notin \Delta'$ for each i, and that $\inf(-a_i) =: \mathbf{1}: - (\sup a_i) =: \mathbf{1}: -: \mathbf{1}$ $= \mathbf{0}: \in \Delta$. It means that Δ is an essential σ-ideal.

Theorem 2. *For each σ-algebra $\mathcal{A}$ of subsets of $[0,1]$ containing all Borel sets, a σ-ideal $\mathcal{I}$ of $\mathcal{A}$ is a maximal σ-ideal in $\mathcal{A}$ if and only if it is of the form*

$$(x) = \{E \subset \mathcal{A} : x \notin E\}$$

for some $x \in [0,1]$.

Proof. $\Rightarrow$ Consider two cases:

1° $\mathcal{I}$ contains all singletons $\{x\}$, $x \in [0,1]$. Then $\mathcal{I}$ is an essential σ-ideal. Indeed, let Δ be an arbitrary maximal ideal Δ of $\mathcal{A}$. We define a descending sequence of intervals as follows. Since Δ is a maximal ideal, therefore either $[0, \frac{1}{2}) \notin \Delta$ or $[\frac{1}{2}, 1] \notin \Delta$. Put $A_1 = [0, \frac{1}{2}]$ in the first case and $A_1 = [\frac{1}{2}, 1]$ in the other case. Suppose that we have defined $A_n = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \notin \Delta$ where $k \in \{0, 1, \dots, 2^n - 1\}$. Consider the pair of intervals $\left[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}\right)$ and $\left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right]$. At least one of these intervals does not belong to Δ. We choose A_{n+1} as that interval. Then the set $\bigcap_{n=1}^{\infty} A_n$ is a singleton, hence it belongs to $\mathcal{I}$. This shows that $\mathcal{I}$ is an essential σ-ideal. So it cannot be maximal, by Theorem 1. Thus the case 1° is impossible.

2° There exists $\{x\} \notin \mathcal{I}$. If there exists $y \neq x$ such that $\{y\} \notin \mathcal{I}$ then $\mathcal{I}$ is not maximal since the σ-ideal $\mathcal{I}_{\{y\}}$ generated by $\mathcal{I}$ and $\{y\}$ is proper and larger than $\mathcal{I}$. So $\{x\}$ is a unique singleton which is not in $\mathcal{I}$ and thus $\mathcal{I} = (x)$ since (x) is the biggest proper σ-ideal which does not contain x.

$\Leftarrow$ Obvious.

Corollary. *Each of the following σ-ideals:*

- *the σ-ideal $\mathcal{L}_0$ of Lebesgue null sets in the σ-algebra of $\mathcal{L}$ of measurable subsets of $[0,1]$.*
- *the σ-ideal $\mathcal{B}_0$ of the first category sets in the σ-algebra $\mathcal{B}$ of subsets of $[0,1]$ with the Baire property,*

is not maximal (in fact, it is an essential σ-ideal).

Remark. Note that $\mathcal{L}_0$ and $\mathcal{B}_0$ can be maximal σ-ideals in some non-trivial subfamilies of the family of all ideals of $\mathcal{L}$ and $\mathcal{B}$, respectively. Namely, consider the family $\mathcal{F}$ of all σ-ideals Δ in $\mathcal{L}$ such that

$$(*) \qquad (\forall A \in \Delta)\,(\forall B \subset A)\,(B \in \mathcal{L})$$

Then $\mathcal{L}_0$ is the greatest σ-ideal in $\mathcal{F}$. Indeed, let $\Delta \in \mathcal{F}$ and suppose that $A \in \Delta \setminus \mathcal{L}_0$. It is known that A contains a nonmeasurable set B (see [1]). Hence $(*)$ is false, which contradicts $\Delta \in \mathcal{F}$. Consequently $\Delta \subset \mathcal{L}_0$. The category case is analogous.

References

[1] J.C. Oxtoby, *Measure and Category*, Springer Verlag, New York, 1971.

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O σ-IDEAŁACH BEZ MAKSYMALNYCH ROZSZERZEŃ

Scharakteryzowano σ-ideały w dowolnej σ-algebrze Boole'a, których nie da się rozszerzyć do σ-ideału maksymalnego w tej algebrze. Podano kilka zastosowań.

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