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OPTIMIZATION PROBLEMS WITH DIFFERENTIAL-INTEGRAL CONSTRAINTS OF VOLTERRA AND FREDHOLM TYPES

In the paper we give theorems on the existence of solutions of linear differential-integral equations as well as necessary conditions for the existence of the extremum for the optimization problems described by these equations.

1. INTRODUCTION

Most papers dealing with optimization theory concern the minimization of integral functionals under additional conditions described by ordinary or partial differential equations. In the present paper we consider the problems of the minimization of linear integral functionals under additional conditions described by differential-integral equations of the form

$$(1.1)\quad \dot{x}(t) = A(t)x(t) + \int_0^t G(t, \tau)x(\tau)d\tau + S(t)u(t)$$

or

$$(1.2)\quad \dot{x}(t) = A(t)x(t) + \int_0^1 G(t, \tau)x(\tau)d\tau + S(t)u(t)$$

as well as the question of the existence and the ways of determining solutions of equations of form (1.1) or (1.2) when $S(t)u(t)$ is an integrable function. Theorems on the existence of solutions of differential-integral equations and the ways of determining them were considered earlier (cf. [4], [5]), but under stronger assumpions than those in our paper.

2. ON A SYSTEM OF DIFFERENTIAL-INTEGRAL EQUATIONS OF VOLTERRA TYPE

Let a differential-integral equation of the form

$$(2.1)\qquad \dot{x}(t) = A(t)x(t) + \int_0^t G(t, \tau)x(\tau)d\tau + u(t)$$

with the initial condition

$$(2.2)\qquad x(0) = 0$$

be given, where $x(\cdot)$ and $u(\cdot)$ are n-dimensional vector functions, where $A(\cdot)$ and $G(\cdot,\cdot)$ are functions with matrix values of degree n.

Assume that the given functions $A(\cdot)$ and $G(\cdot,\cdot)$ are bounded and continuous (i.e. each element of the matrices $A(\cdot)$ and $G(\cdot,\cdot)$ is a bounded and continuous function) in the interval (0, 1) and in the triangle $0 < \tau < t < 1$, respectively, whereas $u(\cdot) \in L_1^n[0, 1]$. About the sought-for solutions $x(\cdot)$ let us assume that they are elements of the space $W_{11}^n[0, 1]$.

Each function $x(\cdot) \in W_{11}^n[0, 1]$ satisfying condition (2.2) can be represented in the form $x(t) = \int_0^t v(\tau)d\tau$ where $v(\cdot) \in L_1^n[0, 1]$. Hence equation (2.1) with condition (2.2) is equivalent to the system of equations

$$(2.3)\qquad v(t) = A(t)\int_0^t v(\tau)d\tau + \int_0^t \left(G(t, \tau)\int_0^\tau v(s)ds\right)d\tau + u(t)$$

and

$$(2.4)\qquad x(t) = \int_0^t v(\tau)d\tau$$

where $v(\cdot) \in L_1^n[0, 1]$. Applying the formula for integration by parts to the second integral occurring on the right-hand side of equality (2.3), we obtain, after simple transformations, the equation

$$(2.5)\qquad v(t) = \int_0^t \left(A(t) + \int_\tau^t G(t, s)ds\right)v(\tau)d\tau + u(t)$$

being a Volterra integral equation of the second kind with the kernel $V(t, \tau) := A(t) + \int_\tau^t G(t, s)ds$ bounded and continuous in the triangle $0 < \tau < t < 1$ and with the function $u(\cdot)$ integrable in the interval $\langle 0, 1\rangle$. Consequently, equation (2.5) possesses exactly one solution $v(\cdot) \in L_1^n[0, 1]$ being the sum of a uniformly convergent Neumann series (cf. [3], II § 5). Having the solution of equation (2.5), we find immediately the solution of equation (2.4), being a solution of equation (2.1) with condition (2.2). Hence

THEOREM 2.1. If $A(\cdot)$ and $G(\cdot,\cdot)$ are bounded and continuous functions in the interval $(0, 1)$ and in the triangle $0 < \tau < t < 1$, respectively, and $u(\cdot)$ is an integrable function in the interval $\langle 0, 1\rangle$, then there exists one and only one absolutely continuous solution $x(\cdot)$ of equation (2.1) with condition (2.2) given by the formula $x(t) = \int_0^t v(\tau)d\tau$ where $v(\cdot)$ is the sum of a uniformly convergent Neumann series of equation (2.5).

A similar theorem can be formulated under the assumption that $G(\cdot,\cdot)$, $A(\cdot)$ and $u(\cdot)$ are square-integrable functions (cf. [3], II. § 10).

3. LINEAR EXTREMUM PROBLEMS WITH DIFFERENTIAL-INTEGRAL CONSTRAINTS OF VOLTERRA TYPE

Suppose that we have

PROBLEM 3.1. Minimize the functional

(3.1) $$I(x, u) = \int_0^1 (a(t)x(t) + b(t)u(t))dt$$

under the conditions

(3.2) $$\dot{x}(t) = A(t)x(t) + \int_0^t G(t, \tau)x(\tau)d\tau + S(t)u(t),$$

(3.3) $$x(0) = x_o = 0,$$

(3.4) $$x(1) = x_1,$$

(3.5) $$u(t) \in U \quad \text{for} \quad t \in \langle 0, 1\rangle \quad \text{a.e.}$$

where we are given:

1^o $a(\cdot)$, $b(\cdot)$ and $S(\cdot)$ are functions with matrix values of dimensions $1 \times n$, $1 \times r$ and $n \times r$, respectively, whose elements belong to $L_1[0, 1]$;

2^o $A(\cdot)$ and $G(\cdot,\cdot)$ are functions with matrix values of degree n, with elements bounded and continuous in the interval (0, 1) and in the triangle $0 < \tau < t < 1$, respectively;

3^o U is a convex set contained in $\mathbb{R}^r$,

and $x_o = 0$ and x_1 are fixed points of the space $\mathbb{R}^n$.

We assume that the sought-for function $u(\cdot)$ called a control, is an element of the space $L^r_\infty[0, 1]$ and its corresponding (by relation (3.2)) function $x(\cdot)$, called a trajectory, is an element of the space $W^n_{11}[0, 1]$.

Let us denote

$$(3.6)\quad f_o(x, u) := \int_0^1 (a(t)x(t) + b(t)u(t))dt,$$

$$(3.7)\quad F(x, u) := \dot{x}(t) - A(t)x(t) - \int_0^t G(t, \tau)x(\tau)d\tau - S(t)u(t),$$

$$(3.8)\quad h_1(x, u) := x(0) - x_o = x(0),$$

$$(3.9)\quad h_2(x, u) := x(1) - x_1,$$

$$(3.10)\quad \mathcal{U} := \{u(\cdot) \in L^r_\infty[0, 1]: u(t) \in U \text{ for } t \in \langle 0, 1\rangle \text{ a.e}\},$$

$$(3.11)\quad X := W^n_{11}[0, 1], \quad Y := L^n_1[0, 1];$$

then

$$f_o: X \times \mathcal{U} \to \mathbb{R}, \quad F: X \times \mathcal{U} \to Y, \quad h_1: X \times \mathcal{U} \to \mathbb{R}^n$$

and $\quad h_2: X \times \mathcal{U} \to \mathbb{R}^n.$

As is known, X and Y are Banach spaces. The convexity of the set U immediately implies the convexity of the set $\mathcal{U}$. With each $x \in X$, the mappings f_o, F, h_1 and h_2 satisfy the condition of the convexity with respect to u in the set $\mathcal{U}$, assumed in the Ioffe--Tikhomirov extremum principle (cf. [2], I § 1.1). It follows from the linearity of these mappings with respect to u and from the convexity of the set $\mathcal{U}$. With each $u \in \mathcal{U}$, the mappings f_o, F, h_1 and h_2 are Fréchet differentiable with respect to x and, for any $\bar{x} \in X$, we have

$$f_{o_x}(x, u)\bar{x} = \int_0^1 a(t)\bar{x}(t)dt,$$

$$F_x(x, u)\bar{x} = \dot{\bar{x}}(t) - A(t)\bar{x}(t) - \int_0^t G(t, \tau)\bar{x}(\tau)d\tau,$$

$$h_{1_x}(x, u)\bar{x} = \bar{x}(0),$$

$$h_{2_x}(x, u)\bar{x} = \bar{x}(1).$$

Since the mappings f_{o_x}, F_x, h_{1_x} and h_{2_x} are constant with respect to x, they are continuous at each point $x \in X$, in the sense of the topology of the space $\mathcal{L}(X, Y)$. Consequently, the mappings $f_o(\cdot, u)$, $F(\cdot, u)$, $h_1(\cdot, u)$ and $h_2(\cdot, u)$ are of the class C^1 with any $u \in \mathcal{U}$. It is known from the considerations of § 2 that, for any $g \in Y$, the equation

$$\dot{\bar{x}}(t) - A(t)\bar{x}(t) - \int_0^t G(t, \tau)\bar{x}(\tau)d\tau = g(t)$$

has a solution $\bar{x} \in X$, which means that the mapping $F(\cdot, u)$ is regular with each $u \in \mathcal{U}$. Consequently, all the assumptions of the Ioffe-Tikhomirov extremum principle are satisfied.

With notations (3,6)-(3,9), the Lagrangian function $\mathcal{L}$ for problem 1 has the form

$$\mathcal{L}(x, u, \lambda_o, \lambda_1, \lambda_2, y^*) = \lambda_o f_o(x, u) + (\lambda_1, h_1(x, u)) + $$
$$+ (\lambda_2, h_2(x, u)) + \langle y^*, F(x, u)\rangle$$

where λ_o, λ_1, λ_2 and y^* are Lagrange multipliers, with that $\lambda_o \in R$, $\lambda_1 \in R^n$, $\lambda_2 \in R^n$, and $y^* \in Y^*$. Since $Y = L_1^n[0, 1]$, therefore $Y^* = L_\infty^n[0, 1]$ and $\langle y^*, F(x, u)\rangle = \int_0^1 (y(t), F(x(t), u(t)))dt$ where $y(\cdot)$ is some function from $L_\infty^n[0, 1]$. Hence and from (3.6), (3.7), (3.8) and (3.9) we have

$$\text{(3.12)}\quad \mathcal{L}(x, u, \lambda_o, \lambda_1, \lambda_2, y^*) = \lambda_o \int_0^1 (a(t)x(t) + b(t)u(t))dt + $$
$$+ (\lambda_1, x(0)) + (\lambda_2, x(1) - x_1) + $$
$$+ \int_0^1 (y(t), \dot{x}(t) - A(t)x(t) - $$
$$- \int_0^t G(t, \tau)x(\tau)d\tau - S(t)u(t))dt.$$

The Fréchet derivative of the Lagrangian function with respect to the variable x has, for any $\bar{x} \in X$, the form

$$(3.13)\quad \mathcal{L}_x(x,\ u,\ \lambda_o,\ \lambda_1,\ \lambda_2,\ y^*)\bar{x} = \lambda_o \int_0^1 a(t)\bar{x}(t)dt + (\lambda_1,\ \bar{x}(0)) + (\lambda_2,\ \bar{x}(1)) + \int_0^1 (y(t),\ \dot{\bar{x}}(t) - A(t)\bar{x}(t) - \int_0^t G(t,\ \tau)\bar{x}(\tau)d\tau)dt.$$

Let a pair $(x^*,\ u^*)$ be a solution of Problem 3.1. In accordance with the Ioffe-Tikhomirov extremum principle, we have

$$(3.14)\quad \mathcal{L}_x(x^*,\ u^*,\ \lambda_o,\ \lambda_1,\ \lambda_2,\ y^*)\bar{x} = 0 \quad \text{for any} \quad \bar{x} \in X$$

and

$$(3.15)\quad \mathcal{L}(x^*,\ u^*,\ \lambda_o,\ \lambda_1,\ \lambda_2,\ y^*) = \min_{u\in\mathcal{U}} \mathcal{L}(x^*,\ u,\ \lambda_o,\ \lambda_1,\ \lambda_2,\ y^*),$$

with that the multipliers λ_o, λ_1, λ_2 and $y(\cdot)$ do not vanish simultaneously and $\lambda_o \geq 0$. From (3.13) and (3.14) we obtain that

$$(3.16)\quad \int_0^1 \lambda_o a(t)\bar{x}(t)dt + (\lambda_1,\ \bar{x}(0)) + (\lambda_2\ \bar{x}(1)) + \int_0^1 (y(t),\ \dot{\bar{x}}(t))dt + \int_0^1 (A^T(t)y(t),\ \bar{x}(t))dt - \int_0^1 (y(t),\ \int_0^t G(t,\ \tau)\bar{x}(\tau)d\tau)dt = 0$$

for each $\bar{x} \in X$.

Denote by $W_{11}^{n0}[0,\ 1]$ the set of functions $\bar{x} \in X$ satisfying the condition $x(0) = 0$. Since (3.16) is to hold for all $\bar{x} \in X = W_{11}^{n}[0,\ 1]$, it also holds for each $\bar{x} \in W_{11}^{n0}[0,\ 1]$ and has the form

$$\int_0^1 (\lambda_o a^T(\tau) - A^T(\tau)y(\tau),\ \bar{x}(\tau))d\tau + (\lambda_2,\ \bar{x}(1)) + \int_0^1 (y(\tau),\ \dot{\bar{x}}(\tau))d\tau + \int_0^1 (y(t),\ \int_0^t G(t,\ \tau)\bar{x}(\tau)d\tau)dt = 0.$$

Changing the succession of integration in the last component and making use of the equality $\bar{x}(1) = \int_0^1 \dot{\bar{x}}(\tau)d\tau$ for $\bar{x} \in W_{11}^{n0}[0,\ 1]$, we get the relationship

$$\int_0^1 (\lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_\tau^1 G^T(t,\ \tau)y(t)dt,\ x(\tau))d\tau + \int_0^1 (\lambda_2 + y(\tau),\ \dot{\bar{x}}(\tau))d\tau = 0$$

for each $\bar{x} \in W_{11}^{n0}[0, 1]$. Hence, having applied to the first integral the formula for integration by parts, we have

$$(3.17) \quad \int_0^1 (\int_\tau^1 (\lambda_o a^T(s) - A^T(s)y(s) - \int_S^1 G^T(t, s)y(t)dt)ds + \lambda_2 + y(\tau), \dot{\bar{x}}(\tau))d\tau = 0$$

for each $\bar{x} \in W_{11}^{n0}[0, 1]$.

Since, for each $\bar{x} \in W_{11}^{n0}[0, 1]$, there exists $v(\cdot) \in L_1^n[0, 1]$ such that $\bar{x}(\tau) = \int_0^\tau v(s)ds$, and conversely, for any $v(\cdot) \in L_1^n[0, 1]$, $\bar{x}(\tau) = \int_0^\tau v(s)ds$ belongs to the space $W_{11}^{n0}[0, 1]$, therefore, denoting $v(\tau) := \dot{\bar{x}}(\tau)$, we have from (3.17) that

$$\int_0^1 (\int_\tau^1 (\lambda_o a^T(s) - A^T(s)y(s) - \int_s^1 G^T(t, s)y(t)dt)ds + \lambda_2 + y(\tau), v(\tau))d\tau = 0$$

for each $v(\cdot) \in L_1^n[0, 1]$. This implies

$$(3.18) \quad y(\tau) = -\lambda_2 - \int_\tau^1 (\lambda_o a^T(s) - A^T(s)y(s) - \int_s^1 G^T(t, s)y(t)dt)ds.$$

From the last equality it follows that $y(\cdot)$ is an absolutely continuous function satisfying the differential-integral equation of Volterra type

$$(*) \quad \dot{y}(\tau) = \lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_\tau^1 G^T(t, \tau)y(t)dt,$$

with that λ_o and $y(\cdot)$ do not vanish simultaneously. If $\lambda_o = 0$ and $y(\cdot) \equiv 0$, then from (3.18) it would follow that $\lambda_2 = 0$ and from (3.16) that $(\lambda_1, \bar{x}(0)) = 0$ for each $\bar{x} \in X$, whence also $\lambda_1 = 0$, which contradicts the extremum principle. From (3.12) and (3.15), after simple transformations, we obtain that

$$(3.19) \quad \int_0^1 (\lambda_o b^T(t) - S^T(t)y(t), u^*(t))dt \le \int_0^1 (\lambda_o b^T(t) - S^T(t)y(t), u(t))dt$$

for each $u \in \mathcal{U}$. Since $\lambda_o b^T(t) - S^T(t)y(t) \in L_1^r[0, 1]$, therefore (cf. [1], § 10) from (3.19) it follows that

$$(\lambda_o b^T(t) - S^T(t)y(t),\ u^*(t)) \le (\lambda_o b^T(t) - S^T(t)y(t),\ u)$$

for each $u \in U$ and for $t \in \langle 0, 1\rangle$ a.e., that is to say,

$$(**)\ (\lambda_o b^T(t) - S^T(t)y(t),\ u^*(t)) = \min_{u \in U} (\lambda_o b^T(t) - S^T(t)y(t),\ u)$$

for $t \in \langle 0, 1\rangle$ a.e.

We have thus proved the following

THEOREM 3.1. If assumptions 1^o-3^o are satisfied and the pair (x^*, u^*), where $x^* \in W_{11}^n[0, 1]$ and $u^* \in L_\infty^r[0, 1]$, is a solution of problem 1, then there exist a function $y \in W_{11}^n[0, 1]$ and a number $\lambda_o \ge 0$, not vanishing simultaneously and such that

$$(*)\ \dot{y}(\tau) = \lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_\tau^1 G^T(t, \tau)y(t)dt$$

and

$$(**)\ \lambda_o b^T(t) - S^T(t)y(t),\ u^*(t)) = \min_{u \in U} (\lambda_o b^T(t) - S^T(t)y(t),\ u)$$

for $t \in \langle 0, 1\rangle$ a.e.

Suppose that in problem 1 the right end of the trajectory is free, i.e. we have

PROBLEM 3.2. Minimize functional (3.1) under conditions (3.2), (3.3) and (3.5). The Lagrangian function has now the form

$$\mathcal{L}(x, u, \lambda_o, \lambda_1, y^*) = \lambda_o f_o(x, u) + (\lambda_1, h_1(x, u)) + $$
$$+ \langle y^*, F(x, u)\rangle.$$

Reasoning analogously as in Problem 3.1, we obtain that the multipliers $\lambda_o \ge 0$ and $y \in L_\infty^n[0, 1]$, not vanishing simultaneously, satisfy

$$(3.20)\quad y(\tau) = -\int_\tau^1 (\lambda_o a^T(s) - A^T(s)y(s) - \int_s^1 G^T(t, s)y(t)dt)ds$$

and equation (**). From (3.20) it follows that $y(\cdot)$ is an absolutely continuous function satisfying the differential-integral equation of the form

$$(*_1)\quad \dot{y}(\tau) = \lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_\tau^1 G^T(t, \tau)y(t)dt,$$

with that $y(1) = 0$. Hence

COROLLARY 3.1. If assumptions 1^o-3^o are satisfied and the pair (x^*, u^*) is a solution of Problem 3.2, then there exists a function $y \in W_{11}^n[0, 1]$ and a number $\lambda_o \geq 0$, not vanishing simultaneously and such that

$$(*_1)\ \dot{y}(\tau) = \lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_\tau^1 G^T(t, \tau)y(t)dt,\ y(1) = 0,$$

and condition (**) is satisfied.

In order to close this section, we shall solve

Example 3.1. Minimize the functional

$$I(x, u) = \int_0^1 (x(t) + 2t \sinh(t - 1)u(t))dt$$

under the conditions

$$(3.21)\quad \dot{x}(t) = \int_0^t x(\tau)d\tau + u(t),\ x(0) = 0,$$

$u(t) \in U = \langle -1, 1\rangle$ for $t \in \langle 0, 1\rangle$ a.e.

Let (x^*, u^*) be a solution of this problem. In accordance with Corollary 3.1, there exist a function $y(\cdot) \in W_{11}[0, 1]$ and a number $\lambda_o \geq 0$, such that

$$(3.22)\quad \dot{y}(\tau) = \lambda_o - \int_\tau^1 y(t)dt,\ y(1) = 0.$$

Let $y(\tau) = -\int_\tau^1 v(s)ds$ where $v(\cdot) \in L_1[0, 1]$. Equation (3.22) takes then the form

$$v(\tau) = \lambda_o + \int_\tau^1 (\int_t^1 v(s)ds)dt$$

and, after the application of the formula for integration by parts,

$$(3.23)\quad v(\tau) = \lambda_o + \int_1^\tau (\tau - t)v(t)dt.$$

The solution of the last equation is the function $v(\tau) = \lambda_o \cosh(\tau - 1)$. Consequently, the solution of equation (3.22) is the function $y(\tau) = \lambda_o \sinh(\tau - 1)$. Since $\lambda_o \geq 0$ and $y(\cdot)$ do not vanish simultaneously, therefore from the form of the function $y(\cdot)$ it follows that $\lambda_o > 0$. Condition (**) has the form

$$2\lambda_o \sinh(t-1)(t-\tfrac{1}{2})u^*(t) = \min_{u \in \langle -1,1\rangle} (2\lambda_o \sinh(t-1)(t-\tfrac{1}{2})u) \quad \text{for} \quad t \in \langle 0, 1\rangle \ \text{a.e.}$$

Hence it appears that the optimal control u^* has the form

$$(3.24)\qquad u^*(t) = \begin{cases} -1 & \text{for} \quad t \in \langle 0, \tfrac{1}{2}\rangle \\ 1 & \text{for} \quad t \in (\tfrac{1}{2}, 1\rangle. \end{cases}$$

Consequently, the optimal trajectory x^* satisfies the equation

$$\dot{x}^*(t) = \int_0^t x^*(\tau)d\tau - 1, \ x^*(0) = 0, \quad \text{for} \quad t \in \langle 0, \tfrac{1}{2}\rangle.$$

Having solved this equation, we obtain that

$$(3.25)\qquad x^*(t) = -\sinh t \quad \text{for} \quad t \in \langle 0, \tfrac{1}{2}\rangle.$$

For $t \in (\frac{1}{2}, 1\rangle$, the trajectory x^* satisfies the equation

$$\dot{x}^*(t) = \int_{\frac{1}{2}}^t x^*(\tau)d\tau + 1 \quad \text{with the condition} \quad x^*(\tfrac{1}{2}) = -\sinh\tfrac{1}{2}.$$

From this, after solving it, we have

$$(3.26)\qquad x^*(t) = (-\sinh\tfrac{1}{2})\cosh(t - \tfrac{1}{2}) + \sinh(t - \tfrac{1}{2}) \quad \text{for} \ t \in (\tfrac{1}{2}, 1\rangle.$$

Making use of (3.24), (3.25) and (3.26), we get

$$\min (I(x, u)) = I(x^*, u^*) = 2(1 - \cosh\tfrac{1}{2})(1 - 2\sinh\tfrac{1}{2}) - \sinh^2 \tfrac{1}{2}.$$

4. ON A SYSTEM OF DIFFERENTIAL-INTEGRAL EQUATIONS OF FREDHOLM TYPE

Let a differential-integral equation of the form

$$(4.1)\qquad \dot{x}(t) = A(t)x(t) + \int_0^1 G(t, \tau)x(\tau)d\tau + u(t)$$

with the initial condition

$$(4.2)\qquad x(0) = 0$$

be given, where $x(\cdot)$ and $u(\cdot)$ are n-dimensional vector functions, while $A(\cdot)$ and $G(\cdot, \cdot)$ are functions with matrix values of degree n.

Assume that the functions $A(\cdot)$ and $G(\cdot, \cdot)$ are bounded and continuous in the interval (0, 1) and in the square $(0, 1) \times (0, 1)$, respectively, with that $||A|| \le \alpha$, $||G|| \le \gamma$ and $\alpha + \gamma < 1$, whereas $u(\cdot) \in L_1^n[0, 1]$.

Solutions $x(\cdot)$ of equation (4.1) are assumed to be elements of the space $W_{11}^n[0, 1]$.

Each function $x(\cdot) \in W_{11}^n[0, 1]$ satisfying (4.2) can be re-presented in the form $x(t) = \int_0^t v(s)ds$ where $v(\cdot) \in L_1^n[0,1]$. Hence equation (4.1) with condition (4.2) is equivalent to the system of equations

$$(4.3)\qquad v(t) = \int_0^1 \Big(A(t) \int_0^t v(s)ds + G(t, \tau) \int_0^\tau v(s)ds\Big)d\tau + u(t),$$

$$(4.4)\qquad x(t) = \int_0^t v(s)ds,$$

where $v(\cdot) \in L_1^n[0, 1]$. By introducing a linear integral operator F defined by the equality

$$Fv(t) = \int_0^1 \Big(A(t) \int_0^t v(s)ds + G(t, \tau) \int_0^\tau v(s)ds\Big)d\tau,$$

equation (4.3) can be written down in the form

$$v(t) = u(t) + Fv(t)$$

and solved by means of the method of successive approximations.

Taking $v_o(t) = 0$ as the zero approximation, we shall obtain a sequence of approximations with the general term

(4.5) $\quad v_n(t) = u(t) + Fu(t) + \dots + F^{n-1}u(t), \quad n = 1, 2, \dots,$

where the iteration F^n of the operation F is expressed by the recurrence formula

$$F^1u(t) = Fu(t),$$

$$F^nu(t) = F(F^{n-1}u)(t) = \int_0^1 (A(t)\int_0^t (F^{n-1}u)(s)ds +$$

$$+ G(t, \tau)\int_0^\tau (F^{n-1}u)(s)ds)d\tau \quad \text{for} \quad n = 2, 3, \dots$$

Sequence (4.5) is a sequence of partial sums with integrable components of a series of functions of the form

(4.6) $\quad u(t) + Fu(t) + F^2u(t) + \dots + F^nu(t) + \dots,$

called a Neumann series.

Let $\beta = \int_0^1 ||u(s)||ds$. It is not difficult to show that, for every positive integer n, the estimate $||F^nu(t)|| \leq (\alpha + \gamma)^n\beta$ is true when $\alpha + \gamma < 1$. So, the series $\sum_{n=0}^{\infty} (\alpha + \gamma)^n\beta$ is the convergent majorant of series (4.6). In consequence, (4.6) is a uniformly convergent series. Hence (cf. [3], II. § 5) its sum $v(t)$ is a unique solution of equation (4.3). Having the solution of (4.3), we find at once a solution of equation (4.4), being a solution of equation (4.1) with condition (4.2). In view of the above, we have

THEOREM 4.1. If $A(\cdot)$ and $G(\cdot, \cdot)$ are bounded and continuous functions in the interval (0, 1) and in the square (0, 1) × × (0, 1), respectively, with that $||A|| + ||G|| < 1$, whereas $u(\cdot)$ is an integrable function in the interval <0, 1>, then there exists one and only one absolutely continuous solution of equation (4.1) with condition (4.2), given by the formula $x(t) = \int_0^t v(s)ds$ where $v(\cdot)$ is the sum of uniformly convergent Neumann series (4.6).

A similar theorem can be formulated under the assumption that $u(\cdot)$, $A(\cdot)$ and $G(\cdot, \cdot)$ are square-integrable functions, with that $||A|| + ||G|| < 1$ (cf. [3], II. § 10).

5. EXTREMUM PROBLEMS WITH DIFFERENTIAL-INTEGRAL CONSTRAINTS OF FREDHOLM TYPE

Replacing constraint (3.2) in Problems 3.1 and 3.2 by the equation

$$\dot{x}(t) = A(t)x(t) + \int_0^1 G(t, \tau)x(\tau)d\tau + S(t)u(t),$$

we shall obtain Problems 3.1′ and 3.2′, respectively, which are called extremum problems with differential-integral constraints of Fredholm type.

Let us adopt assumptions 1^o and 3^o from § 3 and assume that

4^o $A(\cdot)$ and $G(\cdot, \cdot)$ are functions with matrix values of degree n, with elements bounded and continuous in the interval (0, 1) and in the square (0, 1) x (0, 1), respectively, with that $||A|| + ||G|| < 1$.

Reasoning analogously as in § 3, one can prove

THEOREM 5.1. If assumptions 1^o, 3^o and 4^o are satisfied and the pair (x^*, u^*), where $x^* \in W^n_{11}[0, 1]$ and $u^* \in L^r_\infty[0, 1]$, is a solution of Problem 3.1′, then there exist a function $y \in W^n_{11}[0, 1]$ and a number $\lambda_o \geq 0$, not vanishing simultaneously and such that

$$(*) \quad \dot{y}(\tau) = \lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_0^1 G^T(t, \tau)y(t)dt$$

and

$$(**) \quad (\lambda_o b^T(t) - S^T(t)y(t), u^*(t)) = \min_{u \in U} (\lambda_o b^T(t) - S^T(t)y(t), u) \quad \text{for } t \in \langle 0, 1\rangle \text{ a.e.}$$

In an analogous way one can also prove

COROLLARY 5.1. If assumptions 1^o, 3^o and 4^o are satisfied and the pair (x^*, u^*) is a solution of Problem 3.2′, then there exist a function $y \in W^n_{11}[0, 1]$ and a number $\lambda_o \geq 0$, not vanishing simultaneously and such that

$(*_1)\ \dot{y}(\tau) = \lambda_o a^T(\tau) - A^T(\tau)y(\tau) - \int_0^1 G^T(t, \tau)y(t)dt,\ y(1) = 0,$

and condition (**) holds.

E x a m p l e 5.1. Minimize the functional

$$I(x, u) = \int_0^1 (x(t) - u(t))dt$$

under the conditions

$$(5.1)\quad \dot{x}(t) = \int_0^1 \tfrac{1}{2}x(t)dt,\quad x(0) = 0,$$

$u(t) \in U = \langle 0, 1\rangle$ for $t \in \langle 0, 1\rangle$ a.e.

Let (x^*, u^*) be a solution of this problem. Equation $(*_1)$ has, in this case, the form

$$(5.2)\quad \dot{y}(\tau) = \lambda_o - \int_0^1 \tfrac{1}{2}y(t)dt,\quad y(1) = 0.$$

Putting $y(\tau) = -\int_\tau^1 v(s)ds$, from (5.2) we get an integral equation of the form

$$v(\tau) = \lambda_o + \int_0^1 (\tfrac{1}{2}\int_t^1 v(s)ds)dt$$

whose the solution is the function $v(\tau) = \frac{4}{3}\lambda_o$. Consequently, the solution of equation (5.2) is the function $y(\tau) = \frac{4}{3}\lambda_o(\tau - 1)$. Since $\lambda_o \geq 0$ and $y(\cdot)$ do not vanish simultaneously, the form of the function $y(\cdot)$ implies that $\lambda_o > 0$. Condition (**) has here the form

$$\frac{4\lambda_o}{3}(\tfrac{1}{4} - t)u^*(t) = \min_{u\in\langle 0,1\rangle} (\frac{4\lambda_o}{3}(\tfrac{1}{4} - t)u) \text{ for } t\in\langle 0, 1\rangle \text{ a.e}$$

It implies that

$$(5.3)\quad u^*(t) = \begin{cases} 0 & \text{for } t \in \langle 0, \tfrac{1}{4}\rangle \\ 1 & \text{for } t \in (\tfrac{1}{4}, 1\rangle. \end{cases}$$

Hence and from (5.1) we have

$$(5.4)\quad \dot{x}^*(t) = \int_0^1 \tfrac{1}{2}x^*(t)dt \quad \text{for } t \in \langle 0, \tfrac{1}{4}\rangle$$

and

(5.5) $\dot{x}^*(t) = \int_0^1 \frac{1}{2}x^*(t)dt + 1$ for $t \in (\frac{1}{4}, 1\rangle$,

with that $x^*(0) = 0$.

From (5.4) we get

(5.6) $x^*(t) = 0$ for $t \in \langle 0, \frac{1}{4}\rangle$,

and from (5.5), after the condition $x^*(\frac{1}{4}) = 0$ has been taken into account, it follows that

(5.7) $x^*(t) = \frac{2}{7}(4t - 1)$ for $t \in (\frac{1}{4}, 1\rangle$.

Making use of (5.3), (5.6) and (5.7), we obtain that

$$\min(I(x, u)) = I(x^*, u^*) = -\frac{3}{7}.$$

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PROBLEMY OPTYMALIZACYJNE UKŁADÓW DYNAMICZNYCH OPISANYCH RÓWNANIAMI RÓŻNICZKOWO-CAŁKOWYMI TYPU VOLTERRA I FREDHOLMA

W pracy rozważa się twierdzenia o istnieniu rozwiązań liniowych równań różniczkowo-całkowych jak również dowodzi się warunki konieczne istnienia ekstremum dla zadań optymalizacyjnych opisanych przez tego typu równania.