

Tadeusz Gerstenkorn* Joanna Gerstenkorn**

REMARKS ON THE GENERALIZED DOUBLY
TRUNCATED GAMMA DISTRIBUTION

Abstract. In the present paper we discuss a doubly truncated generalized gamma distribution and give formulae for the moments of this distribution and special cases together with examples of calculations.

Key words: truncated probability distributions, moments, generalized gamma distribution.

1. INTRODUCTION

After introducing the generalized gamma distribution to the literature by Stacy (1962), the investigations of this distribution were carried out by many authors such as Śródka (1964, 1966, 1967, 1970), Stacy and Mihran (1965), Malik (1967), Wasilewski (1967), Harter (1967), Lienhard and Meyer (1967), Rosłonek (1968), Hager and Bain (1970), Podolski (1972), Królikowska (1973), Jakuszenkow (1973, 1974, 1976), Lajkó (1977), Lawless (1980), Achcar and Bolafine (1986), M aswadah (1991).

The importance of this distribution lies in the fact it is a generalization of the gamma, Weibull, Rayleigh and X distributions significant in many technical and economic applications.

In the present paper we deal with a generalized doubly truncated gamma distribution and give formulae for the moments of this distribution together with examples of calculations. This will enable us to apply the distribution and its special cases whenever the range of the variable is essentially bounded to some interval of values. In the practice of operation researches, these may be the cases quite frequently encountered.

* University of Łódź, Faculty of Mathematics.

** University of Łódź, Chair of Statistical Methods.

2. THE RESULTS

The density function of random variable X with the generalized gamma distribution is given by

$$f(x) = \begin{cases} \frac{\alpha \lambda^{\rho/\alpha}}{\Gamma(\rho/\alpha)} x^{\rho-1} e^{-\lambda x^\alpha} & \text{for } x > 0; \alpha, \rho, \lambda > 0, \\ 0 & \text{for } x \leq 0. \end{cases} \quad (1)$$

Apart from many articles in journals, distribution (1) is also easy to find in the lexicon Müller *et al.* 1975 and handbook Gerstenkorn, Śródka (1983) listed in the references.

As some special cases of (1) we have:

- 1) gamma distribution if $\alpha = 1$,
 - 2) Weibull distribution if $\alpha = \rho$,
 - 3) Rayleigh distribution if $\alpha = \rho = 2$,
 - 4) χ distribution if $\alpha = 2, \rho = n$,
(most frequently, $\alpha = 1/2$, see Müller *et al.* 1975, p. 34),
- a) $\alpha = 2, \rho = n = 2$ – the Maxwell distribution with 2 components (most frequently, $\lambda = 1/2\sigma^2$).
b) $\alpha = 2, \rho = n = 3$ – the Maxwell distribution with 3 components (most frequently, $\lambda = 1/2\sigma^2$).

We assume that $f(x)$ is a positive density function of random variable X with $P(X \in (c, d)) = 1$. The density function $f_1(u)$ of random variable U , given by

$$f_1(u) = \begin{cases} f(u)/\int_a^b f(x)dx & \text{for } a \leq u \leq b, \\ 0 & \text{for } u < a \text{ or } u > b \end{cases} \quad (2)$$

is called the density of a doubly truncated distribution of random variable X . We consider a truncated distribution if the values of random variable have to be limited to interval $[a, b]$.

Distributions 1–4 are well known and have been investigated in many papers.

For distribution (1), we have from (2)

$$f_1(u) = \begin{cases} u^{\rho-1} e^{-\lambda u^\alpha} / \int_a^b x^{\rho-1} e^{-\lambda x^\alpha} dx & \text{of } 0 < a \leq u \leq b, \\ 0 & \text{of } u < a \text{ or } u > b. \end{cases} \quad (3)$$

Denote

$$W_n = \int_a^b v^{\rho-1+n} e^{-\lambda v^\alpha} dv, \quad n = 0, 1, 2, \dots \quad (4)$$

Lemma. For (4), we have

$$W_n = \frac{1}{\lambda \alpha} [a^{\rho+n-\alpha} e^{-\lambda a^\alpha} - b^{\rho+n-\alpha} e^{-\lambda b^\alpha} + (\rho+n-\alpha) W_{n-\alpha}]. \quad (5)$$

Proof. We can write

$$W_n = -\frac{1}{\lambda \alpha} \int_a^b \lambda \alpha v^{\alpha-1} e^{-\lambda v^\alpha} v^{\rho+n-\alpha} dv.$$

After integration by parts, we have

$$W_n = -\frac{1}{\lambda \alpha} [(v^{\rho+n-\alpha} e^{-\lambda v^\alpha})|_a^b - (\rho+n-\alpha) \int_a^b v^{\rho-1+n-\alpha} e^{-\lambda v^\alpha} dv] = (5)$$

We denote by m_n the moment of order n of random variable U with density function (3), that is,

$$m_n = E(U^n) = \int_a^b u^n f_1(u) du. \quad (6)$$

From (6) and (4) we immediately have

Theorem. If random variable U has density function (3) of the truncated generalized gamma distribution (1), then, for $n \geq 1$,

$$m_n = W_n / W_0 \quad (7)$$

where W_n is given by (4).

Examples.

We calculate m_2 in the case of truncated generalized gamma distribution (3). From (7) and (5) we have

$$m_2 = \frac{1}{\lambda \alpha W_0} [a^{\rho+2-\alpha} e^{-\lambda a^\alpha} - b^{\rho+2-\alpha} e^{-\lambda b^\alpha} + (\rho+2-\alpha) W_{2-\alpha}].$$

We consider the following special cases:

1. Let $\alpha = 1$, $\lambda = 1$; then (gamma distribution)

$$\begin{aligned} m_2 &= (a^{\rho+1} e^{-a} - b^{\rho+1} e^{-b} + (\rho+1) W_1) / W_0 = \\ &= (a^{\rho+1} e^{-a} - b^{\rho+1} e^{-b} + (\rho+1)(a^\rho e^{-a} - b^\rho e^{-b} + \rho W_0)) / W_0. \end{aligned}$$

We notice that

$$W_0 = \int_a^b v^{\rho-1} e^{-v} dv = \Gamma(p, b) - \Gamma(p, a) \quad (8)$$

where $\Gamma(p, x)$ represents the so-called incomplete gamma function

$$\Gamma(p, a) = \int_0^x v^{\rho-1} e^{-v} dv,$$

which means that one can calculate (8) with the aid e.g. K. Pearson's Tables.

2. If $\rho = 1 = \alpha$ (special case of the Weibull distribution), then

$$W_0 = e^{-a} - e^{-b},$$

therefore

$$\begin{aligned} m_2 &= \{a^2 e^{-2} - b^2 e^{-b} + 2[(a+1)e^{-a} - (b+1)e^{-b}]\}/(e^{-a} - e^{-b}) = \\ &= \{(a^2 + 2a + 2)e^{-a} - (b^2 + 2b + 2)e^{-b}\}/(e^{-a} - e^{-b}). \end{aligned}$$

3. Let $\alpha = \rho = 2$ (Rayleigh distribution). In this case

$$m_2 = W_2/W_0 = \frac{1}{2\lambda W_0} (a^2 e^{-\lambda a^2} - b^2 e^{-\lambda b^2} + 2W_0),$$

however,

$$W_0 = \int_a^b v e^{-\lambda v^2} dv = -\frac{1}{2\lambda} (e^{-\lambda v^2})|_a^b = -\frac{1}{2\lambda} (e^{-\lambda b^2} - e^{-\lambda a^2}),$$

therefore

$$m_2 = \frac{(a^2 + \frac{1}{\lambda})e^{-\lambda a^2} - (b^2 + \frac{1}{\lambda})e^{-\lambda b^2}}{e^{-\lambda a^2} - e^{-\lambda b^2}}.$$

4. Let $\alpha = 2, \rho = 3$ (Maxwell distribution with 3 components), then

$$m_2 = \frac{1}{2\lambda W_0} (a^3 e^{-\lambda a^2} - b^3 e^{-\lambda b^2} + 3W_0)$$

where, in this case

$$W_0 = \int_a^b v^2 e^{-\lambda v^2} dv.$$

After integration by parts, we have

$$W_0 = -\frac{1}{2\lambda} [ve^{-\lambda v^2}|_a^b - \int_a^b e^{-\lambda v^2} dv] = -\frac{1}{2\lambda} [be^{-\lambda b^2} - ae^{-\lambda a^2} - \int_a^b e^{-\lambda v^2} dv] \quad (9)$$

and, further,

$$\int_a^b e^{-\lambda v^2} dv = \frac{1}{\sqrt{2\lambda}} \int_{\sqrt{2\lambda}a}^{\sqrt{2\lambda}b} e^{-t^2/2} dt = \sqrt{\frac{\pi}{\lambda}} [\Phi(\sqrt{2\lambda}b) - \Phi(\sqrt{2\lambda}a)].$$

Using the tables of the normal distribution $\Phi(t)$, we obtain the required result.

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Tadeusz Gerstenkorn, Joanna Gerstenkorn

UWAGI O UOGÓLNONYMF PODWÓJNIE UCIĘTYM ROZKŁADZIE GAMMA

W pracy przedstawiony jest podwójnie ucięty uogólniony rozkład gamma i podane są wzory na momenty tego rozkładu oraz jego przypadków szczególnych wraz z przykładami ich wyliczeń.